# COMMUTING IDEMPOTENTS, SQUARE-FREE MODULES, AND THE EXCHANGE PROPERTY

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Abstract. We give a criterion for when idempotents of a ring $R$ which commute modulo the Jacobson radical $J(R)$ can be lifted to commuting idempotents of $R$. If such lifting is possible, we give extra information about the lifts. A "half-commuting" analogue is also proven, and this is used to give sufficient conditions for a ring to have the internal exchange property. In particular, we show that if $R/J(R)$ is an internal exchange ring and idempotents lift modulo $J(R)$, then $R$ is an internal exchange ring. We also clarify some interesting results in the literature by investigating, and ultimately characterizing, the relationships between the finite (internal) exchange property, the $(C_3)$ property, and generalizations of square-free modules. We provide multiple examples delimiting these connections.

## Introduction

In this paper we consider the behavior of direct sum decompositions for modules. More precisely, we study classes of modules which are defined by properties satisfied by their direct summands. Particular emphasis is given to the exchange and internal exchange properties.

The exchange property was introduced in 1964 by Crawley and Jónsson [3] in their study of isomorphic refinements of direct decompositions in algebraic systems. Rather than work with general algebraic systems, we limit ourselves to the category of modules and our exposition follows ideas and definitions found in [11].

Let $\aleph$ be a cardinal. We say that a module $M_R$ has the $\aleph$-*exchange property* if whenever $M$ is a direct summand of another module $A$, and $A = \bigoplus_{i\in I} A_i$ with $|I| \leq \aleph$, then $A = \bigoplus_{i\in I} A'_i \oplus M$ for some submodules $A'_i \subseteq A_i$ for each $i \in I$. When this holds for every cardinal, we say $M$ has the *exchange property*. The 2-exchange property is equivalent to the $n$-exchange property for any integer $n \geq 2$, and so we call that the *finite exchange property*. It is still an open question whether finite exchange implies $\aleph$-exchange for any $\aleph \geq \aleph_0$, although there are many special cases where positive answers are known.

As discovered by Warfield [24], a module $M_R$ has the finite exchange property if and only if $\mathrm{End}(M_R)$ has the finite exchange property as a right (equivalently, as a left) module over itself. Thus, a ring which has the finite exchange property as a right module over itself is called an *exchange ring*. Subsequently, Nicholson [16] characterized the exchange rings as exactly the class of rings for which idempotents lift modulo *every* left and right ideal. This class of rings is very large.

The exchange property is *external* in the sense that one embeds $M$ (as a direct summand) inside a larger module $A$. There is a natural generalization which is *internal*, defined as follows. Let $Y_R$ be a right $R$-module. We say that a direct summand $X \subseteq Y$ *exchanges* in a decomposition $Y = \bigoplus_{i\in I} Y_i$ if $Y = \bigoplus_{i\in I} Y'_i \oplus X$ for some submodules $Y'_i \subseteq Y_i$ for each $i \in I$. If every direct summand of $Y$ exchanges in the decomposition $Y = \bigoplus_{i\in I} Y_i$, then we say that this decomposition is *exchangeable*. Finally, we say that a module $M_R$ has the $\aleph$-*internal exchange property* if every direct sum decomposition $M = \bigoplus_{i\in I} M_i$ with $|I| \leq \aleph$ is exchangeable.

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Just as with the exchange property, it turns out that 2-internal exchange is equivalent to $n$-internal exchange for any integer $n \geq 2$, and so we call that the *finite internal exchange property.*

Every injective module has the (full) exchange property. This was first proved in 1969 by Warfield [23]. Subsequently, this result was generalized to quasi-injective modules [4] and then to continuous modules [12]. (We refer the interested reader to [13] for definitions and basic facts about these types of modules.) The focus then shifted to quasi-continuous modules. These modules do not necessarily have even the finite exchange property, but when they do then they also have full exchange [14, 20]. Surprisingly, all quasi-continuous modules have the internal exchange property [20, Proposition 1.1].

If $M$ is a quasi-continuous module, then we can write $M = M_1 \oplus M_2$ where $M_1$ is quasi-injective and $M_2$ is square-free by [13, Theorem 2.37]. Since the square-free modules form one of the main classes we are interested in, we recall the definition here. A module $M$ is *square-free* if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$, then $Y_1 = Y_2 = (0)$.

Essentially subsuming and expanding all the results above, the second author proved that square-free modules with the finite exchange property have full exchange [19]. Additionally, the second author (in unpublished work) and independently Mohamed [10] proved that square-free modules with finite internal exchange have full internal exchange. Central to these results is the property which has been dubbed $(C_3)$; a module $M$ has the $(C_3)$ *property* if whenever two direct summands $A, B \subseteq M$ satisfy $A \cap B = (0)$, then $A \oplus B$ is also a direct summand of $M$. For the historical origin of this property, and for the definitions of $(C_1)$ and $(C_2)$ which will not be used in this paper, see [13].

Each direct summand $N$ of a module $M$ has the form $\pi(M)$ for a projection $\pi$, by which we just mean an idempotent of the ring $\mathrm{End}(M_R)$ of endomorphisms of $M$. One thus expects to infer important information about direct summands of $M$ by first considering idempotents of $\mathrm{End}(M_R)$. This pattern will be used repeatedly, and to great success, in this paper. It also happens that for the classes of modules we are interested in, a significant role is played by the ability to lift idempotents modulo ideals, and in particular lifting pairs of idempotents to idempotents which commute (see Section 2). This component of our research is interesting in its own right, and can be seen as a counterpart to the issue of lifting idempotents orthogonally, which is one of the main concepts in the context of the Krull-Schmidt Theorem. The exchange property, in connection with square-free modules, has played a central role in the structure theorems of modules invariant under automorphisms, such as in [5].

One accomplishment of this paper is establishing and clarifying inclusion relations between classes of modules. In particular, we will focus on modules which are square-free, have finite (internal) exchange, have the $(C_3)$ property, or possess certain other related properties. We achieve this objective by discovering important new relations between these properties, which enriches the known literature in this area. We also construct many non-trivial examples which show the natural limitations of these results and enable us to achieve in Section 7 a complete classification of connections (see Figure 7.2).

The paper is organized as follows. In Section 1 we recall some natural equivalence relations on the set of idempotents of a given ring $R$, and in Section 2 we examine when one may lift idempotents modulo the Jacobson radical to idempotents which commute. In Section 3 we prove Theorem 3.2 which connects the lifting results of the previous section to the internal exchange property. We then formulate a few useful corollaries. Section 4 contains information about the $(C_3)$ property for modules and the structure of their endomorphism rings, and Example 4.8 shows that the $(C_3)$ property is not symmetric. In Section 5 we investigate square-free modules and some classes closely related to them. The main results there unite two seemingly different generalizations of the square-free property, and we also show that if $E := \mathrm{End}(M_R)$, then the properties "$M_R$ is square-free" and "$E_E$ is square-free" are independent. The main purpose of Section 6 is to construct an example of a module which is square-free and which does not have $(C_3)$, which is accomplished in Example 6.1. Finally, Section 7 summarizes and systematizes all of obtained results. Figure 7.2 is an Euler diagram which fully justifies the claim in the previous paragraph of a complete classification of connections among these properties. We also mention that this section contains the decidedly non-trivial Example 7.3 (which is the final step in our classification) of a ring $R$ where $R_R$ is square-free, the right singular ideal is contained in the Jacobson radical, and $R_R$ does not have finite internal exchange.

## Notations

We will use the following notations in this paper. The letter $R$ will always refer to a ring; all rings in this paper are associative with 1. All modules we consider are unital, and $M$ will always denote a right $R$-module unless context requires otherwise (such as in Lemma 1.1 below). We also let $E$ denote the endomorphism ring of $M$, sometimes written $\mathrm{End}(M_R)$. We write module endomorphisms on the opposite side from the scalar action. To denote that $N \subseteq M$ is a direct summand, we write $N \subseteq^{\oplus} M$.

To denote that $I$ is a two-sided ideal of $R$ we write $I \trianglelefteq R$. We write $a \equiv b \pmod{I}$ for some $a, b \in R$ if $a - b \in I$. Given a ring $R$ we let $U(R)$ be its group of units, we let $\mathrm{idem}(R)$ denote the set of idempotents in $R$, and we let $J(R)$ be the Jacobson radical of $R$. Elements in $R/J(R)$ will usually be written using bar notation. Finally, $\mathbb{M}_n(R)$ is the ring of $n \times n$ matrices over $R$, and by $e_{i,j} \in \mathbb{M}_n(R)$ we mean the matrix with 1 in the $(i, j)$-position and 0 elsewhere.

## 1. Idempotent facts

In this paper we are concerned with direct sum decompositions of modules, so we are naturally led to consider the behavior of idempotents in rings. We begin by recalling a few natural equivalence relations on the set $\mathrm{idem}(R)$. First, we need the following very useful fact.

**Lemma 1.1** ([7, Exercise 21.4] and [17, Lemma 3])**.** *Given $e, f \in \mathrm{idem}(R)$, the following are equivalent:*

(1) $eR = fR$.
(2) $ef = f$ *and* $fe = e$.
(3) $f = eu$ *where* $u = 1 + ex(1 - e) \in U(R)$ *for some* $x \in R$.
(4) $R(1 - e) = R(1 - f)$.

*In case $R$ is an endomorphism ring of a module $M_\Lambda$ (for some ring $\Lambda$), these are also equivalent to the following conditions:*

(5) $eM = fM$.
(6) $\ker(1 - e) = \ker(1 - f)$.

Following the terminology in [18, Definition 2.1], if $eR = fR$ for two idempotents $e, f \in R$, we will say that they are *right associate idempotents*, and write $e \sim_r f$. We define *left associate idempotents* similarly, and write $e \sim_\ell f$. Finally, we will denote conjugate idempotents with the notation $e \sim_c f$. Note that left (or right) associate idempotents are conjugate, which can be seen using the unit $u$ in Lemma 1.1.

Another important relation on idempotents is that of isomorphism. We recall this fundamental definition, but also see [6, Proposition 21.20] for more information.

**Definition 1.2.** Let $R$ be a ring and $e, f \in \mathrm{idem}(R)$. We say that $e$ and $f$ are *isomorphic* exactly when $eR \cong fR$ (as right $R$-modules). This is equivalent to $Re \cong Rf$ (as left $R$-modules), and also equivalent to the existence of some $a, b \in R$ with $e = ab$ and $f = ba$. When $e$ and $f$ are isomorphic we write $e \cong f$.

Another important concept connected to idempotents is *lifting*. This notion is very natural, and holds for many classes of rings. If $x^2 - x \in I$ for some one-sided ideal $I \subseteq R$, so that $x$ behaves like an idempotent modulo $I$, then one says $x$ *lifts modulo* $I$ if there is some *actual* idempotent $e^2 = e \in R$ with $e - x \in I$. If idempotents lift modulo every left (and right) ideal of $R$, then $R$ is called an *exchange ring* (see, for instance, [16] for further properties of such rings). These rings encompass many classes of rings, such as von Neumann regular rings, left or right artinian rings, and local rings, to name a few.

It is well known that idempotents always lift modulo nil ideals of a ring $R$, and they also lift modulo $I \trianglelefteq R$ in the case when $R$ is $I$-adically complete, see [6, Theorems 21.28 and 21.31]. However, the ring

$$R = \mathbb{Z}_{(6)} = \{a/b \in \mathbb{Q} \ : \ a, b \in \mathbb{Z},\ \gcd(b, 6) = 1\}$$

consisting of integers localized away from the ideal (6) is the classic example of a ring where idempotents do *not* lift modulo the Jacobson radical, since $R/J(R) \cong \mathbb{Z}/6\mathbb{Z}$ has four idempotents, but $R$ has only the trivial idempotents.

Recently, Alkan et al. [1] showed that the Jacobson radical and many other ideals have a special property that can be decoupled from lifting, which they call the enabling property. To be exact, an ideal $I \trianglelefteq R$ is *enabling* in $R$ if whenever there exist $x \in R$ and $e \in \mathrm{idem}(R)$ with $e - x \in I$, then there exists $f \in \mathrm{idem}(R)$ with $f - x \in I$ and $f \in xR$. In other words, whenever an element $x \in R$ lifts to an idempotent $e^2 = e \in R$ modulo $I$, then $x$ lifts *strongly*, meaning to an idempotent $f^2 = f \in xR$ which is a right multiple of $x$. Surprisingly, this notion is left-right symmetric [1, Lemma 1]. Any ideal contained in the Jacobson radical is automatically enabling even though, as we saw above, not all idempotents need lift.

The following three lemmas are prime examples of how enabling and equational identities can be used to find other idempotents without assuming any lifting hypothesis. We will also make use of these statements in the remainder of the paper. The first result tells us that if two idempotents are right associate in $R/J(R)$, then they are right associate in $R$ assuming one small fact.

**Lemma 1.3** ([2, Lemma 2.2]). *Let $e, f \in \mathrm{idem}(R)$ and let $I \trianglelefteq R$ with $I \subseteq J(R)$. If $ef = f$ and $(e + I) \sim_r (f + I)$, then $e \sim_r f$.*

The next lemma tells us that right associates of liftable idempotents also lift, and further they lift as right associates. This result generalizes [2, Corollary 3.6] by removing an unnecessary hypothesis.

**Lemma 1.4.** *Let $e \in \mathrm{idem}(R)$, let $x \in R$, and let $I \trianglelefteq R$. If $x + I \in \mathrm{idem}(R/I)$ and $(x + I) \sim_r (e + I)$, then there exists $f \in \mathrm{idem}(R)$ satisfying $f - x \in I$ and $f \sim_r e$.*

*Proof.* Since $(x + I) \sim_r (e + I)$, Lemma 1.1(3) implies that $x = e(1 + er(1 - e)) + i$ for some $r \in R$ and $i \in I$. Put $f := e(1 + er(1 - e)) = x - i$. Clearly $x - f \in I$. By direct computation (using only the fact that $e^2 = e$) we find $f^2 = f$ and again by Lemma 1.1(3) we have $e \sim_r f$. □

We end this section by stating a nice result strengthening the already "strong" lifting condition that enabling ideals possess.

**Lemma 1.5** ([2, Lemma 3.3]). *Let $x, y, s, t \in R$, let $e \in \mathrm{idem}(R)$, and let $I \trianglelefteq R$ be an enabling ideal. If $xs \equiv ty \equiv e \pmod{I}$, then there exists $f \in \mathrm{idem}(R)$ such that $f \equiv e \pmod{I}$ and $f \in xRy$.*

## 2. Lifting commuting idempotents

One of the main reasons to desire that idempotents lift, say modulo the Jacobson radical, is that this allows one to further lift some of the structure inherent in the factor ring $R/J(R)$ up to $R$. From a historical point of view particular emphasis has been placed on lifting orthogonal idempotents to orthogonal idempotents. This has been very effective, for instance, in understanding semiperfect rings. Surprisingly, it appears that other natural situations, such as lifting *commuting* idempotents, have been ignored.

The main goal of this section is to prove that if two idempotents commute modulo the Jacobson radical, then they can be lifted to commuting idempotents under a weak (but necessary) assumption on the Jacobson radical. A surprising consequence of this work is that one of the lifted idempotents can be left unchanged, and the other only conjugated! In the next section we will also connect these results to a generalization of exchange rings.

**Theorem 2.1.** *Let $R$ be a ring, and let $e, f \in \mathrm{idem}(R)$ with $ef - fe \in J(R)$. There exist commuting idempotents $g, h \in \mathrm{idem}(R)$ with $g - e, h - f \in J(R)$ if and only if $\overline{ef} \in \mathrm{idem}(R/J(R))$ lifts modulo $J(R)$. In this case, we can take $g = u^{-1}eu$ and $h = f$, for some unit $u \equiv 1 \pmod{J(R)}$.*

*Proof.* Fix $e, f \in \mathrm{idem}(R)$ with $ef \equiv fe \pmod{J(R)}$.

($\Rightarrow$): Fix $g, h \in \mathrm{idem}(R)$ with $g \equiv e, h \equiv f \pmod{J(R)}$ and $gh = hg$. Note that

$$(gh)(gh) = g(hg)h = g(gh)h = gh$$

so $gh \in \mathrm{idem}(R)$. Further $gh \equiv ef \pmod{J(R)}$, so $ef$ lifts to an idempotent.

($\Leftarrow$): The element $ef$ is automatically an idempotent modulo $J(R)$ from the assumption that $ef \equiv fe \pmod{J(R)}$, by a similar computation as in the previous paragraph. Fix an idempotent $p \in R$ which lifts $ef$ modulo $J(R)$. Since $J(R)$ is an enabling ideal and $p \equiv ef \equiv fe \pmod{J(R)}$, we may assume $p \in eRe$ by Lemma 1.5. Letting $q = e - p$ we compute

$$q^2 = (e-p)(e-p) = e^2 - ep - pe + p^2 = e - p - p + p = e - p = q.$$

Hence, $q \in \mathrm{idem}(R)$ and $q = e - p \equiv e - ef \pmod{J(R)}$. Thus $e - ef$ also lifts modulo $J(R)$. By a similar argument, we see that $f - ef$ lifts modulo $J(R)$.

Using enabling again, fix an idempotent $s \in eRf$ with $s \equiv ef \pmod{J(R)}$. Then since

$$(e - ef)(1 - s) \equiv e - ef \pmod{J(R)},$$

fix an idempotent $r \in eR(1-s)$ with $r \equiv e - ef \pmod{J(R)}$. Finally, since

$$(1-s)(1-r)(f - ef) \equiv f - ef \pmod{J(R)},$$

fix an idempotent $t \in (1-s)(1-r)Rf$ with $t \equiv f - ef \pmod{J(R)}$.

We have $rs = rt = st = 0$. By [2, Proposition 4.4], the element $u := 1 - sr - tr - ts + 2tsr$ is a unit of $R$, the family $\{ur, us, ut\}$ is a pairwise orthogonal family of idempotents, and $\{ru, su, tu\}$ is also a pairwise orthogonal family of idempotents. (It is also a straightforward task to simply check these facts.) We find

$$u = 1 - sr - tr - ts + 2tsr \equiv 1 \pmod{J(R)},$$

since $r$, $s$, and $t$ are pairwise orthogonal modulo the Jacobson radical. Take $g = ur + us \equiv (e - ef) + ef = e \pmod{J(R)}$ and $h = us + ut \equiv ef + (f - ef) = f \pmod{J(R)}$. Since $ur$ and $us$ are orthogonal idempotents, we see that $g^2 = g$; and similarly $h^2 = h$. Finally $gh = us = hg$, so $g$ and $h$ commute.

We have now completed the proof of the biconditional. We will continue working with the idempotents in the previous paragraph in order to prove the last statement of the theorem. Let $g' = ru + su \in eR$, which is an idempotent. Since $eg' = g'$ and $g' \equiv r + s \equiv (e - ef) + ef = e \pmod{J(R)}$, by Lemma 1.3 we have $g' \sim_r e$, and so (by Lemma 1.1) we may write $g' = ev$ with $v = 1 + ex(1-e)$ for some $x \in R$. Note that since $e + ex(1-e) = g' \equiv e \pmod{J(R)}$, we also have $v \equiv 1 \pmod{J(R)}$. By a similar (symmetric) argument, we have $h \sim_\ell f$, so we can write $h = wf$ with $w = 1 + (1-f)yf$, and $w \equiv 1 \pmod{J(R)}$.

Note that $ve = e$ so $v^{-1}e = e$. Similarly, we have $fw = f = fw^{-1}$. From $gh = hg$ we find $(ug'u^{-1})wf = wf(ug'u^{-1})$. Conjugating by $w$, we see

$$w^{-1}uevu^{-1}wf = fuevu^{-1}w.$$

Using $v^{-1}e = e$ twice and $fw^{-1} = f$ once, this yields

$$w^{-1}uv^{-1}evu^{-1}wf = fw^{-1}uv^{-1}evu^{-1}w.$$

Thus, conjugating $e$ by $vu^{-1}w \equiv 1 \pmod{J(R)}$ produces an idempotent that commutes with $f$. $\square$

It is known that there are many situations when given an element $x \in R$ with $x^2 \equiv x \pmod{J(R)}$, then $x$ lifts modulo $J(R)$ to an idempotent of $R$ *without* assuming that all idempotents lift modulo $J(R)$. Usually this happens when $x$ is related in some way to known idempotents of $R$. For instance, as we see in Lemma 1.4, if there is an idempotent $e \in R$ with $\overline{x} \sim_r \overline{e}$ in $R/J(R)$, then $x$ always lifts to a right associate idempotent of $e$. Thus, because the element $ef \in R$ in Theorem 2.1 is defined equationally in terms of actual idempotents of $R$, we might hope that the lifting hypothesis of the theorem is unnecessary. The following example shows that, unfortunately, the lifting condition is not superfluous.

**Example 2.2.** *There exists a ring $R$ and idempotents $e, f \in \text{idem}(R)$ such that $e$ and $f$ commute modulo $J(R)$, but $\overline{ef} \in \text{idem}(R/J(R))$ does not lift modulo $J(R)$.*

*Construction and proof.* Let $\mathbb{Z}_{(6)}$ be the integers localized away from (6), and let $R = \mathbb{M}_2(\mathbb{Z}_{(6)})$. Fix $e = \left(\begin{smallmatrix} -1 & -1 \\ 2 & 2 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 2 & -1 \\ 2 & -1 \end{smallmatrix}\right)$. We compute

$$ef = \begin{pmatrix} -4 & 2 \\ 8 & -4 \end{pmatrix}, \qquad fe = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}$$

so $ef - fe \in J(R)$. Every representative of the coset $ef + J(R)$ has all entries with positive 2-adic valuation, so the only idempotent in that coset is 0. But $ef \not\equiv 0 \pmod{J(R)}$, so $\overline{ef} \in \text{idem}(R/J(R))$ does not lift modulo $J(R)$ to an idempotent. By Theorem 2.1 we know that there are no idempotents $g \equiv e, h \equiv f \pmod{J(R)}$ with $gh = hg$. This fact can also be checked directly, but we leave that involved computation to the motivated reader.

We do note that since $\mathbb{Z}_{(6)}$ is a PID, any two rank 1 idempotents from $R = \mathbb{M}_2(\mathbb{Z}_{(6)})$ must be conjugate, and thus any two idempotents in $R$ have conjugates that commute. In the case above, if we take $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in R$ with $a = d$, $c = -2b - 3d$, and $\gcd(ad - bc, 6) = 1$ then $u^{-1}eu = f$. Such a choice is possible, for instance take $a = d = 3$, $b = -2$, and $c = -5$. However, for such a unit $u$ it is easily checked that $u \not\equiv 1 \pmod{J(R)}$. □

One might also wonder if the final statement of Theorem 2.1 could be improved, by strengthening the conclusion that $g$ can be taken as a conjugate of $e$, by instead choosing $g$ to be a right (or left) associate of $e$. This is also not possible.

**Example 2.3.** *There exists a ring $R$, idempotents $e, f \in R$ which commute modulo the Jacobson radical $J(R)$, and idempotents lift modulo $J(R)$, but no left or right associate of $e$ commutes with any left or right associate of $f$.*

*Construction and proof.* Let $F$ be a field, and let $R = \mathbb{T}_2(F)^4$ which is the direct product of four copies of the $2 \times 2$ upper-triangular matrix ring over $F$. Notice that $J(R)$ is nilpotent of index 2, and so idempotents lift modulo $J(R)$. Put

$$e = \left( \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

and put

$$f = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

It is simple to compute that $e$ and $f$ commute modulo $J(R)$.

Over $S = \mathbb{T}_2(F)$, the only left associate (idempotent) of $e_{1,1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is itself, and similarly the only left associate of $e_{1,1} + e_{1,2} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is itself. Thus there is no left associate of $e$ which commutes with any left associate of $f$; this is seen visually by considering only the first coordinates of $e$ and $f$. A symmetrical calculation shows that the only right associate of $e_{2,2}$ is itself. So no left associate of $e$ commutes with any right associate of $f$, by considering only the second coordinates of $e$ and $f$. The other two cases are dealt with similarly. □

Quite surprisingly, if we weaken the commuting condition $ef = fe$ to "(left) half-commuting," meaning we only have $ef = efe$, it is still possible to characterize when this condition lifts through the Jacobson radical. Further, we can still keep one of the idempotents fixed, and the other can be chosen in a manner which is stronger than using conjugation. We will also see in the next section that half-commuting is deeply related to the internal exchange property for modules.

**Theorem 2.4.** *Let $R$ be a ring and let $e, f \in \mathrm{idem}(R)$ with $ef - efe \in J(R)$. There exist idempotents $g, h \in \mathrm{idem}(R)$ satisfying $gh = ghg$ with $g - e, h - f \in J(R)$ if and only if $\overline{ef} \in \mathrm{idem}(R/J(R))$ lifts modulo $J(R)$. In this case, we can take $g = eu$ and $h = f$, for some unit $u \equiv 1 \pmod{J(R)}$.*

*Proof.* Fix $e, f \in \mathrm{idem}(R)$ with $ef \equiv efe \pmod{J(R)}$. Notice that this condition is sufficient to give $\overline{ef} \in \mathrm{idem}(R/J(R))$.

($\Rightarrow$): This is the same as in Theorem 2.1, *mutatis mutandis*.

($\Leftarrow$): The proof of this implication is similar in spirit to that given in Theorem 2.1, but we depart in some significant ways. First, since $\overline{ef} \in \mathrm{idem}(R/J(R))$ lifts, and $ef \equiv efe \pmod{J(R)}$, by the enabling property of $J(R)$ we can find an idempotent $p \in eRe$ with $p \equiv ef \pmod{J(R)}$. Now $q = e - p$ is an idempotent of $R$, for the same reasons as before, with $q \equiv e - ef = e(1-f) \pmod{J(R)}$. (Note that we do not know that $f - ef$ lifts to an idempotent, since we don't have a symmetry argument any longer.)

By enabling, fix $r \in \mathrm{idem}(R) \cap eR(1-f)$ with $r \equiv e(1-f) \pmod{J(R)}$. Then fix $s \in \mathrm{idem}(R) \cap eRf(1-r)$ with $s \equiv ef(1-r) \equiv ef \pmod{J(R)}$. Note in particular that $sr = 0$. Set $g = r + (1-r)s = r + s - rs \in eR$. We check (dropping all terms involving $sr$) that

$$g^2 = (r + s - rs)(r + s - rs) = r^2 + rs - r^2 s + s^2 - rs^2 = r + s - rs = g$$

and so $g$ is an idempotent. Also $eg = g$ and

$$g = r + s - rs \equiv r + s \equiv (e - ef) + ef = e \pmod{J(R)}$$

so by Lemma 1.3 we have $g \sim_r e$. Hence Lemma 1.1 implies $g = eu$ for some unit $u \equiv 1 \pmod{J(R)}$.

To complete the proof it suffices to show that $gf = gfg$. Write $r = ex(1-f)$ and $s = eyf(1-r)$ for some $x, y \in R$. Noting in particular that $rf = 0$, we calculate

$$gf = (r + (1-r)s)f = (1-r)eyf.$$

Therefore,

$$\begin{aligned} gfg &= (1-r)eyfg = (1-r)eyf(r + (1-r)s) = (1-r)(eyfr + eyf(1-r)s) = (1-r)(eyfr + s^2) \\ &= (1-r)(eyfr + s) = (1-r)(eyfr + eyf(1-r)) = (1-r)(eyf) = gf. \end{aligned}$$

□

**Remark 2.5.** We note the following four interesting facts about the results of this section.

(1) The unit $u$ in Theorem 2.4 can be chosen to have the additional property $ue = e$ by a quick application of Lemma 1.1, and hence $u^{-1}e = e$. Thus $g$ is also a conjugate of $e$.
(2) We can see that the condition "$\overline{ef} \in \mathrm{idem}(R/J(R))$ lifts modulo $J(R)$" in Theorem 2.4 is not superfluous by appealing to Example 2.2, using the same idempotents.
(3) In Theorems 2.1 and 2.4, one can replace $J(R)$ everywhere by any ideal $I \trianglelefteq R$ with $I \subseteq J(R)$.
(4) Besides the fact that $J(R)$ is an enabling ideal, the only other special property of $J(R)$ we used is Lemma 1.3. Thus Theorems 2.1 and 2.4 remain true if we replace $J(R)$ by any enabling ideal as long as we drop the final sentence from the statement of each theorem.

## 3. Internal exchange

We now turn to the internal exchange property, which was defined in the introduction. The following result of Mohamed connects this property to endomorphism rings.

**Proposition 3.1** (cf. [9, Corollary 3])**.** *Let $R$ be a ring, let $M_R$ be a right $R$-module, and let $E = \mathrm{End}(M_R)$. The following are equivalent:*

(1) *$M_R$ has the finite internal exchange property.*
(2) *$E_E$ has the finite internal exchange property.*
(3) *${}_EE$ has the finite internal exchange property.*
(4) *For every $e, f \in \mathrm{idem}(E)$, there exists some $g \in \mathrm{idem}(E)$ such that $g \sim_r e$ and $gfg = gf$.*

Note that condition (4) is left-right symmetric, by applying the condition to the complementary idempotents $1-e, 1-f$. A ring $R$ for which $R_R$ has the finite internal exchange property is called an *internal exchange ring*. The following are just a few examples of internal exchange rings.

- Any ring with only trivial idempotents.
- More generally, any abelian ring (i.e. all idempotents are central) is an internal exchange ring. In particular, commutative rings and local rings are internal exchange rings.
- All exchange rings.
- If $R$ is an internal exchange ring, and $e \in \mathrm{idem}(R)$, then $eRe$ is an internal exchange ring. This is because the $\aleph$-internal exchange property passes to direct summands.
- A matrix ring $\mathbb{M}_n(R)$, with $n \geq 2$, is an internal exchange ring if and only if $R$ is an exchange ring [11, cf. Corollary 4.2]. In particular, while $A = \mathbb{Z}$ (or $\mathbb{Z}_{(6)}$) is an internal exchange ring, $\mathbb{M}_n(A)$ is not an internal exchange ring for any $n \geq 2$.

Recall that a ring $R$ is an exchange ring if and only if $R/J(R)$ is an exchange ring and idempotents lift modulo $J(R)$. In internal exchange rings, such as $\mathbb{Z}_{(6)}$, idempotents do not need to lift modulo the Jacobson radical. However, when they do, it is often possible to lift the internal exchange property as well. Our main goal in this section is to prove the following theorem.

**Theorem 3.2.** *Let $R$ be a ring. Suppose that there is an ideal $I \trianglelefteq R$ with $I \subseteq J(R)$ such that for each pair of idempotents $e, f \in \mathrm{idem}(R)$ we have that*

(1) *if $ef - efe \in I$, then $\overline{ef} \in \mathrm{idem}(R/J(R))$ lifts modulo $J(R)$, and*
(2) *there exists $\varepsilon \in \mathrm{idem}(R/I)$ such that $\varepsilon \sim_r e + I$ and $\varepsilon(f + I) = \varepsilon(f + I)\varepsilon$ in $R/I$.*

*Then $R$ is an internal exchange ring.*

Before we prove this theorem, we wish to describe some important consequences of the result and the assumed conditions. The first corollary is found implicitly in [11, Proposition 5.4], given a very different proof.

**Corollary 3.3.** *Assume $I \trianglelefteq R$, $I \subseteq J(R)$, and idempotents lift modulo $I$. If $R/I$ is an internal exchange ring, then $R$ is an internal exchange ring.*

*Proof.* Condition (1) of Theorem 3.2 holds due to the lifting hypothesis, because if $ef - efe \in I$ then $(ef)^2 \equiv ef \pmod{I}$. Since $R/I$ is an internal exchange ring, condition (2) of Theorem 3.2 holds in the factor ring $R/I$ by Proposition 3.1. □

**Corollary 3.4.** *Let $R$ and $S$ be rings, and let ${}_RM_S$ be an $R$-$S$-bimodule. The rings $R$ and $S$ are both internal exchange rings if and only if the extension $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ is an internal exchange ring.*

*Proof.* ($\Rightarrow$): Idempotents lift modulo $I = \left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right) \trianglelefteq T$, since this is a nilpotent ideal. Also $T/I \cong R \times S$ is a direct product of internal exchange rings, so also is an internal exchange ring. Now one applies Corollary 3.3.

($\Leftarrow$): This direction follows quickly from Proposition 3.1(4), by considering diagonal entries. □

**Corollary 3.5.** *A ring $R$ is an internal exchange ring if and only if the ring of upper triangular matrices $\mathbb{T}_n(R)$ is an internal exchange ring for all $n \geq 1$.*

*Proof.* The forward direction follows from the previous corollary, applied inductively. The backwards direction is a tautological weakening. □

**Corollary 3.6.** *A ring $R$ is an internal exchange ring if and only if the power series ring $R[\![x]\!]$ is an internal exchange ring.*

*Proof.* The forward direction follows from the fact that idempotents lift modulo $(x)$, since the ring $R[\![x]\!]$ is complete with respect to powers of $x$. The reverse implication easily follows by considering the constant terms of power series, and by Proposition 3.1(4). □

We are now ready to give the surprisingly quick proof of the main theorem.

*Proof of Theorem 3.2.* Let $e, f \in \mathrm{idem}(R)$. Note that condition (2) is equivalent to the seemingly weaker assumption that "there exists $p \in \mathrm{idem}(R)$ such that $p \sim_r e$ and $(p+I)(f+I) = (p+I)(f+I)(p+I)$ in $R/I$" by Lemma 1.4. Now by condition (1), the idempotent $\overline{ep} \in \mathrm{idem}(R/J(R))$ lifts, and so Theorem 2.4 gives us an idempotent $g \sim_r p \sim_r e$ with $gf = gfg$. This proves that $R$ is an internal exchange ring. □

## 4. The $(C_3)$ property

We recall the following definitions, two of which appeared in the introduction, and all of which are standard.

**Definition 4.1.** A module $M$ is *square-free* if whenever $N \subseteq M$ and $N = Y_1 \oplus Y_2$ with $Y_1 \cong Y_2$, then $Y_1 = Y_2 = (0)$. More generally, we say that $M$ is *summand-square-free* if the condition above holds under the additional assumption that $Y_1, Y_2 \subseteq^{\oplus} M$ (i.e. any non-intersecting isomorphic summands of $M$ must be zero).

**Definition 4.2.** A module $M$ has the $(C_3)$ *property* if whenever $A, B \subseteq^{\oplus} M$ with $A \cap B = (0)$, then $A \oplus B \subseteq^{\oplus} M$.

These module-theoretic properties arise naturally when studying generalizations of injectivity, especially in concert with the exchange property. One interesting connection between them is the following.

**Lemma 4.3** ([9, Proposition 10])**.** *If $M$ is a summand-square-free module with the finite internal exchange property, then $M$ has $(C_3)$.*

*Proof.* We include a short proof. Fix $A, B \subseteq^{\oplus} M$ with $A \cap B = (0)$. We can then write $M = B \oplus C$ for some submodule $C \subseteq M$. From the internal exchange property, we get $M = A \oplus B' \oplus C'$ for submodules $B' \subseteq B$ and $C' \subseteq C$. From the modular property, since these submodules are summands of $M$ we may write $B = B' \oplus B''$ and $C = C' \oplus C''$. Thus $A \cong B'' \oplus C''$, so we can write $A = A_1 \oplus A_2$ with $A_1 \cong B''$ and $A_2 \cong C''$. The summand-square-free property (applied to $B''$ and $A_1$) implies that $B'' = (0)$, and so $A \oplus B = A \oplus B' \subseteq^{\oplus} M$, with complement $C'$. □

There are many modules which are both square-free and have $(C_3)$. One natural class with these properties is the set of distributive modules; these are the modules whose submodules satisfy the distributive property $A \cap (B + C) = A \cap B + A \cap C$.

**Lemma 4.4.** *If $M_R$ is distributive, then $M_R$ has $(C_3)$, is square-free, and has finite internal exchange.*

*Proof.* It is well known that the endomorphism ring of a distributive module is abelian (see for example [22, Section 2.6, entry 2.50(2)]) and hence $M_R$ has finite internal exchange and $(C_3)$. Furthermore, $M_R$ is square-free by [21, Corollary 1(i)$'$ of Proposition 1.1]. But we can provide a short direct proof of the lemma.

First, to show that $M_R$ has the finite internal exchange property, assume $M = A \oplus B$ for some $A, B \subseteq M$. Also let $C \subseteq^{\oplus} M$, say $M = C \oplus D$. Since $M_R$ is distributive,

$$M = C \oplus (M \cap D) = C \oplus ((A \oplus B) \cap D) = C \oplus (A \cap D) \oplus (B \cap D)$$

which demonstrates that $M$ has finite internal exchange.

Second, to show square-freeness let $A, B \subseteq M$ with $A \cap B = (0)$ and assume there is a right $R$-module isomorphism $f : A \to B$. Then $N = \{a + f(a) \ : a \in A\}$ is a submodule of $M_R$, isomorphic to $A$ and $B$. By distributivity

$$N = N \cap (A + B) = (N \cap A) + (N \cap B) = (0).$$

Thus $A = B = (0)$, which demonstrates that $M_R$ is square-free.

Now the $(C_3)$ property is an immediate consequence of Lemma 4.3. □

In the remainder of this section we give some interesting facts and examples concerning the $(C_3)$ property of a module and information about the endomorphism ring. Unlike (internal) exchange, the $(C_3)$ property is not an endomorphism ring invariant. However, our next two results show that this is almost the case.

**Lemma 4.5.** *A module $M$ has $(C_3)$ if and only if for every pair of idempotents $e, f \in E = \mathrm{End}(M)$ with $eM \cap fM = (0)$ there exist idempotents $g, h \in E$ with $g \sim_r e$, $h \sim_r f$, and $g, h$ are orthogonal.*

*Proof.* $(\Rightarrow)$: Assume $M$ has $(C_3)$. Let $e, f \in E$ with $eM \cap fM = (0)$. Thus $eM + fM$ is a direct summand of $M$, so after fixing a complement $N \subseteq M$ we have $eM \oplus fM \oplus N = M$. Let $g$ be the projection to $eM$ with kernel $fM + N$, and let $h$ be the projection to $fM$ with kernel $eM + N$. We have $gM = eM$, so by Lemma 1.1 we know $g \sim_r e$. Similarly, $h \sim_r f$. Clearly, $g$ and $h$ are orthogonal.

$(\Leftarrow)$: Let $e, f \in E$ with $eM \cap fM = (0)$. Assuming the condition stated in the lemma for pairs of idempotents, fix $g, h \in \mathrm{idem}(E)$ with $g \sim_r e$, $h \sim_r f$, and $g, h$ orthogonal. By Lemma 1.1, we have $eM = gM$ and $fM = hM$. Thus $eM + fM = gM + hM = (g+h)M$ is a summand of $M$ generated by the idempotent $g+h$. (That $g+h$ is idempotent follows quickly from the orthogonality of $g$ and $h$.) □

**Proposition 4.6.** *Let $R$ be a ring, let $M_R$ be a right $R$-module, and let $E = \mathrm{End}(M_R)$. The right regular module $E_E$ has $(C_3)$ if and only if $M_R$ has $(C_3)$ and for every pair of idempotents $e, f \in E$ with $eE \cap fE = (0)$ we have $eM \cap fM = (0)$.*

*Proof.* $(\Rightarrow)$: Suppose $E_E$ has $(C_3)$. Fix idempotents $e, f \in E$ with $eM \cap fM = (0)$. Suppose by way of contradiction that $eE \cap fE \neq (0)$, so we may fix some nonzero $x \in eE \cap fE$. In particular, $ex = x$ and $fx = x$. Then $0 \neq xM \subseteq eM \cap fM$, yielding the necessary contradiction. Thus we have $eE \cap fE = (0)$. By the previous lemma, applied to $E_E$, we may find $g, h \in \mathrm{idem}(E)$ with $g \sim_r e$, $h \sim_r f$, and $g, h$ are orthogonal. But these idempotents demonstrate the $(C_3)$ property for $M_R$ as well, by another application of the previous lemma.

To show the second condition, suppose $e, f \in \mathrm{idem}(E)$ with $eE \cap fE = (0)$. Then since $E_E$ has $(C_3)$ by hypothesis, by Lemma 4.5 we can fix idempotents $g, h \in \mathrm{idem}(E)$ with $g \sim_r e, h \sim_r f$, and $g, h$ orthogonal. By Lemma 1.1, we have $gM = eM$ and $hM = fM$. Thus, since $g$ and $h$ are orthogonal, we obtain $eM \cap fM = gM \cap hM = (0)$.

$(\Leftarrow)$: We work directly, so assume $M_R$ has $(C_3)$ and also assume the other condition in the statement of the proposition. Let $e, f \in \mathrm{idem}(E)$ with $eE \cap fE = (0)$. By hypothesis, we have $eM \cap fM = (0)$. Applying the previous lemma to $M_R$, we obtain idempotents which (by another application of the lemma) demonstrate that $E_E$ has $(C_3)$, □

By Proposition 4.6, if $E_E$ has $(C_3)$, then $M_R$ has $(C_3)$. The following example shows that the converse does not hold.

**Example 4.7.** *There exists a module $M_R$ with $(C_3)$, but the right regular module $E_E$ for the endomorphism ring does not have $(C_3)$.*

*Construction and proof.* As worked out in [8, Example 2.6], the $\mathbb{Z}$-module $\mathbb{Z}^2_{p^\infty}$ works. □

We finish this section with an example showing that a certain left-right asymmetry can hold for the $(C_3)$ property.

**Example 4.8.** *There exists a ring $R$ such that $R_R$ has $(C_3)$ but ${}_RR$ does not.*

*Construction and proof.* Let $F$ be a field (or even a commutative domain). We will show that for the ring

$$R = F\langle x, y, z \;:\; x^2 = x, y^2 = y, xy = y, yx = x, yz = xz, z^2 = 0, zxz = 0\rangle,$$

the right module $R_R$ has $(C_3)$, whereas the left module ${}_RR$ does not have $(C_3)$. Note that every element $\varepsilon \in R$ can be uniquely written in the form

$$\varepsilon = a + bx + cy + dz + exz + fzx + gzy + hxzx + ixzy, \tag{4.9}$$

where $a, b, c, d, e, f, g, h, i \in F$.

By Lemma 4.3, to show that $R_R$ has $(C_3)$ it suffices to show that $R_R$ has finite internal exchange and is summand-square-free. Let $I$ denote the ideal of $R$ generated by $z$. Observe that $I^2 = (0)$ and every element $\varepsilon \in R$ can be uniquely written in the form

$$\varepsilon = a + bx + cy + \omega,$$

where $a, b, c \in F$ and $\omega \in I$. Since $R/I \cong \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ is an exchange ring, and idempotents lift modulo $I \subseteq J(R)$, we see that $R$ is an exchange ring (and hence an internal exchange ring).

Now we show that $R_R$ is summand-square-free. For this, it suffices to show that if $\varepsilon$ and $\delta$ are non-trivial idempotents of $R$, then

$$\varepsilon R \cap \delta R \neq (0) \text{ or } \varepsilon R \ncong \delta R.$$

It is easy to see that any non-trivial idempotent of $R$ is one of the following types:

(A) $x + k(y - x) + \omega$,
(B) $1 - x - k(y - x) + \omega$,

where $\omega \in I$, $k \in F$. Now, if $\varepsilon$ and $\delta$ are both of type (A), then since $yz = xz$ and $I^2 = (0)$, we obtain $0 \neq xz = \varepsilon z = \delta z \in \varepsilon R \cap \delta R$, and thus $\varepsilon R \cap \delta R \neq (0)$. Similarly, if $\varepsilon$ and $\delta$ are both of type (B), then $0 \neq z - xz = \varepsilon z = \delta z \in \varepsilon R \cap \delta R$.

By symmetry, the only case left to consider is when $\varepsilon$ is of type (A) and $\delta$ is of type (B). If $\varepsilon R \cong \delta R$, then these are isomorphic idempotents so there exist $\alpha, \beta \in R$ such that $\varepsilon = \alpha\beta$ and $\delta = \beta\alpha$. Since 1 is in the support of $\delta$, it follows that 1 is in the supports of both $\alpha$ and $\beta$, and thus 1 is in the support of $\varepsilon$. This contradiction shows that $\varepsilon R \ncong \delta R$.

To prove that ${}_RR$ does not have $(C_3)$, we show first that $Rx \cap Ry = (0)$. For this we use the uniqueness of the form (4.9). Let $\lambda = \alpha x = \beta y \in Rx \cap Ry$, where

$$\alpha = a + bx + cy + dz + exz + fzx + gzy + hxzx + ixzy$$

and

$$\beta = a' + b'x + c'y + d'z + e'xz + f'zx + g'zy + h'xzx + i'xzy$$

for some $a, a', b, b', \ldots, i, i' \in F$. Since the products

$$\alpha x = (a + b + c)x + (d + f + g)zx + (e + h + i)xzx$$

and

$$\beta y = (a' + b' + c')y + (d' + f' + g')zy + (e' + h' + i')xzy$$

have disjoint supports, it follows that $\lambda = 0$ and thus $Rx \cap Ry = (0)$.

If ${}_RR$ were to have $(C_3)$, then by Lemmas 1.1 and 4.5 there would exist orthogonal idempotents $\gamma, \delta \in R$ such that

$$x = x\gamma \text{ and } y = y\delta.$$

Since every non-trivial idempotent of $R$ is of type (A) or (B), it would follow that $\gamma = x + \omega$ and $\delta = y + \upsilon$ for some $\omega, \upsilon \in R$. But then $\gamma\delta \neq 0$, which shows that ${}_RR$ does not have $(C_3)$. □

## 5. Square-free modules

It turns out that there is another natural situation where square-free modules have the $(C_3)$ property, without assuming any exchange hypotheses. To motivate this result, we need two more definitions found in the literature.

**Definition 5.1.** If $M_R$ is a module, we set

$$\Delta := \Delta(M) = \{\varphi \in \mathrm{End}(M_R) \ : \ \ker(\varphi) \text{ is essential in } M\}.$$

This is an ideal of $\mathrm{End}(M_R)$. When $M = R_R$ is the right regular module, then $\Delta$ is just the *right singular ideal* $S_r(R) \trianglelefteq R$.

**Definition 5.2.** Given a module $M$, we say that *direct complements are essentially unique* if whenever $M = A \oplus B = A \oplus C$, then $B \cap C$ is essential in $B$ (and hence also in $C$, by symmetry).

It is not difficult to show that for square-free modules, direct complements are always essentially unique. For example, see [12, Part (3) of the proof of Lemma 3]. One connection between the two definitions above is the following result of Mohamed and Müller.

**Lemma 5.3.** *Idempotents in* $\mathrm{End}(M_R)$ *are central modulo* $\Delta(M)$ *if and only if direct complements are essentially unique in* $M$*. Equivalently, all idempotents of* $\mathrm{End}(M_R)$ *commute modulo* $\Delta(M)$*.*

*Proof.* The first equivalence is sketched in [15, Lemma 15]. Further, if idempotents in $E = \mathrm{End}(M_R)$ are central modulo $\Delta$, then it is tautologically weaker to merely say that they commute with each other. Conversely, suppose that all idempotents of $E$ commute modulo $\Delta$, and fix $e \in \mathrm{idem}(E)$ and $x \in E$. Since $e' = e + ex(1-e), e'' = e + (1-e)xe \in \mathrm{idem}(E)$, we have

$$ex - xe = (ex - exe) - (xe - exe) = (ee' - e'e) - (e''e - ee'') \in \Delta.$$

Thus, $e$ is central modulo $\Delta$. □

We can tie this back to the $(C_3)$ property with the following result.

**Proposition 5.4.** *Let* $R$ *be a ring, let* $M$ *be a right* $R$*-module, and let* $E = \mathrm{End}(M_R)$*. If direct complements are essentially unique in* $M$ *(e.g. if* $M$ *is square-free) and* $\Delta \subseteq J(E)$*, then* $M$ *has* $(C_3)$*.*

*Proof.* Fix $A, B \subseteq^{\oplus} M$ with $A \cap B = (0)$. Since $A$ is a summand, there exists an idempotent $e \in E$ with $eM = A$. Similarly, there is an idempotent $f \in E$ with $fM = B$. As idempotents of $E$ are central modulo $\Delta$ by Lemma 5.3, we have $ef - fe \in \Delta$.

Fix any nonzero element $m \in M$. Then, by the definition of $\Delta$, there exists some $r \in R$ with $mr \neq 0$ and $(ef - fe)(mr) = 0$. Thus $ef(mr) = fe(mr)$. But since $A \cap B = (0)$, this implies $ef(mr) = fe(mr) = 0$. Since $m$ was arbitrary, we must have $ef, fe \in \Delta$.

Now, as $\Delta \subseteq J(E)$, we have that $e$ and $f$ are orthogonal modulo the Jacobson radical. By [2, Theorem 4.1], there exists some unit $u \in U(E)$ with $eu$ and $fu$ orthogonal idempotents. Then we have

$$eM + fM = euM + fuM = (eu + fu)M \subseteq^{\oplus} M$$

so $M$ has $(C_3)$. □

The condition "$\Delta \subseteq J(E)$" is quite natural. For instance, it holds if $M$ has the finite exchange property [15, Lemma 11]. However, this condition is not always necessary to derive the conclusion of Proposition 5.4 as the next example shows.

**Example 5.5.** *There exists a square-free module $M$ with $(C_3)$ and the finite internal exchange property, but $\Delta \not\subseteq J(\mathrm{End}(M))$.*

*Construction and proof.* For a prime $p \geq 2$, set

$$R = \left\{ \begin{pmatrix} n & c \\ 0 & n \end{pmatrix} \;:\; n \in \mathbb{Z},\; c \in \mathbb{Z}_{p^\infty} \right\}$$

and put $M = R_R$. Since the quasi-cyclic group $\mathbb{Z}_{p^\infty}$ is divisible, the set of matrices with zero diagonal entries $I = \left(\begin{smallmatrix} 0 & \mathbb{Z}_{p^\infty} \\ 0 & 0 \end{smallmatrix}\right)$ is comparable with all (right) ideals of $R$ (with respect to inclusion). Furthermore, since all subgroups of $\mathbb{Z}_{p^\infty}$ form a chain, the ideals of $R$ contained in $I$ form a chain as well. Moreover, the ring $R/I \cong \mathbb{Z}$ is distributive (as a module over itself) and thus the lattice of ideals containing $I$ is distributive. Hence $M$ is a distributive module and Lemma 4.3 implies that $M$ is square-free, has $(C_3)$, and has finite internal exchange. On the other hand, it is easy to check that $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right) \in \Delta \setminus J(E)$. □

There is another relationship that exists between the hypotheses employed in Lemma 4.3 and Proposition 5.4.

**Theorem 5.6.** *Let $R$ be a ring and let $M_R$ be a right $R$-module. If direct complements are essentially unique in $M$, then $M$ is summand-square-free.*

*Proof.* We work directly, so assume that direct complements are essentially unique in $M$. Let $E = \mathrm{End}(M_R)$ and fix $e, f \in \mathrm{idem}(E)$ such that $eM \cong fM$ and $eM \cap fM = (0)$. It suffices to show that $e = f = 0$.

By [6, Example (B) following Proposition 21.20] we know that $e$ and $f$ are isomorphic in the endomorphism ring $E$, hence by Definition 1.2 there exist $a, b \in E$ with $e = ab$ and $f = ba$. Lemma 5.3 tells us that $e = ab$ and $f = ba$ are central modulo $\Delta(M_R)$. Thus, we find

$$e = e^2 = a(ba)b \equiv (ba)(ab) \equiv b(ab)a = f^2 = f \pmod{\Delta}.$$

On the other hand $ef - fe \in \Delta$. By the same argument as in the second paragraph of the proof of Proposition 5.4 we obtain $ef, fe \in \Delta$. Thus $0 \equiv ef \equiv e^2 = e \pmod{\Delta}$. Only the zero idempotent lives in $\Delta$, and so $e = 0$; and similarly $f = 0$. □

As one might expect, the converse of the previous theorem does not hold. This is true even under the additional assumption that the module has the finite exchange property.

**Example 5.7.** *There exists a module $M_R$ which is summand-square-free and has finite exchange, but does not have essentially unique complements.*

*Construction and proof.* Let $R$ be the ring constructed in Example 4.8. As shown in the construction of that example, the module $M = R_R$ is summand-square-free and has finite exchange. By Lemma 5.3, to complete the proof it suffices to show that $xz - zx \notin \Delta(R_R)(= S_r(R))$. The right annihilator of $xz - zx$ intersects $(x - y)R = (x - y)F$ trivially, so we are done. □

We end this section by showing that the square-free property behaves badly when passing to endomorphism rings.

**Proposition 5.8.** *Let $R$ be a ring, let $M_R$ be a right $R$-module, and let $E = \mathrm{End}(M_R)$. The statements "$M_R$ is square-free" and "$E_E$ is square-free" are logically independent.*

*Proof.* We first prove that "$M_R$ is square-free" does not imply "$E_E$ is square-free". Let $\mathbb{Z}_4$ be the ring of integers modulo 4 and let $R$ be the matrix ring

$$R = \left\{ \begin{pmatrix} s & t \\ 0 & s \end{pmatrix} \ : \ s, t \in \mathbb{Z}_4 \right\}.$$

Notice that the ring $R$ is commutative. Since the set

$$M = \left\{ \begin{pmatrix} 2s & t \\ 0 & 2s \end{pmatrix} \ : \ s, t \in \mathbb{Z}_4 \right\}$$

is the Jacobson radical of $R$, and hence an ideal, we have that $M$ is a right $R$-module. A quick computation demonstrates that the ideal

$$I = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} R = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$$

is the only minimal submodule of $M$, and so $M$ is square-free.

We claim that *the ring* $\mathrm{End}(M_R)$ *is isomorphic to the factor ring* $R/I$. To see this, let $\varphi : M_R \to M_R$ be any endomorphism. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M.$$

Then we have

$$\varphi(A) = \begin{pmatrix} 2a & b \\ 0 & 2a \end{pmatrix}, \quad \varphi(B) = \begin{pmatrix} 2c & d \\ 0 & 2c \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{Z}_4$. Since $0 = \varphi(0) = \varphi(AA) = \varphi(A)A = \left(\begin{smallmatrix} 0 & 2b \\ 0 & 0 \end{smallmatrix}\right)$, we have $b = 2e$ for some $e \in \mathbb{Z}_4$. Also, since $0 = \varphi(0) = \varphi(BB) = \varphi(B)B = \left(\begin{smallmatrix} 0 & 2c \\ 0 & 0 \end{smallmatrix}\right)$, we have $2c = 0$. We also compute

$$\begin{pmatrix} 0 & 2a \\ 0 & 0 \end{pmatrix} = \varphi(A)B = \varphi(AB) = \varphi(BA) = \varphi(B)A = \begin{pmatrix} 0 & 2d \\ 0 & 0 \end{pmatrix}$$

and so $2a = 2d$. Hence $\varphi(A) = \left(\begin{smallmatrix} 2d & 2e \\ 0 & 2d \end{smallmatrix}\right)$ and $\varphi(B) = \left(\begin{smallmatrix} 0 & d \\ 0 & 0 \end{smallmatrix}\right)$ for some $d, e \in \mathbb{Z}_4$.

Given an arbitrary element $X = \left(\begin{smallmatrix} 2s & t \\ 0 & 2s \end{smallmatrix}\right)$ and letting $1_R \in R$ denote the identity matrix, we see that $\varphi(X) = \varphi(A)(s1_R) + \varphi(B)(t1_R) = \left(\begin{smallmatrix} d & e \\ 0 & d \end{smallmatrix}\right)X$. Thus $\varphi$ is the same as multiplication by $\left(\begin{smallmatrix} d & e \\ 0 & d \end{smallmatrix}\right) \in R$. On the other hand, multiplication by $I$ on $M$ is trivial, and so we have a well-defined surjective map $\Psi : R/I \to \mathrm{End}(M_R)$ sending a coset $X + I$ to left multiplication by (any representative) $X$. It is also straightforward to check that $\Psi$ is injective and a ring homomorphism, so $\Psi$ is an isomorphism.

To show that the ring $\mathrm{End}(M_R)$ is not square-free (as a right module over itself), it suffices to show that the ring $R/I$ is not square-free (as a right module over itself). To see the latter, note that

$$I_1 = \begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad I_2 = \left\{ \begin{pmatrix} 2s & 2t \\ 0 & 2s \end{pmatrix} \ : \ s, t \in \mathbb{Z}_4 \right\}$$

are ideals of $R$ properly containing $I$ such that $I_1 \cap I_2 = I$ and the right $R/I$-modules $I_1/I$ and $I_2/I$ are isomorphic (an isomorphism $\psi : I_1/I \to I_2/I$ is given by $\psi\left(\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) + I\right) = \left(\begin{smallmatrix} 2x & 0 \\ 0 & 2x \end{smallmatrix}\right) + I$).

We now prove that "$E_E$ is square-free" does not imply "$M_R$ is square-free". Let $F$ be a field and let $R$ be the matrix ring

$$R = \left\{ \begin{pmatrix} r & s & t \\ 0 & r & u \\ 0 & 0 & v \end{pmatrix} \ : \ r, s, t, u, v \in F \right\}.$$

Let $e = e_{1,1} + e_{2,2} \in \mathrm{idem}(R)$ and set $M_R = eR$. This module is not square-free since $Fe_{1,3}$ and $Fe_{2,3}$ are isomorphic non-intersecting submodules. On the other hand, by [6, Corollary 21.7], we have $E := \mathrm{End}(M_R) = \mathrm{End}(eR) \cong eRe = F(e_{1,1} + e_{2,2}) + Fe_{1,2}$. The only right ideals of $eRe$ are

$$(0) \subsetneq Fe_{1,2} \subsetneq eRe$$

and so $E$ is square-free as a right module over itself. □

## 6. Examples of square-free modules without $(C_3)$

An interesting question raised in personal communication from M. Yousif is whether there exists a square-free module that doesn't have $(C_3)$. The answer is yes, and we devote this section to presenting two examples of modules without $(C_3)$.

**Example 6.1.** *There is a square-free module which does not have $(C_3)$.*

*Construction and proof.* Let $F$ be a field and put

$$R := F[x_1, x_2, \ldots]\langle e, f \ : \ e^2 = e, f^2 = f, ex_1 = fx_1 = 0, efx_2 = fex_2 = 0, efex_3 = fefx_3 = 0, \ldots\rangle.$$

For ease of notation we put $R_n := F[x_n, x_{n+1}, \ldots]$, $e_{2n} = (ef)^n$, $e_{2n+1} = (ef)^n e$, $f_{2n} = (fe)^n$, and $f_{2n+1} = (fe)^n f$ for each $n \geq 1$. (We also set $e_0 = f_0 = 1$.) An element $r \in R$ can be written uniquely as a (finite) sum of the form

$$r = r_0 + \sum_{i=1}^{\infty} (e_i r_i + f_i r_i')$$

where $r_i, r_i' \in R_{i+1}$ for each $i \geq 0$. (We also set $r_0' = 0$.)

**Claim 1:** $eR \cap fR = (0)$ and $eR + fR$ is not a direct summand of $R_R$.

*Proof of Claim 1.* The first statement follows from the relations. To prove the last statement, we show that $\overline{eR} + \overline{fR}$ is not even a direct summand of $\overline{R}_{\overline{R}}$ where $\overline{R} = R/(x_1, x_2, \ldots) \cong F\langle e, f \ : \ e^2 = e, f^2 = f\rangle = S$. We will work over $S$, and drop all bar notation.

Suppose by way of contradiction that $eS + fS = gS$ for some $g \in \text{idem}(S)$. Since $e, f \in gS$ it must be the case that $e = ge$ and $f = gf$. We see that $1 \notin \text{supp}(g)$ since 1 is not in the support of any element in $eS + fS$. Let $t \in \text{supp}(g)$ be a monomial of maximal length, say $\ell \geq 1$. If $t$ ends with $e$, then $tf$ is a monomial of length $\ell + 1$ in $\text{supp}(gf)$, contradicting the fact that $gf = f$. Similarly, one gets a contradiction in the case where $t$ ends with $f$. Since $\ell \geq 1$, this covers all possibilities. □

The fact that the module $R_R$ does not have $(C_3)$ now follows immediately from Claim 1. We note that one can show $\Delta(R_R)$ is the ideal generated by $ef, fe$, but we won't prove this as it isn't needed.

**Claim 2:** The module $R_R$ is square-free.

*Proof of Claim 2.* Assume $rR \cong sR$ for some $r, s \in R$; we may assume that the isomorphism takes $r \mapsto s$, and so $r$ and $s$ have the same right annihilators. We need to show that either $r = s = 0$ or $rR \cap sR \neq (0)$. We may assume $r \neq 0$ and $s \neq 0$. Write

$$r = r_0 + \sum_{i=1}^{\infty} (e_i r_i + f_i r_i')$$

and

$$s = s_0 + \sum_{i=1}^{\infty} (e_i s_i + f_i s_i')$$

with $r_i, r_i', s_i, s_i' \in R_{i+1}$ for each $i \geq 0$ (and $r_0' = s_0' = 0$).

If $r_0 \neq 0$ then $rx_1 \neq 0$, and so $sx_1 \neq 0$ hence $s_0 \neq 0$. In this case $0 \neq rx_1 s_0 = r_0 s_0 x_1 = sx_1 r_0 \in rR \cap sR$. Thus, we may assume $r_0 = s_0 = 0$.

Fix $n \geq 1$ minimal such that either $r_n \neq 0$ or $r_n' \neq 0$. Without loss of generality we may assume that $s_k, s_k' = 0$ for all $k < n$ (switching the roles of $r$ and $s$ if necessary). Now $rx_{n+1} = e_n r_n x_{n+1} + f_n r_n' x_{n+1} \neq 0$, and therefore $sx_{n+1} \neq 0$. Thus, either $s_n \neq 0$ or $s_n' \neq 0$.

**Case 1:** Assume $n = 2k$ is even. Thus $e_n = (ef)^k$ and $f_n = (fe)^k$. If $r_n \neq 0$ we see that $rfx_{n+1} = e_n r_n x_{n+1} \neq 0$. Thus, $0 \neq sfx_{n+1} = e_n s_n x_{n+1}$, and so $s_n \neq 0$. Hence, $0 \neq r(fs_n x_{n+1}) = e_n r_n s_n x_{n+1} = s(fr_n x_{n+1}) \in rR \cap sR$. A similar argument works if $r'_n \neq 0$.

**Case 2:** Assume $n$ is odd. A similar argument as in Case 1 suffices. □

That finishes the proof of Claim 2, and thus $R_R$ is square-free and does not have $(C_3)$ as claimed. □

Notice that by Proposition 5.4, for the ring $R$ in Example 6.1 we have $\Delta(R_R) \nsubseteq J(\mathrm{End}(R_R))$. Also, Lemma 4.3 implies that $R_R$ does not have finite internal exchange.

We now turn to the second example of this section, which should be compared to Proposition 5.4

**Example 6.2.** *There exists a summand-square-free module $M$ without $(C_3)$, but $\Delta(M) \subseteq J(\mathrm{End}(M))$.*

*Construction and proof.* Let $F$ be a field and put

$$R = F[z]\langle e, f \ : \ e^2 = e, f^2 = f, efz = fez = 0\rangle.$$

An arbitrary element $r \in R$ can be written uniquely in the form

$$r = g_0(z) + eg_1(z) + fg_2(z) + \sum_{n=2}^{m} (\alpha_n e_n + \alpha'_n f_n) \tag{6.3}$$

with $g_0(z), g_1(z), g_2(z) \in F[z]$, $\alpha_n, \alpha'_n \in F$ for each $2 \leq n \leq m$ (for some $m \geq 2$), and $e_n$, $f_n$ are defined as in Example 6.1.

By a nearly identical argument as in Claim 1 in the proof of Example 6.1, we have that $M = R_R$ does not have $(C_3)$. To show that $R_R$ is summand-square-free, let us consider any nonzero $g, h \in \mathrm{idem}(R)$ with $g \cong h$. Since for any $a, b \in R$ the element $ab - ba$ belongs to the ideal of $R$ generated by $\{ef, fe\}$, it follows from Definition 1.2 and the relations defining $R$ that $(g - h)z = 0$, i.e. $gz = hz$. Furthermore, it is easy to see that if $r$ in (6.3) is a nonzero idempotent of $R$, then $g_0(z) + eg_1(z) + fg_2(z) \neq 0$ and hence $rz \neq 0$. Applying this fact to $g, h \in \mathrm{idem}(R)$, we have

$$0 \neq gz = hz \in gR \cap hR.$$

This proves that $R_R$ is summand-square-free.

Finally, we need to prove that $\Delta(R_R) \subseteq J(\mathrm{End}(R_R))$. It suffices to show that $\Delta(R_R) = (0)$. Suppose $r \neq 0$ has the form in (6.3). If $g_0(z) \neq 0$, then the right annihilator of $r$ intersects $(z - ez - fz)R$ trivially. So we may assume $g_0(z) = 0$. If $g_1(z) \neq 0$, then the right annihilator of $r$ intersects $ezR$ trivially, and a similar statement holds when $g_2(z) \neq 0$. Thus we may assume $g_1(z) = g_2(z) = 0$. Now we may assume (by symmetry) that a monomial $r' \in \mathrm{supp}(r)$ of maximal length (say $\ell_1$) ends in $e$. The right annihilator of $r$ then intersects $feR$ trivially, since given any nonzero $s \in feR$ and letting $s' \in \mathrm{supp}(s)$ be a monomial of maximal length (say $\ell_2$), we see that $r's'$ does not cancel from the product $rs$ (since it has length $\ell_1 + \ell_2$, and no other monomial matches it). □

## 7. Characterizing all connections

In this paper we have looked at the associations entailed among modules with the finite (internal) exchange property, $\Delta(M) \subseteq J(\mathrm{End}(M))$, the $(C_3)$ property, (summand-)square-freeness, and essentially unique direct complements. The containments (as either proven or cited previously) among modules with these properties are displayed in Figure 7.2. Note that all conditions are given by simple loops, and all but the $(C_3)$ property are displayed using rectangular boxes.

To prove there are no other connections among these properties, we need to list examples of the twenty-nine possible classes of modules available in each of the regions in the diagram. The following lemma aids in that endeavor.

**Lemma 7.1.** *Let $S$ and $T$ be rings, let $R = S \times T$, and let $\mathcal{P}$ be any of the seven properties above. Then $R_R$ has $\mathcal{P}$ if and only if $S_S$ and $T_T$ both have $\mathcal{P}$.*

*Proof.* One proves that the Jacobson radical, the right singular ideal, idempotents, units, right ideals, isomorphisms of right ideals, essential submodules, and so forth, behave well with respect to direct products. These are well-known facts, left to the dedicated reader to verify. □

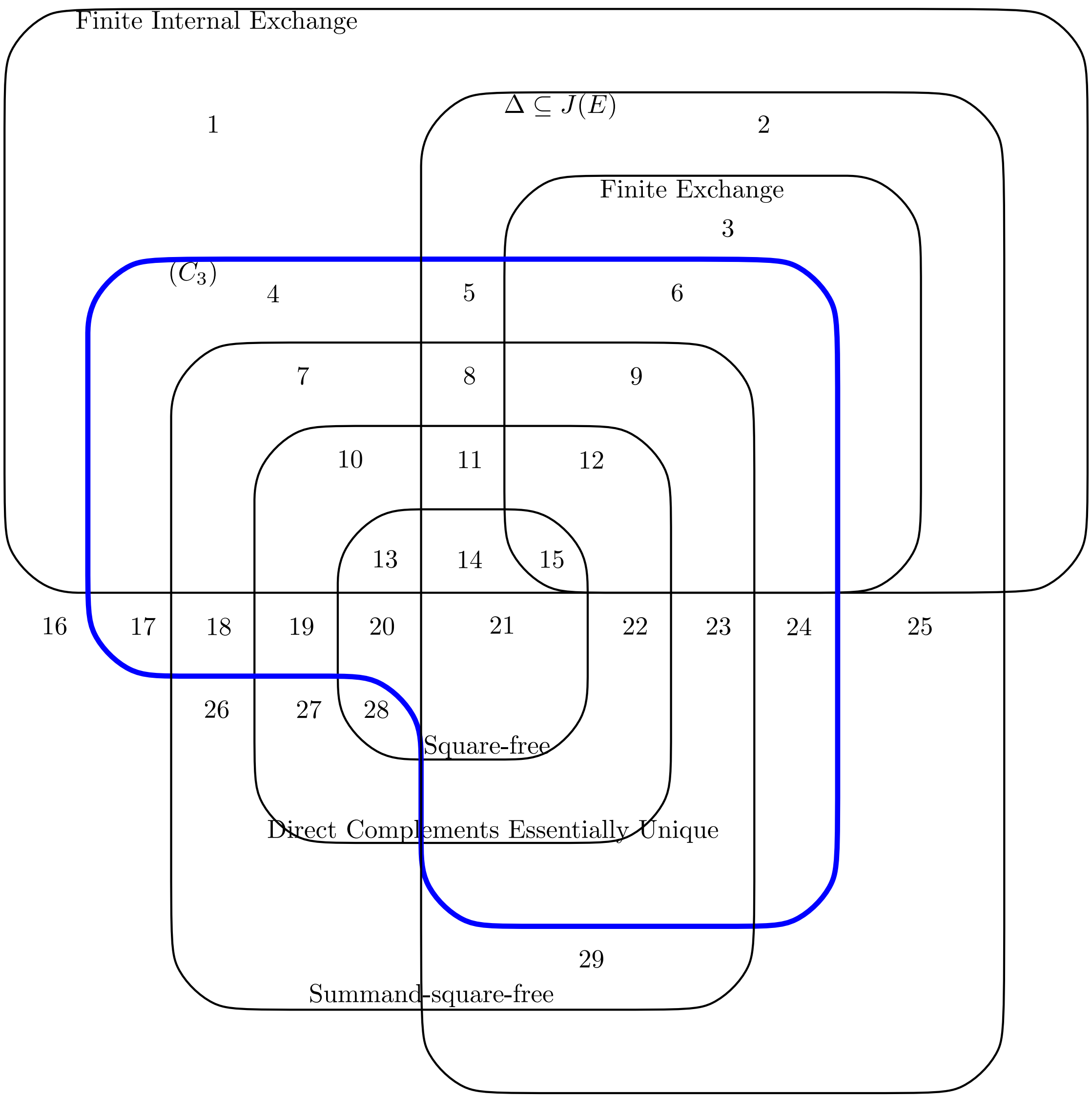


FIGURE 7.2. Euler diagram of the associations between direct summand properties

We are now prepared to present the examples. Throughout, $F$ will denote an arbitrary field. In each case, we will take $M = R_R$ for some ring $R$. We label the rings $R_n$ for $1 \leq n \leq 29$ according to corresponding region in Figure 7.2.

(1) $R_1 = R_2 \times R_4$.
(2) $R_2 = R_3 \times R_5$.
(3) $R_3 = \mathbb{T}_2(F)$.
(4) $R_4 = R_5 \times R_7$.
(5) $R_5 = R_6 \times R_8$.
(6) $R_6 = \mathbb{M}_2(F)$.
(7) $R_7 = R_8 \times R_{10}$.
(8) $R_8 = R_9 \times R_{11}$.
(9) $R_9$ is the ring from Example 5.7.
(10) $R_{10} = R_{11} \times R_{13}$.
(11) $R_{11} = R_{12} \times R_{14}$.
(12) $R_{12} = F[x, y \ : \ x^2 = xy = y^2 = 0]$ so $xR \cong yR$ but $xR \cap yR = (0)$.
(13) $R_{13}$ is the ring from Example 5.5.
(14) $R_{14} = \mathbb{Z}$. (Note that at this point, the first fourteen rings are now defined.)
(15) $R_{15} = F$.
(16) $R_{16} = R_1 \times R_{17}$.
(17) $R_{17} = R_4 \times R_{18}$.
(18) $R_{18} = R_7 \times R_{19}$.
(19) $R_{19} = R_{10} \times R_{20}$.
(20) $R_{20} = R_{13} \times R_{21}$.
(21) $R_{21}$ will be given in Example 7.3 below. (Note that once this ring is in place, the first twenty-one rings are defined.)
(22) $R_{22} = R_{12} \times R_{21}$.
(23) $R_{23} = R_9 \times R_{22}$.
(24) $R_{24} = R_6 \times R_{23}$.
(25) $R_{25} = R_2 \times R_{24}$.
(26) $R_{26} = R_{18} \times R_{27}$.
(27) $R_{27} = R_{19} \times R_{28}$.
(28) $R_{28}$ is the ring from Example 6.1.
(29) $R_{29}$ is the ring from Example 6.2.

We finish this paper by giving an example for $R_{21}$.

**Example 7.3.** *There exists a square-free module $M$ such that $\Delta(M) \subseteq J(\mathrm{End}(M))$, but $M$ does not have finite internal exchange.*

*Construction and proof.* We build this example in stages. We will work over $\mathbb{F}_2$ (the field with two elements) for simplicity. Let

$$S_0 = \mathbb{F}_2\langle e, f \ : \ e^2 = e, f^2 = f\rangle.$$

This is the free construction of an $\mathbb{F}_2$-algebra over the variables $V_0 = \{e, f\}$ subject to the relations $C_0 = \{e^2 = e, f^2 = e\}$. We also let $I_0$ be the ideal generated by the element $ef - fe$. Notice that

$$A_0 := \mathbb{F}_2[e, f \ : \ e^2 = e, f^2 = f] \cong S_0/I_0$$

is a ring all of whose elements are idempotents. Indeed, an $\mathbb{F}_2$-basis of orthogonal idempotents is given by $\{ef, e + ef, f + ef, 1 + e + f + ef\}$. A simpler basis (which we will make more use of), but which does not consist of orthogonal idempotents, is $\{1, e, f, ef\}$.

If for some $n \geq 0$ we are given the free construction of an $\mathbb{F}_2$-algebra $S_n$ generated by variables $V_n$ subject to condition $C_n$, and an ideal $I_n$ with $S_n/I_n$ naturally identified with $A_0$ (by sending $e, f \in S_n$ to the appropriate elements of $A_0$), then we recursively define the next "level" as follows. We first take

$$V_{n+1} = V_n \cup \{v_{s,n} \ : \ s \in I_n\},$$

or in other words, we add a new variable for each element of $I_n$. Our new set of relations is

$$C_{n+1} = C_n \cup \{sv_{s,n} = v_{s,n}s, s + v_{s,n} + s \cdot v_{s,n} = 0 \ : \ s \in I_n\}$$

so that every element $s \in I_n$ now has a quasi-inverse $v_{s,n}$. Let $S_{n+1}$ be the free construction of an $\mathbb{F}_2$-algebra on the variable set $V_{n+1}$ subject to the relations $C_{n+1}$. Finally, take $I_{n+1}$ to be the ideal of $S_{n+1}$ generated by $I_n$ and the new variables $\{v_{s,n} \ : \ s \in I_n\}$. Notice that $S_{n+1}/I_{n+1} \cong S_n/I_n \cong A_0$, where the first isomorphism comes from factoring out the new variables.

Take $V_\infty = \bigcup_{n\geq 0} V_n$, $C_\infty = \bigcup_{n\geq 0} C_n$, $I_\infty = \bigcup_{n\geq 0} I_n$, and let $S_\infty$ be the free construction of an $\mathbb{F}_2$-algebra on the variable set $V_\infty$ subject to the relations $C_\infty$. Notice that $I_\infty$ is an ideal of $S_\infty$, and $S_\infty/I_\infty \cong A_0$ in the natural way. Further, from the relations, we see that every element of $I_\infty$ has a quasi-inverse, so $I_\infty \subseteq J(S_\infty)$. On the other hand, $A_0$ is a Jacobson semisimple ring, and so $I_\infty = J(S_\infty)$.

We claim that $\overline{ef} \in \operatorname{idem}(S_\infty/I_\infty)$ does not lift to an idempotent of $S_\infty$; and thus by Theorem 2.4 (in conjunction with Proposition 3.1) we have that $S_\infty$ is not an internal exchange ring. To see that $\overline{ef}$ does not lift, it suffices to find *any* $\mathbb{F}_2$-algebra with two idempotents which commute modulo the Jacobson radical but their product doesn't lift to an idempotent, for then by virtue of the free construction, this same fact must hold for $S_\infty$. To find such an example we modify Example 2.2. Let $B = \mathbb{F}_2[x]_{(x+x^2)}$ be the localization of the polynomial ring away from the ideal $(x + x^2)$; so this is the subring of the field of rational functions $\mathbb{F}_2(x)$ whose denominators are relatively prime to both $x$ and $1 + x$. In $\mathbb{M}_2(B)$ we take our idempotents to be

$$E = \begin{pmatrix} 1+x & x \\ 1+x & x \end{pmatrix} \qquad \text{and} \qquad F = \begin{pmatrix} x(1+x) & x \\ (1+x)(1+x+x^2) & 1+x+x^2 \end{pmatrix}.$$

It is easy to check that $EF - FE \in J(\mathbb{M}_2(B))$. On the other hand, every entry of $EF$ has positive $x$-valuation but $EF \notin J(\mathbb{M}_2(B))$, thus $\overline{EF}$ does not lift modulo $J(\mathbb{M}_2(B))$ to an idempotent.

Our next goal is to force square-freeness by adding enough zero-divisor conditions. To this end we perform another recursive construction. Take $T_0 = S_\infty$, $W_0 = V_\infty$, $D_0 = C_\infty$, and $J_0 = I_\infty$. Recursively define

$$W_{n+1} = W_n \cup \{w_{s,n} \ : \ s \in J_n\}$$

and

$$D_{n+1} = D_n \cup \{s \cdot w_{s,n} = 0, w_{s,n}T_n w_{s',n} = 0 \ : \ s, s' \in J_n\},$$

then let $T_{n+1}$ be the free construction of an $\mathbb{F}_2$-algebra over the variables $W_{n+1}$ subject to the relations $D_{n+1}$, and let $J_{n+1}$ be the ideal generated by $J_n$ and the new variables $w_{s,n}$. Notice that the ideal generated by just the new variables is nilpotent of index 2, and thus since $T_{n+1}/J_{n+1} \cong T_n/J_n \cong A_0$, we have that $J_n = J(T_n)$ for each $n \geq 0$. Letting $W_\infty, D_\infty, T_\infty$, and $J_\infty$ be defined in the obvious way, we similarly have $J(T_\infty) = J_\infty$ and $T_\infty/J_\infty \cong A_0$ naturally.

Finally, let $P = W_\infty \cup \{z, y_1, y_2, \ldots\}$,

$$Q = D_\infty \cup \{z \text{ is central}, y_n \text{ is central}, z^2 = 0, J_\infty z = 0, y_n z = 0, y_n^{n+1} = 0\},$$

and let $R$ be the free construction of an $\mathbb{F}_2$-algebra generated by the variable set $P$ subject to the relations in $Q$. We have that $J := J(R)$ is the ideal generated by $J_\infty$ and $\{z, y_1, y_2, \ldots\}$; since $\{z, y_1, y_2, \ldots\}$ generates a nil ideal, and once we factor out $z, y_1, y_2, \ldots$ then $J_\infty$ is the Jacobson radical (by the work in the previous paragraph). Our module is $M_R = R_R$ as usual. Notice that $M$ does not have the finite internal exchange property, since $\overline{ef} \in \operatorname{idem}(R/J)$ does not lift to an idempotent (since it doesn't even lift to an idempotent in the factor ring $R/(P \setminus V_\infty) \cong S_\infty$).

Next, fix $r \in R$. We can uniquely write $r = \alpha_1 + \alpha_2 e + \alpha_3 f + \alpha_4 ef + r'$ with $r' \in J$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{F}_2$. We say that $\operatorname{init}(r) = \alpha_1 + \alpha_2 e + \alpha_3 f + \alpha_4 ef$ is the *initial part* of $r$. Assume $r \notin J$, so then $\operatorname{init}(r) \neq 0$. The right annihilator of $r$ intersected with $rzR = \operatorname{init}(r)zR$ is trivial, since $\operatorname{init}(r)$ is an idempotent modulo $I_0$ and $z$ kills $I_0$. It is also the case that $\operatorname{init}(r)zR \neq 0$; perhaps the easiest way to see this is to pass to the factor ring $R/(P \setminus \{e, f, z\}) \cong \mathbb{F}_2[z]\langle e, f \ : \ e^2 = e, f^2 = f, (ef - fe)z = 0, z^2 = 0\rangle$ and note that $\text{init}(r)z$ is nonzero in the factor ring. Thus, we have shown that $(R \setminus J) \cap \Delta(R_R) = \emptyset$, so $\Delta(R_R) \subseteq J(\text{End}(M)) = J(R)$.

All that remains is to show the square-freeness of $R_R$. Letting $r_1, r_2 \in R$, it suffices to show that either the right annihilators of these two elements do not agree, or $r_1R \cap r_2R \neq (0)$. First, if $\text{init}(r_1) = \text{init}(r_2) \neq 0$, then $0 \neq r_1z = r_2z \in r_1R \cap r_2R$. Second, if $\text{init}(r_1) \neq \text{init}(r_2)$ are both nonzero, then either $\text{init}(1-r_1)\text{init}(r_2) \notin J$ or $\text{init}(1-r_2)\text{init}(r_1) \notin J$ or both (since $A_0$ contains only commuting idempotents) and so either $r_1 \cdot (1-r_2)z \neq 0 = r_2 \cdot (1-r_2)z$ or $r_2 \cdot (1-r_1)z \neq 0 = r_1 \cdot (1-r_1)z$. Thus, $r_1$ and $r_2$ have different right annihilators in this case. Third, if $\text{init}(r_1) \neq 0 = \text{init}(r_2)$ then $r_1z \neq 0 = r_2z$, so they again have different right annihilators.

We may thus reduce to the case when $r_1, r_2 \in J$ are nonzero and their right annihilators agree. If $r_1, r_2 \in Rz$ then an argument similar to the one in the previous paragraph shows $r_1 = r_2$; so we may assume $r_1 \notin Rz$. After multiplying both $r_1$ and $r_2$ (on the right) by $y_n$ for sufficiently large $n$ (and using the fact that their right annihilators agree), we may as well assume no monomial in either $r_1$ or $r_2$ contains $z$. Further, after multiplying by the appropriate powers of $y_1, y_2, \ldots, y_n$ (again for sufficiently large $n$) we may assume $r_1 = r_1'(y_1y_2^2 \ldots y_n^n)$ and $r_2 = r_2'(y_1y_2^2 \cdots y_n^n)$ with $r_1', r_2'$ containing no instances of any $y_i$.

Without loss of generality, replacing $r_i$ by $r_i'$ if necessary, we may as well assume that $r_1$ and $r_2$ contain no instances of any $y_i$. Fix $m \geq 0$ such that the monomials appearing in $r_1, r_2$ come from $W_m$. From the equality of right annihilators, we must have $r_2 \cdot w_{r_1,m} = 0$. This equality continues to hold if we impose the *extra* assumptions that all of the variables in $P_1 := P \setminus (W_m \cup \{w_{r_1,m}\})$ are zero. But $R/(P_1)$ is isomorphic to extension of $T_m$ by a single variable $w_{r_1,m}$ along with the new relations $r_1w_{r_1,m} = 0$ and $w_{r_1,m}xw_{r_1,m} = 0$ (for any $x \in T_m$). Thus, the only way to get $r_2w_{r_1,m} = 0$ is that $r_2 = s_1r_1$ for some $s_1 \in T_m$. By a symmetric argument, we have $r_1 = s_2r_2$.

**Case 1:** $r_1w_{r_1+r_2,m} = 0$. By repeating the argument we just gave in the previous paragraph, but for the pair $r_1$ and $r_1 + r_2$, we must have $t_1(r_1 + r_2) = r_1$ for some $t_1 \in T_m$. Thus

$$r_1 = t_1(r_1 + r_2) = t_1(1 + s_1)r_1 = t_1(1 + s_1)s_2r_2 = (t_1(1 + s_1)s_2s_1)r_1. \tag{7.4}$$

Since $\text{init}(s_1)$ and $\text{init}(1 + s_1)$ are orthogonal idempotents, we must have $b := t_1(1 + s_1)s_2s_1 \in J(R)$. But then (7.4) says $(1 - b)r_1 = 0$, which implies $r_1 = 0$, a contradiction.

**Case 2:** $r_1w_{r_1+r_2,m} \neq 0$. Then since $(r_1 + r_2)w_{r_1+r_2,m} = 0$, we have $0 \neq r_1w_{r_1+r_2,m} = r_2w_{r_1+r_2,m} \in r_1R \cap r_2R$. This completes the proof. □

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