Exchange Elements in Rings, and the Equation $XA - BX = I$

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Abstract

The equation $XA - BX = I$ has been well studied in ring theory, operator theory, linear algebra, and other branches of mathematics. In this paper, we show that, in the case where $B^2 = B$, the study of $XA - BX = I$ in a noncommutative ring R leads to several new ways to view and to work with the exchange (or "suitable") elements in R in the sense of Nicholson. For any exchange element $A \in R$, we show that the set of idempotents $E \in R$ such that $E \in RA$ and $I - E \in R(I - A)$ is naturally parametrized by the roots of a certain left-right symmetric "exchange polynomial" associated with A. From the new viewpoints on exchange elements developed in this paper, the classes of clean and strongly clean elements in rings can also be better understood.

§1. Introduction

The origin of this paper lies in the research of Warfield and Nicholson on exchange rings (a.k.a. suitable rings) in the 1970s. By adapting the Crawley-Jónsson theory of direct decompositions of algebraic systems to module categories over rings, Warfield defined in [Wa_1, Wa_2] the notion of modules with the exchange property. In Warfield's theory, a ring R is said to be an *exchange ring* if the left module ${}_RR$ (or equivalently, the right module R_R) has such a property. Exchange rings form a large ring class; for instance, they include all π-regular rings (e.g. (von Neumann) regular rings), all semiperfect rings (e.g. left or right artinian rings), and all (unital) C^*-algebras of real rank zero. In 1977, using the notion of lifting idempotents modulo one-sided ideals, Nicholson defined in [Ni_1] the classes of *left suitable rings* and *right suitable rings,* both of which he proved to be the same as Warfield's class of exchange rings. Shortly thereafter, the study of such rings blossomed into a popular and far-reaching topic for research in ring theory.

While Nicholson only defined suitable rings without taking the "element-wise" viewpoint, it is well known that some of his results in [Ni_1] can be formulated on the level of ring elements. With this adaptation, an element a in an arbitrary ring R is said to be *left suitable* if, for every left ideal $I \subseteq R$ containing $a - a^2$, there exists an idempotent $e \in R$ such that $e - a \in I$. The ring R is then said to be left suitable if all elements in R are left suitable. Right suitable elements and right suitable rings are defined similarly. In both Corner's unpublished work [Cn] (1973) and Nicholson's short paper [Ni_2] (1997), it was proved that left/right suitable elements actually form the same class. Since suitable rings are the same as exchange rings, suitable elements are often called *exchange elements,* especially in places where the word "suitable" can be easily misinterpreted. However, in this paper, we will consistently use the term "suitable elements" (except in the title and in the Abstract); the set of all (left) suitable elements in a ring R will be denoted throughout by $\text{suit}\,(R)$.

The earliest criteria for left suitable elements were given by Monk in [Mk: Theorem 1] and by Corner in [Cn: Proposition 4.6] (although their results were stated only as criteria for exchange rings). Several other criteria were given later by Nicholson in [Ni_1: Proposition 1.1]. Among these, the most popular one is perhaps that of Goodearl (from [GW]) and Nicholson, which states that $a \in \text{suit}\,(R)$ if and only if there exists an idempotent $e \in R\,a$ such that $1 - e \in R\,(1 - a)$. This deeply significant criterion for $a \in \text{suit}\,(R)$ has been used heavily in most work on suitable elements. For instance, it was precisely this criterion which enabled Nicholson to prove in [Ni_2] that left and right suitable elements are the same without any appeal to Warfield's theory of exchange rings. A useful list of the various criteria for (left) suitable elements in a ring can be found in Mohamed's report [Mh: Proposition 2.3].

Ostensibly unrelated to all of the above, linear equations of the form $XA - BX = I$ have attracted much research in many branches of mathematics, including linear algebra, operator theory, and control theory. Since we prefer to take a ring-theoretic viewpoint in this paper, we'll rewrite this equation in the form $xa - bx = 1$, using lower case letters to denote elements in a ring R. In §§2–5, we'll be primarily interested in the equation $xa - bx = 1$ in the case where $b \in R$ is an *idempotent,* which we'll rename f so that its complementary idempotent can be denoted by the standard letter e. It came to us as a total surprise that studying the solvability of the equation $xa - fx = 1$ (for some $f = f^2$) turned out to be *precisely equivalent* to studying the condition $a \in \text{suit}\,(R)$. The following are some of our principal results. More detailed versions of them (and their proofs) will appear, respectively, in (2.1), (3.4), (3.12), (3.14), (4.6), and (4.8).

Theorem A. *A ring element $a \in R$ is left suitable if and only if there exists an idempotent $f \in R$ such that the equation $xa - fx = 1$ is solvable in R.*

In view of this result, a linear equation of the form $xa - fx = 1$ where $f^2 = f$ will be called an **exchange equation** (for $a \in R$) in this paper. The study of such exchange equations will occupy much of our attention in §§2-5.

Our second characterization theorem for left suitable elements consists of a set of *three* additional criteria, the first of which coincides with one that was found by Corner in [Cn: Proposition 4.6]. The first criterion is perhaps logically "the simplest" criterion for left suitable elements so far since it is expressed in a first-order sentence involving only one (existential) variable and one polynomial equation, while both Monk's criterion and the Goodearl-Nicholson criterion involve two (existential) variables and two polynomial equations. The other two criteria for $a \in \text{suit}\,(R)$ are new.

Theorem B. *For any $a, x \in R$, let $P_a(x)$ denote the "exchange polynomial" defined by*

$$P_a(x) = (1 - xa)\,(1 + (1 - a)\,x).$$

Each of the following three conditions is equivalent to the condition that $a \in \operatorname{suit}(R)$:

(1) *The quadratic equation $P_a(x) = 0$ is solvable in R.*
(2) *The quadratic equation $P_a(x)\,a\,(1 - a) = 0$ is solvable in R.*
(3) *The quadratic equation $(1 - a)\,P_a(x)\,a = 0$ is solvable in R.*

From Criterion (1) above, the fact that left suitable elements are right suitable becomes an immediate consequence of the fact that we have a polynomial identity

$$(1.1) \qquad (1 - xa)\,(1 + (1 - a)\,x) = (1 + x\,(1 - a))\,(1 - ax) \quad \text{(in any ring } R\text{)}.$$

Furthermore, for any $a \in \operatorname{suit}(R)$, Theorems A and B lead to a new understanding of the associated idempotent set

$$\mathcal{E}(a) := \{e \in R\,a : \; e^2 = e \text{ and } 1 - e \in R\,(1 - a)\}$$

in terms of the set

$$\mathcal{X}(a) := \{x \in R : \; P_a(x) = 0\},$$

as in the following result.

Theorem C. *For any $a \in \operatorname{suit}(R)$, there is a "fibration" (by which we simply mean a surjective mapping) $\varphi \colon \mathcal{X}(a) \to \mathcal{E}(a)$ where*

$$\varphi(x) = xa + a - xa^2 \quad \textit{for every } x \in \mathcal{X}(a).$$

Using this result and other assorted tools, we prove in §3 the following theorem of a universal nature on the general solution of an exchange equation $xa - fx = 1$ over any ring R in case $a \in R$ is an idempotent.

Theorem D. *For any idempotent $a \in R$, the general solution for an exchange equation $xa - fx = 1$ (where by definition $f = f^2$) is given by $f = 1 - a$ and $x = 2\,(a - ara) + ar + ra - 1$ for any $r \in R$.*

In [Ni_1] and [Ni_3], Nicholson also introduced the notions of clean and strongly clean elements in a ring R. An element $a \in R$ is said to be *clean* if a is the sum of an idempotent and a unit, and a is said to be *strongly clean* if it is the sum of an idempotent and a unit that commute. According to Nicholson [Ni_1: Proposition 1.8(1)], clean elements are always suitable.[1] In §§2–4, the three theorems stated above for suitable elements are shown to have some analogues for clean elements, as follows.

[1] As we have noted earlier in this Introduction, some results of Nicholson from [Ni_1] (and [Ni_2] also) are valid on the element level. Those that are will be quoted and applied in our paper accordingly.

Theorem E. *An element $a \in R$ is clean if and only if there exists an idempotent $f \in R$ and a unit $x \in R$ such that $xa - fx = 1$; or equivalently, there exists a unit $x \in R$ such that $P_a(x) = 0$. For any clean element $a \in R$, there is a bijection*

$$\sigma : \mathrm{U}(\mathcal{X}(a)) \to \mathcal{I}(a),$$

where $\mathrm{U}(\mathcal{X}(a))$ is the set of units in $\mathcal{X}(a)$, and $\mathcal{I}(a)$ is the set of idempotents $e \in R$ such that $a - e$ is a unit. Here, σ is defined by $\sigma(x) = a - x^{-1}$ for every $x \in \mathrm{U}\big(\mathcal{X}(a)\big)$.

There is also an analogue of Theorem E for strongly clean elements, which is given separately in Theorem 2.1(3), Theorem 3.14(B), and Corollary 4.11. With the new understanding and information on suitable and (strongly) clean elements obtained from the above results, a number of basic facts in the theory of suitable rings can be re-obtained, simplified, or extended. To this end, some examples and applications are given in §5. Further applications of the ideas and results of this paper will appear in a forthcoming work.

Throughout this paper, R denotes an arbitrary ring with $1 \in R$, $\mathrm{U}(R)$ denotes the group of units in R, and $Z(R)$ denotes the center of R. We'll write $\mathrm{idem}\,(R)$ and $\mathrm{reg}\,(R)$ for the set of idempotents and the set of (von Neumann) regular elements in R. The sets of clean and strongly clean elements in R will be denoted, respectively, by $\mathrm{cn}\,(R)$ and $\mathrm{scn}\,(R)$. For any $a \in R$, $C_R(a)$ denotes the centralizer of a in R, and $\mathrm{ann}_\ell(a)$, $\mathrm{ann}_r(a)$ denote the left and right annihilators of a. Other standard terminology and notations in ring theory follow those in [La]. Whenever it is more convenient, we'll use the widely accepted shorthand "iff" for "if and only if" in the text.

§2. Characterization of Suitable Elements by Exchange Equations

The main goal of this section is to introduce the idea of an "exchange equation" for a ring element, and to use this notion to give a uniform view of suitable elements, clean and strongly clean elements, as well as regular and strongly regular elements in a ring.

For any ring R, we have trivially $\mathrm{scn}\,(R) \subseteq \mathrm{cn}\,(R)$, but Nicholson's result that $\mathrm{cn}\,(R) \subseteq \mathrm{suit}\,(R)$ is by no means clear from the definitions of clean and suitable elements. The first main result of this section offers a set of parallel membership criteria (1)–(3) below for the three sets $\mathrm{cn}\,(R)$, $\mathrm{scn}\,(R)$ and $\mathrm{suit}\,(R)$, which sheds new light on the inclusion relations $\mathrm{scn}\,(R) \subseteq \mathrm{cn}\,(R) \subseteq \mathrm{suit}\,(R)$ mentioned earlier in this paragraph.

Theorem 2.1. *For any element $a \in R$, the following hold.*
(1) $a \in \mathrm{cn}\,(R)$ *iff there exist* $f \in \mathrm{idem}\,(R)$ *and* $x \in \mathrm{U}(R)$ *such that* $xa - fx = 1$.
(2) $a \in \mathrm{suit}\,(R)$ *iff there exist* $f \in \mathrm{idem}\,(R)$ *and* $x \in R$ *such that* $xa - fx = 1$.
(3) $a \in \mathrm{scn}\,(R)$ *iff there exist* $f \in \mathrm{idem}\,(R)$ *and* $x \in C_R(a)$ *such that* $xa - fx = 1$. *In this case, x is automatically a unit (with inverse $a - f$).*

In particular, $\mathrm{scn}\,(R) \subseteq \mathrm{cn}\,(R) \subseteq \mathrm{suit}\,(R)$.

Proof. (1) This part first appeared in [Št: Lemma 2.1]. For completeness, we give a short proof here. Say $a = \varepsilon + u$, where $\varepsilon \in \text{idem}(R)$ and $u \in \text{U}(R)$. Letting $x := u^{-1}$ gives $1 = x(a - \varepsilon) = xa - fx$ where $f := x\varepsilon x^{-1} \in \text{idem}(R)$. Conversely, if $xa - fx = 1$ as in (1), then $a = x^{-1}fx + x^{-1} \in \text{cn}(R)$.

(2) Suppose $1 = xa - fx$ as in (2). Left-multiplying by $e := 1 - f \in \text{idem}(R)$ yields $e = e(xa) \in Ra$. Also, the given equation can be written as $1 = x(a-1) + ex$, so left-multiplying by f yields $f = (fx)(a-1) \in R(1-a)$. These two steps show that $a \in \text{suit}(R)$. Conversely, if $a \in \text{suit}(R)$, there exist $r, s \in R$ such that $e := ra$ and $f := s(1-a)$ are complementary idempotents. After replacing r by er and s by fs, we may assume that $er = r$ and $fs = s$. These amount to $fr = es = 0$. Letting $x := r - s \in R$, we have

$$xa - fx = (r-s)a - f(r-s) = ra - sa + s = e + s(1-a) = e + f = 1,$$

which gives the desired equation in (2).

(3) The "only if" part follows from the same proof given for (1), after adding the condition that $ua = au$ (which implies that $xa = ax$ where $x = u^{-1}$). Conversely, assume f, x exist as in (3). Since $xa = ax$, the first part of the proof of (2) gives

$$f = (fx)(a-1) = (1 - xa)(1 - a) \in C_R(x).$$

With all this information, $xa - fx = 1$ implies that $x \in \text{U}(R)$ with inverse $a - f$, so now $a = f + x^{-1} \in \text{scn}(R)$. $\square$

In view of the fact that $a \in \text{suit}(R)$ is characterized by the solvability of $xa - fx = 1$ (for some $f = f^2$), we'll henceforth refer to $xa - fx = 1$ as a (left) *exchange equation* for a, and we'll call x a **suitabilizer** for a (via the idempotent f).

Remark 2.2. It is easy to see that the criterion in (1) remains valid if we replace the condition $xa - fx = 1$ by $xa - fx \in \text{U}(R)$. Indeed, if $w := xa - fx \in \text{U}(R)$, then $(w^{-1}x)a - (w^{-1}fw)(w^{-1}x) = 1$. Since $w^{-1}x \in \text{U}(R)$ and $w^{-1}fw \in \text{idem}(R)$, we get back the condition in (1). Similarly, the criterion in (2) remains valid if we replace the condition $xa - fx = 1$ by $xa - fx \in \text{U}(R)$, and the criterion in (3) remains valid if we replace the condition $xa - fx = 1$ by $xa - fx \in \text{U}(R) \cap C_R(a)$.

Remark 2.3. If we fix an exchange equation $xa - fx = 1$ (where $f^2 = f$) for a given $a \in \text{suit}(R)$, it is also easy to verify directly the characteristic "lifting property" of the suitable element a; namely, there exists $e \in \text{idem}(R)$ such that $e - a \in R(a - a^2)$. (See [Ni$_1$: Proposition 1.1(2)].) Indeed, if we let $e := 1 - f$, then

$$(*) \qquad x(a - a^2) = (1 + fx)(1 - a) = 1 - a + f\big(x - (1 + fx)\big) = e - a.$$

This formula gives a new interesting property of the suitabilizer x. (Later, we'll see that the idempotent f is actually uniquely determined by a and x.)

Remark 2.4. If S is any (left) suitable ring, it is well known (from [Wa$_2$] and [Ni$_1$]) that any matrix ring $R = \mathbb{M}_n(S)$ is also (left) suitable. Thus, it follows from part (2) of

Theorem 2.1 that, for any matrix $A \in R$, there exist matrices $X, F \in R$ with $F^2 = F$ such that $XA - FX = I_n$. Similarly, if S is a clean ring, then so is R by [HN]. In this case, X above can even be chosen to be in $\mathrm{U}(R) = \mathrm{GL}_n(S)$.

Next, we note that Theorem 2.1(2) leads to a simple (but completely different) proof of the following result of Nicholson from [Ni_1: Proposition 1.8].

Corollary 2.5. *If R is an abelian ring (in the sense that every idempotent in R is central), then* $\mathrm{scn}\,(R) = \mathrm{cn}\,(R) = \mathrm{suit}\,(R)$.

Proof. It suffices to prove that any $a \in \mathrm{suit}\,(R)$ is strongly clean. By Theorem 2.1(2), there exists an exchange equation $xa - fx = 1$ where $x \in R$ and $f \in \mathrm{idem}\,(R)$. Here f is central, so we have $x\,(a - f) = 1$. This implies that $x \in \mathrm{U}(R)$ since the abelian ring R must be Dedekind-finite. Therefore, $a = f + x^{-1} \in \mathrm{scn}\,(R)$. □

In the study of suitable elements in rings, it is well known that the set $\mathrm{suit}\,(R)$ has the "n-th root property", in the sense that, if some n-th power a^n is suitable in R (where $n \geq 1$), then a is suitable in R. This can be seen quickly, for instance, from the Goodearl-Nicholson characterization of suitable elements that was mentioned in the Introduction. This "n-th root property" for suitable elements led us to the formulation of the following curious property on the solvability of exchange equations.

Proposition 2.6. *Let $a \in R$, $f \in \mathrm{idem}\,(R)$, and let n be any positive integer. If $ya^n - fy = 1$ has a solution $y \in R$, then $xa - fx = 1$ has a solution $x \in R$ that is a universal (noncommutative) polynomial in a, f, and y.*

Proof. Starting with $ya^n - fy = 1$, we let

$$e := 1 - f, \quad \text{and} \quad x := eya^{n-1} + fy\,(a^{n-1} + \cdots + 1) \in R.$$

Then $fx = fy\,(a^{n-1} + \cdots + 1)$, so we have

$$\begin{aligned} xa - fx &= eya^n + fy\,(a^{n-1} + \cdots + 1)\,(a - 1 + 1) - fy\,(a^{n-1} + \cdots + 1) \\ &= eya^n + fy\,(a^n - 1) = ya^n - fy = 1, \end{aligned}$$

as desired. (In particular, applying Theorem 2.1(2) twice, we recapture the fact that $a^n \in \mathrm{suit}\,(R) \Rightarrow a \in \mathrm{suit}\,(R)$.) □

From the proof of [Ni_1: Proposition 1.6], it is well known that $\mathrm{reg}\,(R)$, the set of (von Neumann) regular elements of R, is contained in $\mathrm{suit}\,(R)$. In the proposition below, we'll retrieve and extend this classical result by providing a couple of new characterizations for regular elements in a ring R in terms of the exchange equations.

Proposition 2.7. *For any $a \in R$, the following three statements are equivalent:*

(1) $a \in \mathrm{reg}\,(R)$.

(2) *There exist $x, x' \in R$ and $f \in \mathrm{ann}_r(a)$ such that $xa - fx' = 1$. Here, xa and $-fx'$*

are necessarily a pair of complementary idempotents in R.

(3) *There exist* $x \in R$ *and* $f \in \mathrm{idem}\,(R) \cap \mathrm{ann}_r(a)$ *such that* xa *and* $-fx$ *are a pair of complementary idempotents in* R. (*In particular,* $xa - fx = 1$.)

Finally, every $a \in \mathrm{reg}\,(R)$ *has a suitabilizer* x *that is an inner inverse of* a (i.e. *such that* $axa = a$); *in particular,* a *must be suitable.*

Proof. To begin with, (3) $\Rightarrow$ (2) is a tautology.

(2) $\Rightarrow$ (1). For x, x', f as in (2), left multiplying the equation $xa - fx' = 1$ by a gives $axa = a$, so (1) holds. Also, since $a = axa \Rightarrow (xa)^2 = xa$, it follows that xa and $1 - xa = -fx'$ are a pair of complementary idempotents.

(1) $\Rightarrow$ (3). Assuming (1), take $s \in R$ such that $a = asa$. Let $p := sa \in \mathrm{idem}\,(R)$, $q := 1 - p$, and $f := q\,(1 - a)$. Then $aq = (ap)\,q = 0$ implies that $af = 0$ as well as $f^2 = q\,[(1-a)\,q]\,(1-a) = q^2(1-a) = f$; that is, $f \in \mathrm{idem}\,(R) \cap \mathrm{ann}_r(a)$. Letting $x := sas - q\,(1 - as)$, we have $xa = sa = p$, so $axa = asa = a$. Also,

$$fx = (q - qa)\,(sas - q + qas) = qsas - q + qas - qas = -q,$$

since $aq = 0$ and $qsa = qp = 0$. Therefore, $xa = p$ and $-fx = q$ are complementary idempotents, which proves (3).

The last part of the theorem now follows from the two equations $axa = a$ and $xa - fx = 1$. □

Remark 2.8. While we have shown that every $a \in \mathrm{reg}\,(R)$ has an inner inverse that is a suitabilizer, "being an inner inverse of a" and "being a suitabilizer of a" turn out to be logically independent properties. Examples illustrating this will be given later in (5.9). (However, see also Example 5.3.)

Combining the two propositions above leads immediately to the fact that every π-regular element (i.e. an element with a positive power in $\mathrm{reg}\,(R)$) is suitable.

Corollary 2.9. *If* $a^n \in \mathrm{reg}\,(R)$ *for some* $n \geq 1$, *then* $a \in \mathrm{suit}\,(R)$.

We take this opportunity to also record some characterizations of strongly regular elements in terms of exchange equations. Recall that a ring element $a \in R$ is called *strongly regular* if $a \in R\,a^2 \cap a^2 R$. It is well known that this is equivalent to the existence of some $e \in \mathrm{idem}\,(R)$ and some $u \in \mathrm{U}(R)$ such that $a = eu = ue$.

Proposition 2.10. *For any* $a \in R$, *the following statements are equivalent*:

(1) a *is strongly regular.*

(2) *There exist* $x \in C_R(a)$, $x' \in R$, *and* $f \in \mathrm{ann}_r(a)$ *such that* $xa - fx' = 1$. (*Here,* xa *and* $-fx'$ *are necessarily a pair of complementary idempotents in* R.)

(3) *There exist* $x \in C_R(a)$ *and* $f \in \mathrm{idem}\,(R) \cap \mathrm{ann}_r(a)$ *such that* $xa - fx = 1$.

Proof. As in the proof of Proposition 2.7, (3) $\Rightarrow$ (2) is a tautology, and (2) $\Rightarrow$ (1) is easily proved by left multiplying $xa - fx' = 1$ by a.

(1) $\Rightarrow$ (3). Write $a = eu = ue$ where $e \in \text{idem}(R)$ and $u \in \text{U}(R)$. For $f := 1 - e \in \text{idem}(R)$, $a - f = eu - f$ has inverse $x := eu^{-1} - f \in C_R(a)$. Therefore,

$$1 = (a - f)\, x = ax - fx = xa - fx,$$

with $af = (ue)\, f = 0$. This checks (3). (From (2), we expect that xa and $-fx$ are a pair of complementary idempotents. Indeed, a quick computation shows that they are simply the complementary idempotents e and f!) $\square$

In generalization of the notion of strongly regular elements, an element $a \in R$ is said to be *strongly π-regular* if there exists an integer $n \geq 1$ such that $a^n \in R\, a^{n+1} \cap a^{n+1} R$. We note in passing that the following result of Nicholson in [Ni$_3$] can be seen very quickly from the proof of Proposition 2.10.

Corollary 2.11. *Let $a \in R$ be a strongly π-regular element. Then $a \in \text{scn}(R)$.*

Proof. By Fitting's decomposition theory (or by [Ni$_3$: Proposition 1]), there exists an integer $n \geq 1$ such that $a^n = eu = ue$ where $e \in \text{idem}(R)$ and $u \in \text{U}(R)$ both commute with a. Let $f := 1 - e \in \text{idem}(R) \cap C_R(a)$. The proof of (1) $\Rightarrow$ (3) in Proposition 2.10 (applied to a^n) shows that $a^n - f^n = a^n - f \in \text{U}(R)$. Since $af = fa$, we have $a - f \in \text{U}(R)$ and hence $a \in \text{scn}(R)$. (Of course, exchange equations were implicitly at work in this argument!) $\square$

§3. Further Characterizations of Suitable Elements

In preparation for giving a new set of characterizations for left suitable elements, we first present an elementary lemma below on the two sets $\text{reg}(R)$ and $\text{idem}(R)$ which holds for any ring R. This lemma will serve as the workhorse for the entire section. For the rest of the paper, we associate to each pair of elements $a, x \in R$ the following important "**exchange polynomial**"[2]

$$P_a(x) := (1 - xa)\,(1 + (1 - a)\, x) \in R, \tag{3.1}$$

which arises naturally from the first step of the proof of this lemma.

Lemma 3.2. *For any two elements $a, x \in R$, the following hold.*

(1) $1 - xa$ *is regular with inner inverse* $1 - a$ *iff* $P_a(x)\, a = 0$.

(2) $(1 - xa)\,(1 - a) \in \text{idem}(R)$ *iff* $P_a(x)\, a\, (1 - a) = 0$.

(3) $(1 - a)\,(1 - xa) \in \text{idem}(R)$ *iff* $(1 - a)\, P_a(x)\, a = 0$. *In this case, there exists $y \in R$ such that* $(1 - ya)\,(1 - a) \in \text{idem}(R)$.

Proof. Note that $1 - xa - (1 - xa)\,(1 - a)\,(1 - xa)$ can be factored into

$$(1 - xa)\,[1 - (1 - a - xa + axa)] = (1 - xa)\,(1 + (1 - a)\, x)\, a = P_a(x)\, a.$$

[2]More precisely speaking, of course, $P_a(x)$ is just a polynomial expression in $x \in R$, with $a \in R$ playing the role of a (non-commuting) parameter.

This proves (1). Right multiplying this calculation by $1-a$ then gives (2). Similarly, left multiplying the same calculation by $1-a$ gives the first statement in (3). To prove the second statement in (3), assume that $(1-a)(1-xa) \in \text{idem}(R)$. Recall the nice fact that if a product pq is an idempotent, so is $qpqp$. Thus,

$$(\dagger) \qquad (1-xa)(1-a)(1-xa)(1-a) \in \text{idem}(R).$$

By expansion, we have $(1-xa)(1-a)(1-xa) = 1-ya$ for some $y \in R$. Now (†) gives $(1-ya)(1-a) \in \text{idem}(R)$, as desired. □

Next, we derive a number of useful algebraic consequences of an exchange equation $xa - fx = 1$ (again for any ring R).

Proposition 3.3. *Let $f \in \text{idem}(R)$, and let $a, x \in R$ be such that $xa - fx = 1$. Then the following hold.*

(A) *The suitabilizer x for a (via f) is in $\text{reg}(R) + \text{reg}(R)$.*

(B) *$-x$ is a suitabilizer for $1-a$ (via the idempotent $1-f$).*

(C) *$f = (1-xa)(1-a)$ (so $f \in \text{idem}(R)$ is uniquely determined by a and x).*

(D) *$P_a(x) = 0$; that is, x is a root of the exchange polynomial.*

(E) *$1 - xa = -fx \in \text{reg}(R)$, with inner inverse $1-a$.*

Proof. (A) Let $e := 1-f$. By the proof of (2) in Theorem 2.1, we have $e = exa$, and $f = -(fx)(1-a)$. Let $r = ex$, and $s = -fx$. Then $r = er = rar \in \text{reg}(R)$. Similarly, we can show that $s \in \text{reg}(R)$, so $x = (e+f)x = r - s \in \text{reg}(R) + \text{reg}(R)$.

(B) We check directly that $(-x)(1-a) - (1-f)(-x) = xa - fx = 1$.

(C) In the proof of the "if" part of Theorem 2.1(2), we have already shown that

$$f = fx(a-1) = (1-xa)(1-a).$$

(D) From (C), $1 - xa = -fx = -(1-xa)(1-a)x$. Transposition gives then $P_a(x) = 0$.

(E) This follows from (D) and Lemma 3.2(1). But (E) can also be proved directly from the exchange equation, as follows. Since

$$(fx)(a-1)(fx) = [f(1+fx) - fx]\, fx = f^2 x = fx,$$

we see that $1 - xa = -fx$ is regular, with inner inverse $1-a$. □

We come now to our second characterization theorem ("Theorem B" in §1) for left suitable elements, which consists of *three* different criteria each amplifying the theme of part (2) of Theorem 2.1. After we discovered the first criterion (in (1) below, through the use of exchange equations), we realized that it had also appeared in Corner's 1973 unpublished manuscript [Cn], which had remained totally unknown to the ring theory community until it was cited in 2010 in Mohamed's work [Mh]. (Actually, Corner's result was stated in [Cn: Proposition 4.6] only for exchange rings. The fact that this result holds already at the element level was pointed out by Mohamed in [Mh: Proposition 2.3].) The second and third criteria for left suitable elements ((2) and (3) below) are our further refinements of the first criterion.

Theorem 3.4. *For any* $a \in R$, *each of the following three conditions is equivalent to the condition that* $a \in \text{suit}\,(R)$:

(1) *The quadratic equation* $P_a(x) = 0$ *is solvable in* R.
(2) *The quadratic equation* $P_a(x)\,a\,(1-a) = 0$ *is solvable in* R.
(3) *The quadratic equation* $(1-a)\,P_a(x)\,a = 0$ *is solvable in* R.

The roots of the equation $P_a(x) = 0$ *in* R *constitute exactly the set of suitabilizers for* a.

Proof. First assume that $a \in \text{suit}\,(R)$. Then there exists, by Theorem 2.1(2), an exchange equation $xa - fx = 1$ (with $f = f^2$). By part (D) of Proposition 3.3, we have $P_a(x) = 0$, so the three statements (1), (2), (3) all hold. For the rest, it suffices to show that each one of (2) and (3) implies $a \in \text{suit}\,(R)$. First assume (2); that is, $P_a(x)\,a\,(1-a) = 0$ for some $x \in R$. By Lemma 3.2(2), $f := (1-xa)\,(1-a)$ is an idempotent in $R\,(1-a)$. Its complementary idempotent

$$1 - f = 1 - (1-xa)\,(1-a) = a + xa - xa^2$$

lies in $R\,a$, so we have $a \in \text{suit}\,(R)$. Finally, assume (3); that is, $(1-a)\,P_a(x)\,a = 0$ for some $x \in R$. By Lemma 3.2(3) there exists $y \in R$ such that

$$f_1 := (1-ya)\,(1-a) \in \text{idem}\,(R) \cap R\,(1-a).$$

As above, we have $1 - f_1 \in \text{idem}\,(R) \cap R\,a$, so again $a \in \text{suit}\,(R)$.

For the last statement of the theorem, we have already remarked at the beginning of this proof that any suitabilizer x for a satisfies $P_a(x) = 0$. Conversely, note that for any $a, x \in R$, we have the following useful expression for the exchange polynomial:

$$(3.5) \qquad P_a(x) = (1-xa)\,(1+(1-a)\,x) = 1 - xa + fx, \text{ where } f := (1-xa)\,(1-a).$$

If $P_a(x) = 0$, then $f \in \text{idem}\,(R)$ again by Lemma 3.2(2). In this case, (3.5) shows that $xa - fx = 1$, so x is a suitabilizer for a (via the idempotent f). □

For illustration, we offer below two quick consequences of Theorem 3.4.

Corollary 3.6. *If* $a \in \text{suit}\,(R)$, *there exists* $e = e^2 \in R\,a$ *such that* $1 - e \in (1-a)\,R$.

Proof. By Theorem 3.4, $P_a(x) = 0$ for some $x \in R$. Applying Lemma 3.2(3), we have $(1-a)\,(1-xa) \in \text{idem}\,(R) \cap (1-a)\,R$, with complementary idempotent $a + xa - axa \in R\,a$. (This result was first obtained in [KLN] by a different method.[3] Indeed, in [KLN], it was proved that the *converse* of this corollary is also true.) □

[3] In the case where $a^2 \in \text{suit}\,(R)$ (which is a stronger assumption than $a \in \text{suit}\,(R)$), Corollary 3.6 can also be deduced from [NZ: Lemma 1].

Corollary 3.7. *Let* $R = \mathbb{M}_2(S)$ *where* S *is any ring, and let* $a = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in R$. *If* $q \in \mathrm{U}(S)$ *or* $r \in \mathrm{U}(S)$, *then* $a \in \mathrm{suit}\,(R)$.

Proof. Say $r \in \mathrm{U}(S)$ (the case $q \in \mathrm{U}(S)$ being similar). For $x := \begin{pmatrix} 0 & r^{-1} \\ 0 & 0 \end{pmatrix} \in R$, a direct calculation shows that $1 - xa$ has first column zero, and $1 + (1-a)\,x$ has second row zero. Thus, $P_a(x) = 0$, so x is a suitabilizer of a. By Theorem 3.4, $a \in \mathrm{suit}\,(R)$. (In fact, it is even true that $a \in \mathrm{cn}\,(R)$, since taking the idempotent matrix $\varepsilon := \begin{pmatrix} 1 & q-1-(p-1)\,r^{-1}s \\ 0 & 0 \end{pmatrix}$ leads to $a - \varepsilon = \begin{pmatrix} p-1 & 1+(p-1)\,r^{-1}s \\ r & s \end{pmatrix} \in \mathrm{U}(R)$. By the proof of Theorem 2.1(1), a has a *unit* suitabilizer $(a-\varepsilon)^{-1}$, which is, however, not as simple in form as the non-unit suitabilizer $x = \begin{pmatrix} 0 & r^{-1} \\ 0 & 0 \end{pmatrix}$.) $\square$

We note in passing that Corollary 3.7 is a special result for 2×2 matrix rings. If $R = \mathbb{M}_n(S)$ where $n \geq 3$ and $a \in R$ has an off-diagonal unit entry, it does not follow that $a \in \mathrm{suit}\,(R)$. For instance, for $n = 3$, the matrix $a = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ can be shown to be non-suitable in $\mathbb{M}_3\big(\mathbb{Z}\big)$ by Theorem 5.12 below. The case $n \geq 4$ is similar.

Over a general ring R, if $P_a(x)\,a\,(1-a) = 0$, we may not have $P_a(x) = 0$, so x may not be a suitabilizer for a. However, we do know from Theorem 3.4 that $a \in \mathrm{suit}\,(R)$, so a must have a suitabilizer. The following supplement to Theorem 3.4 gives an explicit construction for such a suitabilizer (as well as an exchange equation for a) in this case. Remarkably, the suitabilizer obtained below is just x modulo a right multiple of the "error term" $P_a(x)$.

Theorem 3.8. *Let* $a, x \in R$ *be such that* $P_a(x)\,a\,(1-a) = 0$. *Then* $y := x + P_a(x)\,(2a-1)$ *is a suitabilizer for* a, *satisfying the exchange equation* $ya - fy = 1$ *where* $f := (1-xa)\,(1-a) \in \mathrm{idem}\,(R)$.

Proof. We know from Lemma 3.2(2) that $f \in \mathrm{idem}\,(R) \cap R\,(1-a)$, with complementary idempotent $e = 1-(1-xa)\,(1-a) = (1+x-xa)\,a \in R\,a$. According to the proof of Theorem 2.1(2), an *explicit* suitabilizer for a is given by

(3.9) $y_0 := e\,(x+1-xa) - f(1-xa) = ex + (e-f)\,(1-xa) = (1-f)\,x + (2e-1)\,(1-xa)$,

and we have an accompanying exchange equation $y_0 a - f y_0 = 1$. Here,

$$e\,(1-xa) = (1+x-xa)\,(a-axa) = (1+x-xa)\,(1-ax)\,a = P_a(x)\,a$$

in view of (1.1) (which will be proved in more detail in Lemma 4.1 below). With this information, (3.9) and (3.5) show that

$$y_0 = x - fx - 1 + xa + 2\,P_a(x)\,a = x + P_a(x)\,(2a-1) = y,$$

as desired. (We note incidentally that, by Proposition 3.3(C) applied to $ya-fy=1$, we must have $f=(1-ya)\,(1-a)$ too. Since f was originally defined to be $(1-xa)\,(1-a)$, the new interpretation for f simply means that $y-x\in \mathrm{ann}_\ell\, a\,(1-a)$, which is also predictable from the equation $y-x=P_a(x)\,(2a-1)$.) □

Corollary 3.10. *Let $a,x\in R$ be such that $P_a(x)\,a=0$. Then $z:=x-P_a(x)$ is a suitabilizer for a, satisfying the exchange equation $za-fz=1$ where $f:=(1-xa)\,(1-a)\in \mathrm{idem}\,(R)$. However, we may have $a\,(1-a)\,P_a(x)\neq 0$ (even in the case where R is a simple artinian ring).*

Proof. The first conclusion is just a special case of Theorem 3.8. For the second conclusion, consider a matrix ring $R=\mathbb{M}_2(S)$ over any nonzero ring S of characteristic $\neq 2$ (e.g. $S=\mathbb{Q}$). For the matrices $a=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $x=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ in R, a quick computation shows that $P_a(x)\,a=0$, but $a\,(1-a)\,P_a(x)=\begin{pmatrix}0&2\\0&0\end{pmatrix}\neq 0$. □

Remark 3.11. In the special case where $a\in \mathrm{reg}\,(R)$, any inner inverse s for a satisfies

$$P_a(s)\,a=(1-sa)\,(1+s-as)\,a=(1-sa)\,sa=0$$

(as well as $a\,P_a(s)=0$). Therefore, Corollary 3.10 applies in this case to show that $s-P_a(s)$ is an explicit suitabilizer for a. Indeed, it is easy to check that this is precisely the suitabilizer for $a=asa\in \mathrm{reg}\,(R)$ constructed in the proof of Proposition 2.7(3). In this way, the rather *ad hoc* proof of Proposition 2.7(3) is now placed in the more general framework of Theorem 3.8 and Corollary 3.10.

We should like to point out that in the special case where $a\in \mathrm{idem}\,(R)$, Theorem 3.8 leads to a general parametric solution for all (left) exchange equations $ya-fy=1$ over any ring. For applications in more familiar contexts, for instance, the following consequence of Theorem 3.8 can be profitably applied to any ring of operators R, or any matrix ring $R=\mathbb{M}_n(S)$ over a classical ring S (such as $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Z}$).

Theorem 3.12. *Let $a,f\in \mathrm{idem}\,(R)$, and $y\in R$. The following two statements are equivalent:*

(1) *The exchange equation $ya-fy=1$ holds.*
(2) *$f=1-a$, and $y=2\,(a-axa)+ax+xa-1$ for some $x\in R$.*

Proof. We first prove (2) ⇒ (1). For any $x\in R$, the equation $P_a(x)\,a\,(1-a)=0$ is *trivially* satisfied (since $a^2=a$). Therefore, Theorem 3.8 yields the following suitabilizer for a via the idempotent $f=(1-xa)\,(1-a)=1-a$:

(3.13) $y:=x+P_a(x)\,(2a-1)=x+(1+x-ax-xa)\,(2a-1)=2\,(a-axa)+ax+xa-1.$

This proves that $ya-fy=1$. (For instance, for $x=0$, (3.13) gives the special unit solution $y=2a-1$ for the exchange equation.) Conversely, if $y\in R$ satisfies an exchange equation $ya-fy=1$ (where by definition $f=f^2$), then Proposition 3.3(C) implies that $f = (1-ya)(1-a) = 1-a$. In addition, $P_a(y) = 0$ by Proposition 3.3(D), so $y = y + P_a(y)(2a-1) = 2(a-aya)+ay+ya-1$ by the computation in (3.13). Given hindsight, this equation for y can also be seen more directly as follows. Left multiplying $ya-(1-a)y = 1$ by a gives $aya = a$, so $2(a-aya)+ay+ya-1 = ay+ya-1 = y$. (We note however that, in general, not every inner inverse of a would solve the equation $ya-(1-a)y = 1$.) □

Returning to Theorem 3.4, our next goal is to show that a somewhat modified form of the reasoning behind the equivalence (1) $\Leftrightarrow$ $a \in \mathrm{suit}(R)$ in that theorem can be applied to handle the case of clean and strongly clean elements. This quickly leads to the result below.

Theorem 3.14. *For any ring element $a \in R$, the following hold.*

(A) $a \in \mathrm{cn}(R)$ *iff there exists a unit* $x \in R$ *such that* $P_a(x) = 0$.

(B) *Each of the following two conditions is equivalent to* $a \in \mathrm{scn}(R)$:

(1)$'$ *There exists* $x \in C_R(a)$ *such that* $P_a(x) = 0$.

(2)$'$ *There exists* $x \in C_R(a)$ *such that* $P_a(x)\,a(1-a) = 0$.

Any element x *as in* (1)$'$ *is necessarily a unit in* R.

Proof. (A) follows from Theorem 2.1 and the last statement in Theorem 3.4.

(B) If $a \in \mathrm{scn}(R)$, the same argument used to prove (A) also proves (1)$'$, which clearly implies (2)$'$. To prove that (2)$' \Rightarrow a \in \mathrm{scn}(R)$, suppose there exists $x \in C_R(a)$ such that $P_a(x)\,a(1-a) = 0$. Applying Theorem 3.4 to the *commutative* subring $S \subseteq R$ generated by $\{a, x\}$, it follows that $a \in \mathrm{suit}(S)$. But $\mathrm{suit}(S) = \mathrm{scn}(S)$ by [Ni$_1$: Proposition 1.8(2)],[4] so $a \in \mathrm{scn}(R)$. Finally, the last statement of the theorem follows by expanding the left-hand side of the equation $P_a(x) = 0$ (in the case $x \in C_R(a)$). □

Remark 3.15. In view of the criteria (2), (3) for suitable elements in Theorem 3.4 and the criterion (2)$'$ for strongly clean elements in Theorem 3.14(B), it is natural to ask whether the existence of a *unit* $x \in R$ such that $P_a(x)\,a(1-a) = 0$ (without the commuting condition $ax = xa$) would imply that $a \in \mathrm{cn}(R)$. Somewhat surprisingly, the answer to this question is "no". Indeed, let $R = \mathbb{M}_2(\mathbb{Z})$, and consider a matrix $a = \begin{pmatrix} m & n \\ 0 & 0 \end{pmatrix} \in R$ where $m, n \in \mathbb{Z}$ satisfy a unimodular equation $mn' + nm' = 1$. Then a has a unit inner inverse $x = \begin{pmatrix} n' & -n \\ m' & m \end{pmatrix}$. By Remark 3.11, we have then $P_a(x)\,a = 0$ (as well as $a\,P_a(x) = 0$). Thus, (2)$'$ in Theorem 3.14(B) is certainly satisfied for some unit x *without* the requirement that $ax = xa$. According to [KL: (4.5)], $a \notin \mathrm{cn}(R)$ for many choices of (m, n) (e.g. $(m, n) = (12, 5)$, $(12, 7)$, or $(13, 5)$).

Remark 3.16. In summary of Theorem 3.4 and Theorem 3.14, it is useful to keep track of the number of variables and equations that were used in them for characterizing the

[4]The use of this result of Nicholson on abelian rings from [Ni$_1$] can easily be avoided. We just used it here for a quick finish.

elements in $\operatorname{suit}(R)$, $\operatorname{cn}(R)$, and $\operatorname{scn}(R)$. For clean elements, it is two variables and three equations. For strongly clean elements, it is one variable and two equations. But for suitable elements, quite remarkably, it is one variable and one equation!

§4. Left-Right Symmetry, and a Fibration Map

For any pair of elements a, x in a ring R, we have introduced in §3 the exchange polynomial $P_a(x) := (1 - xa)(1 + (1 - a)x) \in R$. If we think of a, x as two (noncommuting) variables, then $P_a(x)$ is an element in the free ring $\mathbb{Z}\langle a, x\rangle$. We begin this section by noting a special identity ((4.2) below), which holds in $\mathbb{Z}\langle a, x\rangle$, and hence in any ring containing the two elements a and x.

Lemma 4.1. *In the free ring $\mathbb{Z}\langle a, x\rangle$, we have*

$$(1 - xa)(1 + (1 - a)x) = (1 + x(1 - a))(1 - ax). \tag{4.2}$$

Equivalently, $P_a(x) = P_{1-a}(-x)$.

Proof. Both sides of the equation (4.2) expand into $1 + x - ax - xa - xax + xa^2x$. (For a more "conceptual" proof, note that, upon subtracting $(1 - xa)(1 - ax)$ from either side, we get $(1 - xa)x = x(1 - ax)$.) The last statement follows since

$$P_{1-a}(-x) = \big(1 - (-x)(1 - a)\big)\big(1 - x - (1 - a)(-x)\big) = (1 + x(1 - a))(1 - ax),$$

which is just the right-hand side of (4.2). □

After the writing of this paper, George Bergman informed us that, like all identities of the type $pq = rs$ in a free algebra, the particular polynomial equation (4.2) can be obtained from a "leapfrog construction" in the sense of P. M. Cohn: see [Co: p. 148]. For instance, the equation $(1 - xa)x = x(1 - ax)$ that showed up in the (second) proof above is one of the simplest identities obtainable from Cohn's leapfrog constructions.

Example 4.3. In the case where $x \in R$ and $a \in \operatorname{idem}(R)$ in any ring R, $P_a(x)$ boils down to a linear expression $1 + x - ax - xa$. (This is a fact we have used already in the proof of Theorem 3.12.) If a is a *central* idempotent, $P_a(x)$ further simplifies to $1 - (2a - 1)x$. As special cases of this, we have $P_0(x) = 1 + x$, and $P_1(x) = 1 - x$.

With what we have done in §3, we note that the issue of left-right symmetry for the (element-wise) notion of suitability is already at hand. This is because, by Theorem 3.4, the left suitability of $a \in R$ is characterized by the solvability of the equation $(1-xa)(1+(1-a)x) = 0$, while (similarly) the right suitability of a is characterized by the solvability of $(1+x(1-a))(1-ax) = 0$, which happens to be the very same equation according to the identity (4.2)! From this viewpoint, the erstwhile possibly mysterious left-right symmetry of suitable elements now follows basically "for free", purely from the new structural information on the set $\operatorname{suit}(R)$ obtained in this paper. Also, a quick consequence of the equation $P_a(x) = P_{1-a}(-x)$ is that some of the "obvious" variations of (1), (2) and (3) in Theorem 3.4 are also valid criteria for $a \in \text{suit}(R)$; e.g. the solvability of the equation $a\,P_a(x)\,(1-a) = 0$, etc.

Since we have full symmetry between left suitable elements and right suitable elements, we should expect a similar kind of symmetry between left exchange equations and right exchange equations. Given a left exchange equation $xa - fx = 1$ (where $f^2 = f$), how does one construct a right exchange equation $ax - xg = 1$ (where $g^2 = g$), *with hopefully the same* x? The result below shows that such a construction is indeed possible. As the reader will see, it is again due to the somewhat magical identity (4.2) that, in this "left to right" passage, we *do not* have to change the suitabilizer x.

Theorem 4.4. *Let $xa - fx = 1$ (with $f^2 = f$) be a left exchange equation over R. Then $g := (1-a)(1-ax) \in \text{idem}(R)$, and we have a right exchange equation $ax - xg = 1$. (Of course, from this viewpoint too, the element a is necessarily right suitable.)*

Proof. By Proposition 3.3(D), we know that $P_a(x) = 0$, so Lemma 4.1 gives

$$(1 + x\,(1-a))\,(1-ax) = 0.$$

It follows that, in the opposite ring $R^* = \{r^* : r \in R\}$,

$$(1^* - x^*a^*)\,(1^* + (1^* - a^*)\,x^*) = 0^*. \tag{4.5}$$

Let $g := (1-a)\,(1-ax) \in R$. Applying the proof of Theorem 3.4 to the element $a^* \in R^*$, we see from (4.5) above that $g^* = (1^* - x^*a^*)\,(1^* - a^*)$ is an idempotent in R^* with $x^*a^* - g^*x^* = 1^*$. Therefore, $g \in \text{idem}(R)$ too, with $ax - xg = 1 \in R$. $\square$

For any $a \in R$, let $\mathcal{E}(a)$ be the set of idempotents $e \in R\,a$ such that $1 - e \in R\,(1-a)$. (Recall that $\mathcal{E}(a)$ is nonempty if and only if $a \in \text{suit}(R)$.) Next, let

$$\mathcal{X}(a) = \{x \in R :\ P_a(x) = (1-xa)\,(1 + (1-a)\,x) = 0\},$$

which we will call the *suitabilizer set* for a. For any $a \in \text{suit}(R)$, the theorem below shows that the set $\mathcal{E}(a)$ is "parametrized" by $\mathcal{X}(a)$; that is, there is a natural fibration map from $\mathcal{X}(a)$ onto $\mathcal{E}(a)$. Following Jacobson's "circle product" notation in [Ja: p. 8], we'll write $r \circ s$ to denote $r + s - rs$ for any $r, s \in R$.

Theorem 4.6. *For any $a \in \text{suit}(R)$, the map $\varphi : R \to R$ defined by $\varphi(x) = xa \circ a$ maps $\mathcal{X}(a)$ onto $\mathcal{E}(a)$. If $\text{ann}_\ell(a - a^2) = 0$, then $\varphi : R \to R$ is an injection, and $\varphi : \mathcal{X}(a) \to \mathcal{E}(a)$ is a bijection.*

Proof. For any $x \in \mathcal{X}(a)$, we saw in the proof of Theorem 3.4 that $f := (1-xa)\,(1-a) \in \text{idem}(R) \cap R\,(1-a)$ satisfies the exchange equation $xa - fx = 1$. Moreover,

$$e := 1 - f = xa + a - xa^2 = xa \circ a \in R\,a, \tag{4.7}$$

which we also knew from $(*)$ in Remark 2.3. Therefore, we have $\varphi(x) = xa \circ a = e \in \mathcal{E}(a)$; that is, φ maps the suitabilizer set $\mathcal{X}(a)$ into $\mathcal{E}(a)$. This map is surjective by the second half of the proof of Theorem 2.1(2) (applied in conjunction with Proposition 3.3). Finally, assume that $\mathrm{ann}_\ell(a-a^2)=0$. If $x, y \in R$ are such that $\varphi(x)=\varphi(y)$, then $xa+a-xa^2 = ya+a-ya^2$ implies that $(x-y)\,(a-a^2)=0$, so $x=y$. This checks that $\varphi: R \to R$ is an injection. From this, it follows that $\varphi: \mathcal{X}(a) \to \mathcal{E}(a)$ is a bijection. $\square$

Next, we give the analogous information on the set $\mathrm{cn}\,(R)$ of clean elements in R. For any $a \in \mathrm{cn}\,(R)$, we define

$$\mathcal{I}(a) := \{\varepsilon \in \mathrm{idem}\,(R) :\ a-\varepsilon \in \mathrm{U}(R)\},$$

which we'll call the set of *cleansing idempotents* for the clean element a. (The idea here is that this set $\mathcal{I}(a)$ is in one-one correspondence with the different ways in which a can be written as a sum of an idempotent and a unit in R.) Also, we define

$$\mathrm{U}\big(\mathcal{X}(a)\big) := \mathcal{X}(a) \cap \mathrm{U}(R).$$

Theorem 4.8. *For any $a \in \mathrm{cn}\,(R)$, the following hold.*

(1) *There is a bijection $\sigma: \mathrm{U}\big(\mathcal{X}(a)\big) \to \mathcal{I}(a)$ defined by $\sigma(x) = a - x^{-1}$ for every $x \in \mathrm{U}\big(\mathcal{X}(a)\big)$.*

(2) *For any $x, y \in \mathrm{U}\big(\mathcal{X}(a)\big)$, $x^{-1}\varphi(x)\,x = y^{-1}\varphi(y)\,y \Rightarrow x = y$ (where φ is as in* (4.6)*).*

(3) *There is a unique map $\psi: \mathcal{I}(a) \to \mathcal{E}(a)$ making the following diagram commutative:*

(4.9)
$$\begin{array}{ccc} \mathrm{U}(\mathcal{X}(a)) & \xrightarrow{\ \sigma\ } & \mathcal{I}(a) \\ & \searrow^{\varphi} \quad \swarrow_{\psi} & \\ & \mathcal{E}(a) & \end{array}$$

For every $\varepsilon \in \mathcal{I}(a)$, we have $\psi(\varepsilon) = 1 - x\varepsilon x^{-1}$ where $x := (a-\varepsilon)^{-1}$.

Proof. (1) For any $x \in \mathrm{U}\big(\mathcal{X}(a)\big)$, we have an exchange equation $xa - fx = 1$ where $f := 1 - \varphi(x)$. Since x is a unit, we have $a - x^{-1}fx = x^{-1}$, so $a - x^{-1}$ is a cleansing idempotent for a. Therefore, the map σ defined in (1) maps $\mathrm{U}\big(\mathcal{X}(a)\big)$ into $\mathcal{I}(a)$, and it is clearly injective. To show that it is *surjective,* consider any $\varepsilon \in \mathcal{I}(a)$, and let $u := a - \varepsilon \in \mathrm{U}(R)$. In the proof of Theorem 2.1(1), we have an equation $xa - fx = 1$ where $x := u^{-1}$ and $f := x\varepsilon x^{-1}$. We know from Proposition 3.3(D) that $x \in \mathcal{X}(a)$. Therefore, $x \in \mathrm{U}\big(\mathcal{X}(a)\big)$, and so $\varepsilon = a - u = a - x^{-1} \in \mathrm{im}\,(\sigma)$.

(2) From the first two lines of the proof of (1), we see that, for every $x \in \mathrm{U}\big(\mathcal{X}(a)\big)$,

(4.10)
$$\sigma(x) = 1 - x^{-1}\varphi(x)\,x.$$

In view of this formula, the injectivity of the map σ proves (2).

(3) Since σ is a bijection, there is a unique map $\psi: \mathcal{I}(a) \to \mathcal{E}(a)$ making the diagram (4.9) commutative; namely, $\psi = \varphi \circ \sigma^{-1}$. To prove the last statement of (3), consider any $\varepsilon = \sigma(x) \in \mathcal{I}(a)$, where $x \in \mathrm{U}\big(\mathcal{X}(a)\big)$. From (4.10), we have

(4.10)′
$$1 - \varphi(x) = x\,\sigma(x)\,x^{-1} = x\varepsilon x^{-1},$$

so by definition, $\psi(\varepsilon) = \varphi(x) = 1 - x\varepsilon x^{-1}$ (where $x = (a - \varepsilon)^{-1}$). □

For any element $a \in \text{cn}\,(R)$, the cardinality of the set of cleansing idempotents $\mathcal{I}(a)$ is called the *clean index* of a as motivated by the paper of Lee and Zhou [LZ$_2$]. According to Theorem 4.8(1), this clean index is exactly the (cardinal) number of *unit* roots of the quadratic equation $P_a(x) = 0$ in R.

We now complete this section by pointing out other possibilities of constructing bijections. For the first one, we work with the subset $\text{scn}\,(R) \subseteq \text{cn}\,(R)$. To handle $a \in \text{scn}\,(R)$, we simply replace $\mathcal{I}(a)$ by the set of *strongly cleansing idempotents* for a; that is, the set $\mathcal{I}(a) \cap C_R(a)$. The Corollary below follows by combining Theorem 4.8(1) with Theorem 3.14, keeping in mind that $\mathcal{X}(a) \cap C_R(a)$ consists of units.

Corollary 4.11. *For any $a \in \text{scn}\,(R)$, the bijection σ in* Theorem 4.8(1) *restricts to a bijection $\sigma_0 : \mathcal{X}(a) \cap C_R(a) \to \mathcal{I}(a) \cap C_R(a)$.*

More generally, if we consider any subset $\mathcal{P} \subseteq \text{U}(R)$, we may define the notion of $a \in R$ being $\mathcal{P}$*-clean* by requiring that $a = \varepsilon + u$ for some $\varepsilon \in \text{idem}\,(R)$ and some $u \in \mathcal{P}$. Any ε that shows up in this way may be called a $\mathcal{P}$*-cleansing idempotent* for a $\mathcal{P}$-clean element a. In view of the bijective nature of the map $\sigma : \text{U}(\mathcal{X}(a)) \to \mathcal{I}(a)$, it follows that the set of the $\mathcal{P}$-cleansing idempotents for a is in one-one correspondence (under σ) with the set $\text{U}(\mathcal{X}(a)) \cap \mathcal{P}^{-1} = \mathcal{X}(a) \cap \mathcal{P}^{-1}$. In the "strongly clean" example, $\mathcal{P}$ is the set $\text{U}(R) \cap C_R(a)$. This set happens to be closed under the inverse map, so that is why Corollary 4.11 holds (granted that $\mathcal{X}(a) \cap C_R(a) \subseteq \text{U}(R)$). Another example would be, for instance, where $\mathcal{P}$ is the set of units of finite order (or units of order ≤ 2) in $\text{U}(R)$. One more example is where $\mathcal{P}$ is the set of *unipotent elements*[5] in R. In each of these cases, $\mathcal{P}$ is still inverse-closed, so the set of $\mathcal{P}$-cleansing idempotents for a can be "identified" (by the above method) with the set $\mathcal{X}(a) \cap \mathcal{P}$. The unipotent case can be applied for instance to study Diesl's class of nil-clean elements, as follows. In [Ds], Diesl defined an element $a \in R$ to be *nil-clean* if $a = \varepsilon + r$ where $\varepsilon \in \text{idem}\,(R)$ and $r \in R$ is nilpotent. This amounts to $1 + a$ being $\mathcal{P}$-clean where $\mathcal{P}$ is the set of unipotents. Thus, from the above discussions, we may conclude that the set of $\varepsilon \in \text{idem}\,(R)$ for which $a - \varepsilon$ is nilpotent is in one-one-correspondence with the set $\mathcal{X}(1 + a) \cap \mathcal{P}$.

§5. Examples and Applications

In this section, we give some examples and offer a number of applications of the new theory of $\mathcal{E}(a)$, $\mathcal{X}(a)$, and the exchange polynomial $P_a(x)$ developed in this paper.

For certain types of suitable elements $a \in \text{suit}\,(R)$, the suitabilizer set $\mathcal{X}(a)$ may consist entirely of units. The proposition below collects a number of such cases. In each of these cases, it turns out that a is strongly clean, and that the fibration map $\varphi : \mathcal{X}(a) \to \mathcal{E}(a)$ is a bijection. In part (2) of this proposition, the first conclusion

[5] As usual, "unipotent" means "$1 +$ nilpotent".

that $a \in \operatorname{scn}(R)$ in the abelian ring case was covered by Corollary 2.5, but the other conclusions are new.

Proposition 5.1. (1) *If* $a \in \operatorname{rad}(R)$ *(the Jacobson radical of* R*), then* $a \in \operatorname{scn}(R)$*,* $\mathcal{E}(a) = \{0\}$*, and* $\mathcal{X}(a) = \{(a-1)^{-1}\}$*.*

(2) *Let* $a \in \operatorname{suit}(R)$*. If either* $a \in Z(R)$ *(the center of* R*) or* R *is an abelian ring, then* $a \in \operatorname{scn}(R)$*,* $\mathcal{X}(a) \subseteq \mathrm{U}(R)$*, and the map* $\varphi : \mathcal{X}(a) \to \mathcal{E}(a)$ *is a bijection.*

Proof. (1) For $a \in \operatorname{rad}(R)$, we have $a - 1 \in \mathrm{U}(R)$. The decomposition $a = 1 + (a-1)$ shows that $a \in \operatorname{scn}(R)$, and $\mathcal{E}(a) = \{0\}$ since $\operatorname{rad}(R) \cap \operatorname{idem}(R) = \{0\}$. Next, for any $x \in \mathcal{X}(a)$, we have an exchange equation $xa - fx = 1$ where $f = 1 - \varphi(x)$. This implies that $fx = xa - 1 \in \mathrm{U}(R)$. Since f is an idempotent, we must have $f = 1$. Thus, $x\,(a-1) = 1$, and so $x = (a-1)^{-1}$.

(2) *First assume that* $a \in Z(R)$. Then $\mathcal{X}(a) \subseteq \mathrm{U}(R)$ by the last statement of Theorem 3.14. For any $x \in \mathcal{X}(a)$, let $e := \varphi(x)$ and $f := 1 - e \in \operatorname{idem}(R)$. We have an exchange equation $1 = xa - fx = (a - f)\,x$, so $a - f \in \mathrm{U}(R)$ (which implies that $a \in \operatorname{scn}(R)$), and $x = (a-f)^{-1}$. The latter means that x is *uniquely* determined by a and $e = \varphi(x)$, so $\varphi : \mathcal{X}(a) \to \mathcal{E}(a)$ is a bijection. Next, *assume that* R *is abelian* (instead of $a \in Z(R)$). For any $x \in \mathcal{X}(a)$, letting $f := 1 - \varphi(x)$ in the proof of Corollary 2.5 shows that $x = (a-f)^{-1} \in \mathrm{U}(R)$, so we are done as before. □

We shall next give an application of our results. Note that, in the following result, there is no assumption on the lifting of idempotents modulo the ideal I in question.

Theorem 5.2. *Let* $\overline{R} = R/I$ *where* $I \subseteq \operatorname{rad}(R)$ *is an ideal of* R*, and let* $a \in \operatorname{suit}(R)$*. If* $\overline{R}$ *is an abelian ring, or if* $\overline{a} \in Z\big(\overline{R}\big) \cup \operatorname{rad}\big(\overline{R}\big)$*, then* $\mathcal{X}(a) \subseteq \mathrm{U}(R)$*, and* $a \in \operatorname{cn}(R)$*.*

Proof. Consider any element $x \in \mathcal{X}(a)$. By functoriality, $a \in \operatorname{suit}(R)$ maps to $\overline{a} \in \operatorname{suit}\big(\overline{R}\big)$, and $x \in \mathcal{X}(a)$ maps to $\overline{x} \in \mathcal{X}\big(\overline{a}\big)$. Under any one of the assumptions made above, we see from Proposition 5.1 that $\overline{x} \in \mathrm{U}\big(\overline{R}\big)$. Since $I \subseteq \operatorname{rad}(R)$, it follows that $x \in \mathrm{U}(R)$, and so $a \in \operatorname{cn}(R)$ by Theorem 3.14(A). □

The result above in the case where $\overline{R}$ is abelian is due to Lee and Zhou [LZ_1]. Even in this case, our proof is quicker and more conceptual. Note that this result implies that the equation $\operatorname{suit}(R) = \operatorname{cn}(R)$ holds for any left quasi-duo ring (i.e. a ring in which all maximal left ideals are two-sided ideals) since for such a ring R, $R/\operatorname{rad}(R)$ is a reduced ring and hence an abelian ring.

Before stating our next theorem, we give two illustrative examples.

Example 5.3. *What happens in the special case* $a \in \operatorname{idem}(R)$? In this case, the only idempotent $e \in Ra$ with $1 - e \in R\,(1-a)$ is $e = a$ (since $R = Ra \oplus R\,(1-a)$). Thus, $\mathcal{E}(a)$ is the singleton set $\{a\}$. (Indeed, the map $\varphi : R \to R$ in Theorem 4.6 has image $\{a\}$ since $\varphi(r) = ra \circ a = ra + a - ra^2 = a$ for every $r \in R$.) Here, $P_a(x) = 0$ linearizes to an exchange equation $xa - (1-a)\,x = 1$ (see Example 4.3). Left multiplying this equation by a shows that $axa = a$, so we have $\mathcal{X}(a) \subseteq \mathrm{Inn}\,(a)$ (the set of all inner inverses of a). However, this is in general *not* an equality. For instance, if $a \in \mathrm{idem}\,(R)$ is central, the linear equation above simplifies to $(2a-1)\,x = 1$, so $\mathcal{X}(a) = \{(2a-1)^{-1}\} = \{2a-1\}$, which is only a small part of $\mathrm{Inn}\,(a)$. If a is not central, then $\mathcal{X}(a)$ can be larger. For an explicit example, let $R = \mathbb{M}_2(S)$ where S is a nonzero ring, and let $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{idem}\,(R)$. An easy computation shows that $\mathcal{X}(a) = \begin{pmatrix} 1 & S \\ S & -1 \end{pmatrix}$. On the other hand, $\mathrm{Inn}\,(a)$ turns out to be $\begin{pmatrix} 1 & S \\ S & S \end{pmatrix}$, which is considerably larger than $\mathcal{X}(a)$. (Both sets contain units as well as non-units.)

Example 5.4. It is easy to give examples of $a \in \mathrm{suit}\,(R)$ for which the set $\mathcal{E}(a)$ is fairly large. For instance, let $a \in \mathrm{U}(R)$. Since $R\,a = R$ here, we have $\mathcal{E}(a) = \big(1 + R\,(1-a)\big) \cap \mathrm{idem}\,(R)$. For a concrete example, take again $R = \mathbb{M}_2(S)$, with $a = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{U}(R)$. A quick computation shows that $\mathcal{E}(a)$ consists of all (idempotent) matrices of the form $\begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix}$ where $s \in \mathrm{idem}\,(S)$ and $rs = 0 \in S$. The set $\mathcal{X}(a)$ here, again, contains both units and non-units of R.

Next, we would like to say something about how the set $\mathcal{E}(a)$ is related to its right-side analogue $\mathcal{E}'(a)$, by which we mean the set of idempotents $e \in aR$ such that $1 - e \in (1-a)\,R$. Just as in Theorem 4.6, we have a fibration map $\varphi' : \mathcal{X}(a) \to \mathcal{E}'(a)$, defined by $\varphi'(x) = a \circ ax$ for every $x \in \mathcal{X}(a)$. (The domain of this map is still the same $\mathcal{X}(a)$ since $\mathcal{X}(a)$ is a left-right symmetric set, thanks to Lemma 4.1.) Nicholson was the first author to have come up with an *ad hoc* proof (in [Ni$_2$]) for the fact that $\mathcal{E}(a)$ is nonempty iff $\mathcal{E}'(a)$ is nonempty (i.e. a is left suitable iff it is right suitable). However, Nicholson's work did not amount to the construction of a "canonical mapping" from $\mathcal{E}(a)$ to $\mathcal{E}'(a)$. In the following result, we'll show that, under a certain assumption on the suitable element $a \in R$, we can construct such a (surjective) mapping.

Theorem 5.5. *For any $a \in \mathrm{suit}\,(R)$ such that $\mathrm{ann}_\ell(a - a^2) \subseteq \mathrm{ann}_r(a - a^2)$, there exists a unique map $\tau : \mathcal{E}(a) \to \mathcal{E}'(a)$ such that the following diagram commutes*:

(5.6)
$$\begin{array}{ccc} & \mathcal{X}(a) & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\varphi'} \\ \mathcal{E}(a) & \xrightarrow{\quad\tau\quad} & \mathcal{E}'(a) \end{array}$$

Moreover, this map τ is a surjection.

Proof. Let $e \in \mathcal{E}(a)$. To define $\tau(e) \in \mathcal{E}'(a)$, take *any* $x \in \mathcal{X}(a)$ such that $e = \varphi(x) = xa \circ a$. We would like to define $\tau(e) := \varphi'(x) = a \circ ax \in \mathcal{E}'(a)$. If $\tau(e)$ is well-defined (i.e. it does not depend on the particular choice of x above), then we will have constructed the commutative diagram (5.6) with a unique map τ. To see that $\tau(e)$ is well-defined, consider any $x, y \in \mathcal{X}(a)$ such that $\varphi(x) = \varphi(y)$. As we have observed before in the proof of Theorem 4.6, this implies that $x - y \in \mathrm{ann}_\ell(a - a^2)$. If $\mathrm{ann}_\ell(a-a^2)) \subseteq \mathrm{ann}_r(a-a^2)$, then $(a-a^2)\,(x-y) = 0$ too, and we have $a \circ ax = a \circ ay$; that is, $\varphi'(x) = \varphi'(y)$, as desired. Since φ' is surjective, clearly so is τ. □

Corollary 5.7. *If $a \in \mathrm{suit}\,(R)$ is such that $\mathrm{ann}_\ell(a - a^2) = \mathrm{ann}_r(a - a^2)$, then there is a unique map $\tau : \mathcal{E}(a) \to \mathcal{E}'(a)$ which makes the diagram* (5.6) *commutative, and τ is necessarily a bijection.*

Proof. The existence and uniqueness of a map τ making the diagram (5.6) commutative is guaranteed by Theorem 5.5. Since we are also assuming $\mathrm{ann}_r(a - a^2) \subseteq \mathrm{ann}_\ell(a - a^2)$, we have similarly a unique map $\tau' : \mathcal{E}'(a) \to \mathcal{E}(a)$ such that $\tau' \circ \varphi' = \varphi$. Clearly, τ and τ' are mutually inverse bijections. □

Example 5.8. In general, there may not exist a map τ making the diagram (5.6) commutative. To see this, all we need is to produce an example of $a \in \mathrm{suit}\,(R)$ with $x, y \in \mathcal{X}(a)$ such that $\varphi(x) = \varphi(y)$, but $\varphi'(x) \neq \varphi'(y)$. This is easily done by choosing for instance $R = \mathbb{M}_2(S)$ (with $S \neq 0$), and taking $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (recalling that a nilpotent element is always strongly clean, and hence suitable). It is routine to check that all matrices of the form $x_s := \begin{pmatrix} 0 & s \\ 1 & s \end{pmatrix}$ are in $\mathcal{X}(a)$, and are mapped by φ to the idempotent $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{E}(a)$. However, the idempotents $\varphi'(x_s) = \begin{pmatrix} 1 & s+1 \\ 0 & 0 \end{pmatrix} \in \mathcal{E}'(a)$ are in one-one correspondence with the elements $s \in S$. Here, expectedly, there are no inclusion relationships between the left and the right annihilators of $a - a^2 = a$. A quick calculation shows that $\mathcal{E}(a)$ consists of all idempotent matrices with first column zero, while $\mathcal{E}'(a)$ consists of all idempotent matrices with second row zero.

Example 5.9. If $a \in R$ is a general suitable element, the two sets $\mathcal{E}(a)$ and $\mathcal{E}'(a)$ may not have the same cardinality (so there may not be a bijection from one to the other). For instance, let R be the (suitable) ring of 2×2 upper triangular matrices over a field K of characteristic $\neq 2$, and let a be the strongly regular element $\mathrm{diag}\,(0, -1) \in R$. The suitabilizer set for a is easily seen to be

$$\mathcal{X}(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1/2 \end{pmatrix} \cup \begin{pmatrix} -1 & K \\ 0 & -1 \end{pmatrix} \subseteq \mathrm{U}(R). \tag{5.10}$$

On the other hand, the set $\mathrm{Inn}\,(a)$ of inner inverses of a is $\begin{pmatrix} K & K \\ 0 & -1 \end{pmatrix}$. Here, in testimony to Remark 2.8, there are no inclusion relations between $\mathrm{Inn}\,(a)$ and $\mathcal{X}(a)$ (although it is true that *only one* element of $\mathcal{X}(a)$ fails to be in $\mathrm{Inn}\,(a)$).

To "compare" $\mathcal{E}(a)$ with $\mathcal{E}'(a)$, note that $1 - a \in \mathrm{U}(R)$ implies that $R\,(1 - a) = R = (1 - a)\,R$. Thus, $\mathcal{E}(a)$ and $\mathcal{E}'(a)$ are simply the sets of idempotents in $R\,a$ and aR respectively. A straightforward computation shows that

$$\mathcal{E}(a) = \{0\} \cup \begin{pmatrix} 0 & K \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \mathcal{E}'(a) = \{0, -a\} \subseteq \mathcal{E}\,(a). \tag{5.11}$$

Therefore, $|\mathcal{E}(a)| = 1 + |K| \geq 4$, while $|\mathcal{E}'(a)| = 2$. In particular, *no bijection can exist between* $\mathcal{E}(a)$ *and* $\mathcal{E}'(a)$. Recalling that $\varphi(x) = xa \circ a$, φ sends $\begin{pmatrix} -1 & 0 \\ 0 & -1/2 \end{pmatrix}$ to the zero matrix, and sends $\begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}$ (for every $k \in K$) to $\begin{pmatrix} 0 & -2k \\ 0 & 1 \end{pmatrix} \in \mathcal{E}(a)$. Since $\text{char}\,(K) \neq 2$, $\varphi : \mathcal{X}(a) \to \mathcal{E}(a)$ is visibly a *bijection.* On the other hand, the fibration φ' is necessarily "highly collapsing": it sends $\begin{pmatrix} -1 & 0 \\ 0 & -1/2 \end{pmatrix}$ again to the zero matrix, and sends the entirety of $\begin{pmatrix} -1 & K \\ 0 & -1 \end{pmatrix}$ to $-a = \text{diag}\,(0, 1) \in \mathcal{E}'(a)$. Since $\varphi : \mathcal{X}(a) \to \mathcal{E}(a)$ is a bijection, of course the commutative diagram (5.6) does exist. (This is also predictable from the easily verifiable fact that $\text{ann}_\ell(a - a^2) \subseteq \text{ann}_r(a - a^2)$ holds here.) Finally, a is (strongly regular and hence) strongly π-regular, so certainly $a \in \text{scn}\,(R)$. An easy application of Theorem 4.8(1) and Corollary 4.11 shows that the set of cleansing idempotents for a is $\mathcal{I}(a) = \{I\} \cup \begin{pmatrix} 1 & K \\ 0 & 0 \end{pmatrix}$, while the set $\mathcal{I}(a) \cap C_R(a)$ of strongly cleansing idempotents for a consists only of the identity matrix and $\text{diag}\,(1, 0)$.

Next, we present an application of the method of the exchange polynomial $P_a(x)$ to the study of corner rings. This application is prompted by a paradigm suggested by a result of Lam and Murray in [LM], which states that, for a corner ring $S = eRe$ in any ring R (where $e^2 = e$), an element $a \in S$ is unit-regular in S iff $a + (1 - e)$ is unit-regular in R. Theorem 5.12 below is an analogue of this result, in which "unit-regular" is replaced by "suitable" or by "strongly clean". Moreover, the term $1 - e$ is generalized to any idempotent element ε in the corner ring fRf where $f = 1 - e$. The use of the exchange polynomial $P_a(x)$ leads easily to a quick and natural proof of this result.

Theorem 5.12. *Let* e, f *be complementary idempotents in a ring* R*, and let* $a = b + \varepsilon$ *where* $b \in S := eRe$ *and* $\varepsilon \in \text{idem}\,(fRf)$*. Then* (1) $a \in \text{suit}\,(R)$ *iff* $b \in \text{suit}\,(S)$*, and* (2) $a \in \text{scn}\,(R)$ *iff* $b \in \text{scn}\,(S)$*.*

Proof. By an easy direct product argument in $eRe \times fRf$, the "if" parts in (1) and (2) are true assuming only, respectively, that $\varepsilon \in \text{suit}\,(fRf)$ and $\varepsilon \in \text{scn}\,(fRf)$. For the "only if" part in (1), assume that $a \in \text{suit}\,(R)$, and let $x \in R$ be a suitabilizer for a; that is, $P_a(x) = 0$. We claim that $y := exe$ *is a suitabilizer for* b in S; that is, $P_b(y) = 0$, where $P_b(y)$ is calculated in the corner ring S, using e as the identity element for S. By Theorem 3.4 (applied to S), this will show that $b \in \text{suit}\,(S)$. To prove our claim, we left and right multiply $P_a(x) = 0$ by e, getting

$$\begin{aligned} 0 &= e\,(1 + x - ax - xa - xax + xa^2x)\,e \\ &= e + exe - e\,(b + \varepsilon)\,xe - ex\,(b + \varepsilon)\,e - ex\,(b + \varepsilon)\,xe + ex\,(b^2 + \varepsilon^2)\,xe \\ &= e + y - by - yb - yby - ex\varepsilon xe + yb^2y + ex\varepsilon^2 xe. \end{aligned}$$

Since $\varepsilon = \varepsilon^2$, this gives $P_b(y) = 0$, as claimed. The same proof is also good for the "only if" part in (2), noting that, if we assume that $a \in \text{scn}\,(R)$, then the suitabilizer x above can be chosen to satisfy $ax = xa$. By a straightforward computation using the Peirce decompositions of a and x, we see easily that $b\,(exe) = (exe)\,b$; that is, $by = yb \in S$. By Theorem 3.14(B), this together with $P_b(y) = 0 \in S$ imply that $b \in \operatorname{scn}(S)$. □

Note that in the case where $\varepsilon = 0$, (1) and (2) above retrieve the well known facts that $S \cap \operatorname{suit}(R) = \operatorname{suit}(S)$ and $S \cap \operatorname{scn}(R) = \operatorname{scn}(S)$. (See, respectively, [Ni$_1$: (2.6)] and [Ch: (2.5)].) The generalization of these facts to the case $\varepsilon \in \operatorname{idem}(fRf)$ is, however, rather nontrivial; in fact, we do not know of any proof of (the necessity parts of) Theorem 5.12 without using the techniques of this paper. The subtle nature of the conclusions (1) and (2) in that theorem can also be seen from the two remarks below.

Remark 5.13. The analogue of (1) and (2) in Theorem 5.12 *does not* hold for clean elements; namely, if $a \in \operatorname{cn}(R)$, we may not have $b \in \operatorname{cn}(S)$, even in the special cases where $\varepsilon = 0$ or $\varepsilon = f$. This is shown by the following explicit examples. Let $R = \mathbb{M}_3(\mathbb{Z})$, $e = \operatorname{diag}(1,1,0)$, $f = 1 - e = \operatorname{diag}(0,0,1)$. and $b = \begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix} \in S = eRe$ (upon identifying S with $\mathbb{M}_2(\mathbb{Z})$). For $\varepsilon = 0$ and $\varepsilon = f$ respectively, $a := b + \varepsilon$ is in $\operatorname{cn}(R)$ since it has the following clean decomposition:

$$\begin{pmatrix} 12 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 3 \\ 7 & 1 & -7 \\ -2 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 14 & 5 & -3 \\ -7 & -1 & 7 \\ 2 & 0 & -3 \end{pmatrix};$$

$$\begin{pmatrix} 12 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -2 & -4 \\ 5 & 2 & 4 \\ 5 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 17 & 7 & 4 \\ -5 & -2 & -4 \\ -5 & -2 & -3 \end{pmatrix}.$$

However, as we have noted in Remark 3.15, $b \notin \operatorname{cn}(S)$ by [KL: (4.5)].

Remark 5.14. The analogue of (1) and (2) in Theorem 5.12 also fails to hold in general if the hypothesis $\varepsilon \in \operatorname{idem}(fRf)$ is weakened to $\varepsilon \in \operatorname{reg}(fRf) \subseteq \operatorname{suit}(fRf)$. Indeed, if we take again $R = \mathbb{M}_3(\mathbb{Z})$, and let $e = \operatorname{diag}(1,0,0)$, $f = 1 - e = \operatorname{diag}(0,1,1)$, $b = 3 \in \mathbb{Z}$, and $\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z})$ (upon identifying $S = eRe$ with $\mathbb{Z}$ and fRf with $\mathbb{M}_2(\mathbb{Z})$), we have $b \notin \operatorname{suit}(\mathbb{Z})$ and $\varepsilon \in \operatorname{reg}(fRf) \subseteq \operatorname{suit}(fRf)$. However, $a := b + \varepsilon = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ turns out to be in $\operatorname{suit}(R)$. In fact, with some help from *Mathematica*, one finds that a has a suitabilizer $\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 1 & 4 & 1 \end{pmatrix}$ in R.

We close this paper by recording a useful application of some of our results to C^*-algebras. The fact (proved in [AG: Theorem 7.2]) that a C^*-algebra is a suitable ring iff it has real rank zero leads to the following conclusions in view of what we have proved in §§3–4. Here again, $P_a(x)$ denotes the exchange polynomial $(1 - xa)\,(1 + (1 - a)\,x)$.

Theorem 5.15. *For any unital C^*-algebra R, the following statements are equivalent*:

(1) R *is of real rank zero.*
(2) *For every* $a \in R$*, there exist* $x, f \in R$ *with* $f^2 = f$ *such that* $xa - fx = 1$*.*
(3) *For every* $a \in R$*, the equation* $P_a(x) = 0$ *is solvable in* R*.*
(4) *For every* $a \in R$*, the equation* $P_a(x)\, a = 0$ *is solvable in* R*.*
(5) *For every* $a \in R$*, the equation* $P_a(x)\, a\,(1-a) = 0$ *is solvable in* R*.*
(6) *For every* $a \in R$*, the equation* $(1-a)\, P_a(x)\, a = 0$ *is solvable in* R*.*

In case R *is commutative,* (3) *amounts to the solvability of* $(a^2-a)\,x^2-(2a-1)\,x+1=0$ (*for every* $a \in R$).

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