HALF-ORTHOGONAL SETS OF IDEMPOTENTS

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Abstract. We improve several results in the literature focused on lifting idempotents, by either removing the lifting hypothesis or weakening other assumptions. We prove that countable sets of idempotents, which are orthogonal modulo an enabling ideal, lift to orthogonal idempotents. Left associates of liftable idempotents also lift modulo the Jacobson radical. Additionally, we exhibit situations when half-orthogonal sets of idempotents can be orthogonalized by multiplying by a unit. We finish by proving a number of results on directs sums of modules with the exchange property.

1. Introduction

Let I be a one-sided ideal of R. Given $x \in R$ with $x - x^2 \in I$, one says that *x lifts to an idempotent* modulo I if there exists an idempotent $e^2 = e \in R$ with $x - e \in I$. When idempotents lift modulo the Jacobson radical, $J(R)$, there are many further properties that follow. For instance, countable sets of orthogonal idempotents in $R/J(R)$ can be lifted orthogonally. Furthermore, idempotents can be lifted strongly [10], meaning that (in the notation above) one can choose $e \in xRx$.

It turns out that many of these properties do not depend, *per se*, on the ability to lift idempotents. Rather, they rely on special properties of the Jacobson radical that can be decoupled from any sort of lifting hypothesis. Indeed, that was the main idea of [1], where strong lifting modulo $J(R)$ was shown to follow from the more basic "enabling" property. Similarly, the process of taking idempotents of R which are orthogonal modulo $J(R)$, and finding other lifts which are orthogonal over R, often does not rely on any lifting assumptions. Perhaps even more astonishingly, some of these facts were known more than thirty years ago by Corner, and are contained in his unpublished manuscript on the exchange property [2]. Part of this work recovers and extends those results.

The present paper also streamlines many of the idempotent lifting machines in the literature. In particular, we are able to remove the assumption "idempotents lift" from some well-known results. We make use of the following basic facts repeatedly:

(i) $U(R) \cap \mathrm{idem}(R) = \{1\}$.
(ii) $J(R) \cap \mathrm{idem}(R) = \{0\}$.
(iii) If ab is an idempotent, then $baba$ is an (isomorphic) idempotent.
(iv) A sum of orthogonal idempotents is an idempotent.
(v) In a direct sum decomposition ${}_RR = \bigoplus_{i=1}^{n} A_i$, the unit $1 \in R$ decomposes (uniquely) as a sum of orthogonal idempotents.

The following is a brief outline of the paper. In §2 we recall some well-known equivalence relations on idempotents, and investigate how the relations behave modulo ideals contained in the Jacobson radical. Next, in §3 we delve further into the consequences of the enabling hypothesis, and develop a strong lifting lemma which makes it easy to lift idempotents related to ones already lifted. We prove in §4-5 that finite, and then infinite, sets of idempotents which are "half-orthogonal" can be orthogonalized by multiplication by a single unit, often with very few additional hypotheses. We also prove that a countable set of idempotents can be lifted orthogonally if they are orthogonal modulo an enabling ideal.

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We end in §6 by investigating the exchange property for direct sums of modules. We are able to classify when the endomorphism ring of a decomposable module $M = \bigoplus_{i\in\Gamma} M_i$ is naturally isomorphic to the direct product of the endomorphism rings $\mathrm{End}(M_i)$, modulo Jacobson radicals. This leads to a new proof that Harada modules have the exchange property. Additionally, new information on the structure of the endomorphism ring of a Harada module is revealed.

Throughout, R is an associative ring with 1, and $J(R)$ is the Jacobson radical. We let $U(R)$ denote the group of units in R, and $\mathrm{idem}(R)$ is the set of idempotents. We write $I \trianglelefteq R$ to mean that I is a two-sided ideal of R.

2. Equivalence relations on idempotents

The simplest equivalence relation on idempotents is equality. Our first result explores what it means for idempotents to be equal modulo the Jacobson radical.

Lemma 2.1. *Let $e, f \in \mathrm{idem}(R)$ and assume $I \trianglelefteq R$ with $I \subseteq J(R)$. We have $e \equiv f \pmod I$ if and only if there exist units $u, v \in U(R)$ with $u \equiv v \equiv 1 \pmod I$ satisfying $ue = fv$.*

Proof. One direction is trivial. For the other, assume that we have $e - f = r \in I$. Multiplying on the left by f we have $fe - f = fr$, hence $fe = f(1 + r)$. If instead we multiply the right by e, then we have $e - fe = re$, hence $fe = (1 - r)e$. Setting $u = 1 - r$ and $v = 1 + r$ we are done. □

Note: The units u and v satisfy many more properties than the ones listed in the lemma. For instance, they commute with each other.

Perhaps even more natural than the notion of equality is the idea of isomorphism. Recall, by [5, Proposition 21.20] that two idempotents $e, f \in R$ are said to be *isomorphic*, written $e \cong f$, if one of the following equivalent conditions holds:

(I1) $eR \cong fR$ as right R-modules.
(I2) $Re \cong Rf$ as left R-modules.
(I3) There exist $a, b \in R$ with $e = ab$ and $f = ba$.
(I4) There exist $a \in eRf$, $b \in fRe$ with $e = ab$ and $f = ba$.

Isomorphism of idempotents is respected by ideals contained in the Jacobson radical. Formally, let $e, f \in \mathrm{idem}(R)$ and let $I \trianglelefteq R$ with $I \subseteq J(R)$. We have $e \cong f$ in R if and only if $\overline{e} \cong \overline{f}$ in R/I (see [5, Proposition 21.21]).

As we will need it in the sequel, we now introduce a slight strengthening of the notion of isomorphism. Following [11], given $e, f \in \mathrm{idem}(R)$ we say that e and f are *left associates*, written $e \sim_\ell f$, if one of the following equivalent conditions holds:

(L1) $ef = e$ and $fe = f$.
(L2) $Re = Rf$.
(L3) $f = e + (1 - e)re$ for some $r \in R$.
(L4) $f = ue$ for some $u \in U(R)$.
(L5) $f = ue$ with $u = 1 + (1 - e)re$ for some $r \in R$.
(L6) $(1 - e)(1 - f) = (1 - f)$ and $(1 - f)(1 - e) = (1 - e)$.

In case $R = \mathrm{End}(M_\Lambda)$ is an endomorphism ring, for some right Λ-module M, these are also equivalent to the following extra properties:

(L7) $\ker(e) = \ker(f)$
(L8) $\mathrm{im}(1 - e) = \mathrm{im}(1 - f)$

We define *right associate idempotents* similarly, denoted by $e \sim_r f$. Left (or right) associate idempotents are isomorphic to each other, and idempotents which are both left and right associate are equal. These relations are each equivalence relations.

The notion of left associate idempotents is especially useful when studying the exchange property. Recall that a ring R is an *exchange ring* if idempotents lift modulo every one-sided ideal. Equivalently, R is an exchange ring in case for every pair of elements $x, y \in R$ satisfying $x + y = 1$ (so $y = 1 - x$) there exist idempotents $e \in Rx$ and $f \in R(1 - x)$ with $e + f = 1$. (In particular, e and f are complementary idempotents, and thus orthogonal.) This notion turns out to be left-right symmetric. However, when working with such rings one often focuses on one side or the other. Left (and right) associate idempotents naturally arise in this context. While we will focus on left associates, the reader should keep in mind that many of the results we prove are left-right symmetric, and thus apply to right associates as well.

The next few results describe the behavior of this relation with respect to the Jacobson radical. The first lemma tells us that when idempotents are left associate modulo the Jacobson radical, to show that they are left associate in R it suffices to have only one of the two defining conditions in (L1).

Lemma 2.2. *Let $e, f \in \mathrm{idem}(R)$ and assume $I \trianglelefteq R$ with $I \subseteq J(R)$. If $ef = e$ and $\overline{e} \sim_\ell \overline{f}$, then $e \sim_\ell f$.*

Proof. First notice that fe is an idempotent since $fefe = fe^2 = fe$. Further $e \sim_\ell fe$ since $fee = fe$ and $efe = e^2 = e$.

Next, write $fe - f = r \in I$. Multiplying on the right by f we have $rf = fef - f^2 = fe - f = r$. We then find $fe = f + r = (1 + r)f \sim_\ell f$ by (L4) since $1 + r \in U(R)$. By transitivity of left associates, $e \sim_\ell f$. □

We proved the lemma above using an element-wise definition of left associates. For those who favor a more module-theoretic viewpoint, the lemma is a quick corollary of the following stronger result.

Lemma 2.3. *Let $A \subseteq B$ be modules and suppose B is finitely generated. If $A + \mathrm{rad}(B) = B$, then $A = B$.*

Proof. We show the contrapositive. Fix a finite generating set $x_1, x_2, \ldots, x_n \in B$. Assume $A \neq B$, then by Zorn's lemma there exists a maximal submodule $M \subsetneq B$ which contains A but not all of the x_i. Hence, $A + \mathrm{rad}(B) \subseteq M \subsetneq B$. □

The next proposition tells us what is possible when we don't assume either $ef = e$ or $fe = f$.

Proposition 2.4. *Let $e, f \in \mathrm{idem}(R)$ and assume $I \trianglelefteq R$ with $I \subseteq J(R)$. If $e \sim_\ell f$ in R, then $\overline{e} \sim_\ell \overline{f}$ in R/I. Conversely, if $\overline{e} \sim_\ell \overline{f}$ in R/I, then there exists $g \in \mathrm{idem}(R)$ with $e \equiv g \pmod{I}$, $e \sim_r g$, and $f \sim_\ell g$.*

Proof. The forward direction is trivial. Conversely, suppose $\overline{e} \sim_\ell \overline{f}$. Then $r := fe - f = fr \in I$ and hence $fe(1 + r)^{-1} = f$. Let $g := e(1 + r)^{-1}f$. That g is an idempotent follows from the fact that if $fx = f$, then $xfxf = (xf)f = xf$ is an idempotent. Further $gf = g$ and $fg = fe(1 + r)^{-1}f = f(1+r)(1+r)^{-1}f = f$. Therefore, by (L1) we have $f \sim_\ell g$. Finally, as $eg = g$ and $g \equiv ef \equiv e \pmod{I}$, Lemma 2.2 implies $e \sim_r g$. □

One can view this proposition as a strengthening of Lemma 2.1. To see this first note that if $e \equiv f \pmod{I}$, then $\overline{e} \sim_\ell \overline{f}$. Thus, by Proposition 2.4 we can find an idempotent $g \in R$ satisfying $g \equiv e \pmod{I}$ and $e \sim_r g \sim_\ell f$. Writing $g = ev^{-1}$ and $g = u^{-1}f$ (by (L4), for some units $u, v \in R$) we recover the first half Lemma 2.1. By (L5), we can pick $u^{-1} = 1 + (1 - f)rf$ for some $r \in R$, but then $(1 - f)rf = u^{-1}f - f = g - f \in I$, implying that $u^{-1} \equiv 1 \pmod{I}$. Similarly, $v^{-1} \equiv 1 \pmod{I}$, and we have the second half of Lemma 2.1.

On the other hand, the hypotheses in Proposition 2.4 cannot be weakened so that e and f are only *equivalent* (i.e. $ue = fv$ for some units $u, v \in R$).

Example 2.5. There exists a ring R and idempotents $e, f \in \mathrm{idem}(R)$ such that e and f are conjugate (and hence equivalent), but there is no idempotent $g \in \mathrm{idem}(R)$ for which $e \sim_r g \sim_\ell f$.

Construction. Let F be a field and let $R = \mathbb{M}_2(F)$. Let $e = e_{1,1}$ and $f = 1 - e = e_{2,2}$ be the standard matrix idempotents. We see that e and f are conjugate via the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We also compute that if $e \sim_r g$, then g is of the form $\left(\begin{smallmatrix} 1 & x \\ 0 & 0 \end{smallmatrix}\right)$ with $x \in F$ arbitrary. However, $g \not\sim_\ell f$ for any such g. □

It is also a simple matter to construct an example showing that when $\overline{e} \sim_\ell \overline{f}$ in $R/J(R)$ we do not necessarily obtain $e \sim_\ell f$ in R.

Example 2.6. There exists a ring R and idempotents $e, f \in \text{idem}(R)$ for which $e \equiv f \pmod{J(R)}$ but $e \not\sim_\ell f$ and $e \not\sim_r f$.

Construction. Let F be a field, and let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right) \times \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ be the product of two copies of the 2×2 upper-triangular matrix ring over F. Set

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } f = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

We see that $e \equiv f \pmod{J(R)}$. On the other hand, there is no unit $u \in R$ such that $ue = f$, and similarly $eu = f$ does not happen. □

Question 2.7. Suppose $e, f \in \text{idem}(R)$ are conjugate. Does there exist a finite set of idempotents $e_1, e_2, \ldots, e_{2n-1} \in \text{idem}(R)$ such that $e \sim_\ell e_1 \sim_r e_2 \sim_\ell e_3 \sim_r \cdots \sim_\ell e_{2n-1} \sim_r f$?

In the next section, we take the seemingly *ad hoc* methods used in the proof of Proposition 2.4, and refine them into a nice lifting machine.

3. Strong lifting

We start this section with a fundamental result about elements which are congruent (modulo the Jacobson radical) to idempotents in R, due to Corner [2, Addendum Lemma 9.3]. By now the idea should look familiar.

Lemma 3.1. *Let $x \in R$, let $e \in \text{idem}(R)$, and let $I \trianglelefteq R$ with $I \subseteq J(R)$. If $exe \equiv e \pmod{I}$, then $exe = ue$ where $u \in U(R)$ commutes with e, and $u \equiv 1 \pmod{I}$.*

Proof. As $exe \equiv e \pmod{I}$ write $exe = e + r$ with $r \in eIe$. Then $exe = e(e + r)e = (1 + r)e$. Putting $u = 1 + r$ we have $u \in U(R)$ since $u \equiv 1 \pmod{I}$. Clearly $eu = ue$. □

Note: It will be useful to observe that u^{-1} also commutes with e, and $u^{-1} \equiv 1 \pmod{I}$.

Following [1], an ideal $I \trianglelefteq R$ is said to be *enabling* if given $x \in R$ and $e \in \text{idem}(R)$ with $x \equiv e \pmod{I}$, then there exists an idempotent $f^2 = f \in Rx$ with $f \equiv e \pmod{I}$. This is a left-right symmetric notion, and f may be chosen so that $f \in xRx$. As we will use this fact repeatedly, we reprove it here. Given $x \equiv e \pmod{I}$, then $x^2 \equiv e^2 = e \pmod{I}$. So, from enabling there exists an idempotent $g \in Rx^2$ with $g \equiv e \pmod{I}$. Writing $g = rx^2$, let $f = xgrx \in xRx$. We compute

$$f^2 = xgrx^2grx = xg^3rx = xgrx = f.$$

Furthermore, since $x \equiv x^2 \pmod{I}$ we find

$$f = xgrx \equiv xgrx^2 = xg^2 \equiv e^3 = e \pmod{I}.$$

With Lemma 3.1 in hand, we immediately recover the fact that ideals contained in the Jacobson radical are enabling (see [1, Proposition 5]).

Corollary 3.2. *If $I \trianglelefteq R$ with $I \subseteq J(R)$, then I is enabling.*

Proof. Assume $x \in R$, $e \in \text{idem}(R)$, and $x \equiv e \pmod{I}$. By Lemma 3.1 we can find $u \in R$ so that $exe = ue$, $ue = eu$, and $u \equiv 1 \pmod{I}$. Since $(u^{-1}ex)e = e$, we know that $f = e(u^{-1}ex) \in Rx$ is an idempotent. Finally, $f = eu^{-1}x \equiv e^2 = e \pmod{I}$. □

The following result doesn't seem to have been noticed in the literature. As it introduces a nice idea, we record it here.

Lemma 3.3. *Let $x, y, s, t \in R$, let $e \in \operatorname{idem}(R)$, and let $I \trianglelefteq R$ be an enabling ideal. If $xs \equiv ty \equiv e \pmod{I}$, then there exists $f \in \operatorname{idem}(R)$ such that $f \equiv e \pmod{I}$ and $f \in xRy$.*

Proof. Notice that $xsty \equiv e^2 = e \pmod{I}$ and so by the enabling property we can find an idempotent $f \in xstyRxsty \subseteq xRy$ such that $f \equiv e \pmod{I}$. ☐

We are now equipped with the machinery to prove the main result of this section.

Proposition 3.4. *Let $a, b \in R$, let $e \in \operatorname{idem}(R)$, and let $I \trianglelefteq R$ be an enabling ideal. If $ab \equiv e \pmod{I}$, then there exists $f \in \operatorname{idem}(R)$ with $f \in bRa$ and $f \equiv bea \pmod{I}$. If we additionally assume $a \in \operatorname{idem}(R)$ with $a \equiv ea \pmod{I}$ and that $I \subseteq J(R)$, then $f \sim_\ell a$.*

Proof. By enabling, pick some $g \in \operatorname{idem}(R)$ with $g \in eabeRe$ and $g \equiv e \pmod{I}$. (The only reason for the first e on the left side of g is to force $eg = g$, and so by Lemma 2.2 we have $g \sim_r e$ when $I \subseteq J(R)$, if we need it.) Writing $g = eaber$ for some $r \in R$, set $f := bergea \in bRa$ which is an idempotent isomorphic to g (by item (iii) in the list of basic facts given in §1). It also happens that $e \equiv g = eaber \equiv er \pmod{I}$. Thus we find $f = b(er)gea \equiv begea \equiv bea \pmod{I}$.

Suppose further that $a \in \operatorname{idem}(R)$, $a \equiv ea \pmod{I}$, and $I \subseteq J(R)$. We get $fa = f$ since $f \in Ra$, and we also calculate

$$af \equiv abea \equiv e^2 a \equiv a \pmod{I}.$$

By Lemma 2.2, we have $f \sim_\ell a$. ☐

Lemma 1 of [8] now follows immediately from these ideas, and we obtain more.

Corollary 3.5. *Let $x \in R$, let $e \in \operatorname{idem}(R)$, and let $I \trianglelefteq R$ with $I \subseteq J(R)$. If $exe \equiv e \pmod{I}$, then there exists $f \in \operatorname{idem}(R)$ with $f \in xRe$, $f \sim_\ell e$, and $f \equiv xe \pmod{I}$.*

Proof. Take $a = e$, take $b = xe$, and apply the previous proposition. ☐

As another corollary, we obtain the fact that left associates in $R/J(R)$ of liftable idempotents also lift, and they lift as left associates.

Corollary 3.6. *Let $e \in \operatorname{idem}(R)$, let $x \in R$, and let $I \trianglelefteq R$ with $I \subseteq J(R)$. If $\overline{x} \in \operatorname{idem}(R/I)$ and $\overline{x} \sim_\ell \overline{e}$, then there exists an idempotent $f \in \operatorname{idem}(R)$ satisfying $f \sim_\ell e$ and $f \equiv x \pmod{I}$.*

Proof. Note that $ex \equiv e \pmod{I}$ and $xe \equiv x \pmod{I}$. By Proposition 3.4 with $a = e$ and $b = x$, there exists an idempotent $f \in xRe$ with $f \sim_\ell e$ and $f \equiv bea = xe \equiv x \pmod{I}$. ☐

The same result is true of isomorphic and conjugate idempotents.

Corollary 3.7. *Let $x \in R$, let $e \in \operatorname{idem}(R)$, and let $I \trianglelefteq R$ with $I \subseteq J(R)$. If $\overline{x} \in \operatorname{idem}(R/I)$ and $\overline{x} \cong \overline{e}$, then there exists $f \in \operatorname{idem}(R)$ such that $f \cong e$ and $f \equiv x \pmod{I}$. Moreover, if $\overline{e}$ and $\overline{x}$ are conjugate, then so are e and f.*

Proof. As $\overline{e} \cong \overline{x}$, fix $a, b \in R$ so that $e \equiv ab \pmod{I}$ and $x \equiv ba \pmod{I}$. In the proof of Proposition 3.4 we constructed idempotents $f, g \in R$ satisfying $f \cong g$, $g \sim_r e$, $g \equiv e \pmod{I}$, and $f \equiv bea \pmod{I}$. As right associate idempotents are isomorphic, we have $e \cong g \cong f$. Further, $\overline{x}$ is an idempotent so we find $ba \equiv baba \equiv bea \pmod{I}$. Thus,

$$f \equiv bea \equiv ba \equiv x \pmod{I}.$$

To prove the last statement, assume further that $\overline{e}$ and $\overline{x}$ are conjugate. Recall that idempotents are conjugate if and only if they are isomorphic and their complements are isomorphic (see [6, Exercise 21.16]). So $\overline{1-f} = \overline{1-x} \cong \overline{1-e}$, and applying [5, Proposition 21.21] we have $1-f \cong 1-e$. As $e \cong f$ and the same is true of their complements, we have that e and f are conjugate. ☐

Note: Most of the results in this section do not hold in general for enabling ideals. Taking $I = R$ (which is enabling) it is a simple matter to construct counter-examples. However, we leave it to the interested reader to show that Corollary 3.6 holds for enabling ideals if we remove the conclusion that $f \sim_\ell e$. Thus left associates of liftable idempotents also lift, but not necessarily as left associates. A similar statement holds true for Corollary 3.7.

4. Orthogonal lifting

We now focus on the problem of lifting idempotents which are orthogonal modulo the Jacobson radical to orthogonal idempotents. Our goal is to do so as often as possible without any assumption that idempotents lift modulo $J(R)$. Moreover, usually we will do it with only half of the orthogonality relations. This generalizes [8, Theorem 5]. The main result of this section is the following theorem.

Theorem 4.1. *Assume $e_1, e_2, \ldots, e_n$ are idempotents in R for which $e_ie_j \in J(R)$ whenever $i < j$. Then there exists a unit $u \in U(R)$ for which $\{ue_i\}$ and $\{e_iu\}$ are orthogonal families of idempotents.*

As a preface, consider what happens if we assume half-orthogonality (not merely modulo the Jacobson radical) for two idempotents. In other words, suppose $e, f \in \text{idem}(R)$ with $ef = 0$. How do we make these idempotents orthogonal? Consider the element $g := (1-f)e$. It is an idempotent, and orthogonal to f. Moreover, $eg = e(1-f)e = e$ and $ge = g$. Hence $e \sim_\ell g$. *A fortiori*, by (L4) we know that $g = ue$ for some unit $u \in U(R)$. After some computation, we find that $u = 1 - fe$ suffices. Furthermore, we compute $uf = f$. Thus, there is a single unit u which, after left multiplication, "twists" both e and f to left associate, orthogonal idempotents.

This is not accidental. As the intuition behind this result is fairly straightforward, we sketch it now. First, ${}_RA = Re + Rf$ is a direct sum (since $ef = 0$). Second, one can show that ${}_RA = Re \oplus Rf$ is a direct summand of ${}_RR$, and so we write $R = Re \oplus Rf \oplus Rh$ with $h^2 = h$. Third, if we are careful, we can choose h so that $eh = fh = 0$. With further work, we also have that $v := e + f + h$ is a unit. Thus $1 = v^{-1}e + v^{-1}f + v^{-1}h$. Now comes the clincher: the element 1 can only decompose as a sum of orthogonal idempotents, when written in terms of a direct summand decomposition! Thus $\{v^{-1}e, v^{-1}f, v^{-1}h\}$ is a full set of orthogonal idempotents.

We are now ready to prove the theorem in stages. We start with a standard fact (see [2, Corollary 9.4] and [8, Lemma 2]). The short proof is included for completeness.

Lemma 4.2. *If $e_1, \ldots, e_n \in \text{idem}(R)$ and $e_ie_j \in J(R)$ whenever $i < j$, then $\sum_{i=1}^n Re_i$ is a direct sum, and a direct summand of ${}_RR$.*

Proof. As $e_1(1-e_2)e_1 \equiv e_1 \pmod{J(R)}$, by Corollary 3.5 there exists an idempotent $f_1 \in (1-e_2)Re_1$ with $f_1 \sim_\ell e_1$. Similarly, there exists an idempotent $f_2 \in (1-e_1)Re_2$ with $f_2 \sim_\ell e_2$. (Here we could also assume $f_2 \equiv e_2 \pmod{J(R)}$ since $(1-e_1)e_2 \equiv e_2 \pmod{J(R)}$, but we don't need this extra information.) These new idempotents are obviously orthogonal. Since $e_i \sim_\ell f_i$ we have that $Re_1 + Re_2 = Rf_1 + Rf_2$ is a direct sum, and a direct summand of ${}_RR$ generated by the idempotent $f_1 + f_2$.

Since $(f_1 + f_2)e_j \in J(R)$ when $j > 2$, we can replace e_1, e_2 by $f_1 + f_2$, and finish by recursion. The base case, when $n = 1$, is trivial. □

Corollary 4.3. *If $e_1, \ldots, e_n \in \text{idem}(R)$, $e_ie_j \in J(R)$ whenever $i < j$, and $v := \sum_{i=1}^n e_i \in U(R)$, then $\{v^{-1}e_i\}_{i=1}^n$ is an orthogonal set of idempotents.*

Proof. As $\sum_{i=1}^n v^{-1}e_i = 1$ is a decomposition of 1 into direct summands (by the previous lemma), the pieces in the decomposition are orthogonal idempotents (by basic fact (v) in §1). □

The following proposition records what is possible when working with half-orthogonal elements in a ring. One can find an explicit orthogonalizing unit. While the unit u defined below appears complicated, it simply arises by generalizing what was done above with two idempotents.

Proposition 4.4. *Let $e_1, \ldots, e_n \in \operatorname{idem}(R)$, and assume $e_i e_j = 0$ whenever $i < j$. Put*

$$u := 1 + \sum_{\substack{m \geq 2 \\ i_1 < i_2 < \ldots < i_m}} (-1)^{m-1}(m-1) e_{i_m} e_{i_{m-1}} \cdots e_{i_1}.$$

The element u is a unit. Putting $f_i := ue_i$ for each $i \geq 1$, we have that $\{f_i\}_{i=1}^n$ is a set of orthogonal idempotents with $e_i \sim_\ell f_i$ for each $i \geq 1$. Furthermore, $e_n = f_n$ and putting $f' := 1 - \sum_{i=1}^n f_i$ we have $uf' = f'$. In particular, $\sum_{i=1}^n e_i + f' = u^{-1}$.

Proof. Given any sequence of indices $i_1 < i_2 < \ldots < i_m$, with $m \geq 2$, then

$$e_i e_{i_m} e_{i_{m-1}} \cdots e_{i_1} e_i = 0$$

since either $i < i_m$ or $i \geq i_m > i_1$. Thus $e_i u e_i = e_i$ for each $i \geq 1$. In particular, $f_i := ue_i$ is an idempotent and $f_i \sim_\ell e_i$ since (L1) holds. An easy computation yields $e_n = f_n$. We now need to prove $f_i f_j = 0$ for $i \neq j$. First, we perform an auxiliary computation which describes f_i more concretely, with some explanatory comments following the calculation. We find

$$\begin{aligned}
f_i = ue_i = e_i &+ \sum_{\substack{m \geq 2 \\ i_1 < \ldots < i_m}} (-1)^{m-1}(m-1) e_{i_m} \cdots e_{i_1} e_i \\
&= e_i + \sum_{\substack{m \geq 2 \\ i = i_1 < \ldots < i_m}} (-1)^{m-1}(m-1) e_{i_m} \cdots e_{i_1} + \sum_{\substack{m \geq 2 \\ i = i_0 < i_1 < \ldots < i_m}} (-1)^{m-1}(m-1) e_{i_m} \cdots e_{i_1} e_{i_0} \\
&= e_i + \sum_{\substack{m \geq 2 \\ i = i_1 < \ldots < i_m}} (-1)^{m-1}(m-1) e_{i_m} \cdots e_{i_1} + \sum_{\substack{m \geq 3 \\ i = i_1 < \ldots < i_m}} (-1)^{m-2}(m-2) e_{i_m} \cdots e_{i_1} \\
&= \sum_{\substack{m \geq 1 \\ i = i_1 < \ldots < i_m}} (-1)^{m-1} e_{i_m} \cdots e_{i_1}.
\end{aligned}$$

When going from the first line to the second, the sum breaks into two pieces according to whether or not $i_1 = i$. The change from the second line to the third line consists of reindexing the last sum, so that m now measures the number of idempotents in the product $e_{i_m} \cdots e_{i_1}$. The final equality is obtained by adding the two sums together when $m \geq 3$, and noting that the remaining summands give the $m = 1, 2$ terms in the final expression.

Now, to show that $f_i f_j = 0$ when $i \neq j$ it suffices to prove $e_i f_j = 0$ when $i \neq j$. We compute

$$\begin{aligned}
e_i f_j &= \sum_{\substack{m \geq 1 \\ j = i_1 < \ldots < i_m}} (-1)^{m-1} e_i e_{i_m} \cdots e_{i_1} \\
&= \sum_{\substack{m \geq 1 \\ j = i_1 < \ldots < i_m = i}} (-1)^{m-1} e_{i_m} \cdots e_{i_1} + \sum_{\substack{m \geq 1 \\ j = i_1 < \ldots < i_m < i_{m+1} = i}} (-1)^{m-1} e_{i_{m+1}} e_{i_m} \cdots e_{i_1} = 0
\end{aligned}$$

The last equality follows from the fact that there are no terms in the first sum when $m = 1$ since $i \neq j$, and then (after reindexing) the second sum is the negative of the first.

Next, set $f := \sum_{i=1}^n f_i$ so its complementary idempotent is $f' := 1 - \sum_{i=1}^n f_i$. Multiply by e_j on the left to obtain

$$e_j f' = e_j - \sum_{i=1}^n e_j f_i = e_j - \sum_{i=1}^n \delta_{i,j} e_j = e_j - e_j = 0.$$

In particular, $uf' = f'$. Thus we see

$$u\left(\sum_{i=1}^{n} e_i + f'\right) = \sum_{i=1}^{n} f_i + f' = 1$$

and so u is right invertible.

All that remains is to prove that u is a unit. There are three straightforward ways to do this. First, one can show directly that u is *unipotent* (i.e. $u = 1 + x$ where x is nilpotent). Second, one can simply appeal to left-right symmetry, which proves that u is left invertible. Third, one can show directly that a left inverse of u is $\sum_{i=1}^{n} e_i + f'$. By far the easiest route is the second, and we are done. □

From the left-right symmetry of the construction, the set $\{e_i u\}_{i=1}^{n}$ is also an orthogonal family of idempotents, right associated with the e_i. Other statements follow similarly. We also note that an alternate characterization of the idempotent f_i constructed in the proposition is

$$f_i = (1 - e_n)(1 - e_{n-1}) \cdots (1 - e_{i+1})e_i.$$

Similarly, one finds that a compact expression for f' is given by the formula

$$f' = \sum_{\substack{m \geq 0 \\ i_1 < \ldots < i_m}} (-1)^m e_{i_m} \cdots e_{i_1}.$$

We are now ready to prove the Theorem 4.1.

Proof of Theorem 4.1. Let $e_1, \ldots, e_n \in \text{idem}(R)$ and assume $e_i e_j \in J(R)$ whenever $i < j$. By Lemma 4.2, we know that there exist orthogonal idempotents $g_1, g_2, \ldots, g_n, h \in \text{idem}(R)$ with $R = Rg_1 \oplus Rg_2 \oplus \cdots \oplus Rg_n \oplus Rh$ and $Rg_i = Re_i$ for each $i \geq 1$. From the equivalence of (L2) and (L4) we know that $g_i = u_i e_i$ for units $u_i \in U(R)$, and thus $e_i h = 0 \in J(R)$ for all $i \geq 1$.

Working modulo $J(R)$, by Proposition 4.4 there exists a single unit $\overline{u} \in U(R/J(R))$ such that $\overline{ue}_1, \ldots, \overline{ue}_n, \overline{uh}$ are orthogonal idempotents in $R/J(R)$. But as $\overline{R}\overline{e}_1 \oplus \cdots \oplus \overline{R}\overline{e}_n \oplus \overline{R}\overline{h} = \overline{R}$, we have $ue_1 + ue_2 + \cdots + ue_n + uh \equiv 1 \pmod{J(R)}$. Hence $e_1 + e_2 + \cdots + e_n + h \equiv u^{-1} \pmod{J(R)}$, and in particular $e_1 + e_2 + \cdots + e_n + h$ is a unit. We finish by appealing to Corollary 4.3. □

A priori, the assumption in Corollary 4.3 that the sum of the half-orthogonal idempotents is a unit might seem extravagant. However, from the proof above we see that any finite set of half-orthogonal idempotents is extendable to a unit-summable set.

Corollary 4.5. *If $e_1, \ldots, e_n \in \text{idem}(R)$ and $e_i e_j \in J(R)$ whenever $i < j$, then there exists $e_{n+1} \in \text{idem}(R)$ for which $e_i e_{n+1} = 0$ for all $i < n + 1$, and $v = e_1 + e_2 + \cdots + e_n + e_{n+1}$ is a unit.*

Proof. Take $e_{n+1} = h$ in the proof above. □

Throughout this section, we have worked with linearly ordered half-orthogonality. One might wonder about more exotic situations. In such cases it is impossible (in general) to get similar results.

Example 4.6. There exists a ring R with idempotents $e_1, e_2, e_3 \in \text{idem}(R)$ satisfying $e_1 e_2 = e_2 e_3 = e_3 e_1 = 0$, but no unit $u \in U(R)$ exists for which ue_1, ue_2, ue_3 are orthogonal idempotents.

Construction. Let F be a field and let $R = \mathbb{M}_3(F)$. Let

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \text{ and } e_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We compute $e_1 e_2 = e_2 e_3 = e_3 e_1 = 0$.

Setting $u = \begin{pmatrix} a & b & c \\ * & * & * \\ * & * & * \end{pmatrix} \in U(R)$, we calculate $f_1 = ue_1 = \begin{pmatrix} a-b & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$. For f_1 to be a non-zero idempotent we must have $a - b = 1$. Set $f_2 = ue_2$ and $f_3 = ue_3$. If we further require $f_1 f_2 = f_1 f_3 = 0$, then we find $b - c = 0$ and $-a + c = 0$. These equalities are inconsistent with $a - b = 1$. (It may be of interest to note that $e_1 + e_2 + e_3$ is not a unit.) □

5. Infinite collections of idempotents

When working with an infinite collection of idempotents $\{e_i\}_{i\in\Gamma} \subseteq \mathrm{idem}(R)$, where Γ is some indexing set, there are well-known and immediate problems. For instance, if idempotents lift modulo $J(R)$, then *countable* sets of orthogonal idempotents lift orthogonally, but not necessarily *uncountable* sets. On the positive side, once again, orthogonalizing is not a consequence of the lifting hypothesis, but rather the enabling property of $J(R)$.

Proposition 5.1. *Let $e_1, e_2, \ldots \in \mathrm{idem}(R)$ be a countable collection of idempotents, let $I \trianglelefteq R$ be an enabling ideal, and assume $e_i e_j \in I$ for all $i \neq j$. Then there exist idempotents $f_1, f_2, \ldots \in \mathrm{idem}(R)$ which are orthogonal and satisfy $f_n \equiv e_n \pmod{I}$ for each $n \geq 1$.*

Proof. We work by induction. Set $f_1 = e_1$. Suppose that we have defined orthogonal idempotents $f_1, \ldots, f_n$, with $f_i \equiv e_i \pmod{I}$ for $1 \leq i \leq n$. Notice that

$$\left(1 - \sum_{i=1}^{n} f_i\right) e_{n+1} \equiv e_{n+1} \equiv e_{n+1} \left(1 - \sum_{i=1}^{n} f_i\right) \pmod{I}.$$

Applying Proposition 3.4 (or even just Lemma 3.3), we can find an idempotent

$$f_{n+1} \in \left(1 - \sum_{i=1}^{n} f_i\right) e_{n+1} R e_{n+1} \left(1 - \sum_{i=1}^{n} f_i\right)$$

with $f_{n+1} \equiv e_{n+1} \pmod{I}$. Finally, note that an idempotent is orthogonal to $\sum_{i=1}^{n} f_i$ if and only if it is orthogonal to each of $f_1, \ldots, f_n$, as the f_i are pairwise orthogonal. Thus, f_{n+1} is orthogonal to each of $f_1, \ldots, f_n$. □

We will see a little later that better orthogonal lifting is possible modulo $J(R)$, in certain cases. First, we need to describe another dissimilarity between the finite and infinite cases.

Lemma 5.2. *Let Γ be a totally ordered set, and suppose $\{e_i\}_{i\in\Gamma} \subseteq \mathrm{idem}(R)$ is a collection of idempotents where $e_i e_j \in J(R)$ whenever $i < j$ (for $i, j \in \Gamma$). The left ideal $\sum_{i\in\Gamma} Re_i$ is a direct sum, and the family $\{Re_i\}_{i\in\Gamma}$ is a local direct summand of ${}_RR$ (meaning, $\sum_{i\in F} Re_i$ is a direct summand of ${}_RR$ for each finite subset $F \subseteq \Gamma$). Furthermore, $\sum_{i\in\Gamma} Re_i$ is a direct summand of ${}_RR$ if and only if $e_i = 0$ for all but finitely many $i \in \Gamma$.*

Proof. Applying Lemma 4.2, only the forward direction of the last statement remains. Suppose $\sum_{i\in\Gamma} Re_i$ is a direct summand of ${}_RR$. We can then write $R = \bigoplus_{i\in\Gamma} Re_i \oplus C$, where C is some complement summand. As ${}_RR$ is finitely generated (by the element 1_R), there can only exist finitely many nonzero terms in the sum. □

To overcome the obstacle presented in the previous lemma, we need a way to speak of infinite sums of idempotents. We will accomplish this by endowing R with a left linear, Hausdorff topology, which is occasionally also Σ-complete (see [11] for more details, examples, and definitions). The reader who finds this notion too general is welcome to think of R as the endomorphism ring of a right Λ-module M (with endomorphisms written on the left) given the finite topology (where annihilators of finite subsets of M are open left ideals). A collection of endomorphisms $\{\varphi_i\}_{i\in\Gamma}$ is summable if and only if $X_m := \{i \in \Gamma : \varphi_i(m) \neq 0\}$ is a finite set for each $m \in M$, and in that case we define $\left(\sum_{i\in\Gamma} \varphi_i\right)(m) := \sum_{i\in X_m} \varphi_i(m)$. As we will need it throughout the remainder of the paper, we make the following definition, following Corner.

Definition 5.3. A family of elements in a topological ring R is *unisummable* if it is both summable (in the topology) and sums to $1_R \in R$.

Readers who want to explore non-topological generalizations of summability are directed to [2] where the notion of an *agreeable* category is developed. It should be emphasized, however, that in any case these notions are *not* left-right symmetric. Thus, there are some subtleties which arise, and we will take pains to point them out.

In some cases, results which are true with a finite number of idempotents remain true for infinite sets merely by adding the hypothesis that the idempotents are summable. The following is one such example.

Lemma 5.4. *Let R be a ring with a left linear, Hausdorff topology. Let Γ be a totally ordered index set, let $\{e_i\}_{i\in\Gamma} \subseteq \mathrm{idem}(R)$ be a summable family, and assume $e_ie_j \in J(R)$ whenever $i < j$. If $v := \sum_{i\in\Gamma} e_i \in U(R)$, then $\{v^{-1}e_i\}$ and $\{e_iv^{-1}\}$ are each collections of orthogonal, unisummable idempotents.*

Proof. This is [8, Lemma 8]. □

Unfortunately, many of the other results about half-orthogonality apparently do not carry over. For example, Corollary 4.5 does not hold for arbitrary countable collections of (summable) idempotents.

Example 5.5. There exists a ring R with a left linear, Hausdorff topology, and summable idempotents $e_0, e_1, \ldots \in \mathrm{idem}(R)$ with $e_ie_j = 0$ when $i < j$, and with $v = \sum_{i=0}^{\infty} e_i$ a non-unit. Furthermore, $J(R) = 0$ and there does not exist any nonzero idempotent $e \in R$ which when inserted into the sequence $e_0, e_1, \ldots$ preserves the half-orthogonality relations.

Construction. Let F be a field and let $R = \mathrm{CFM}(F)$ be the ring of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ column-finite matrices. We view R as the endomorphism ring of the countable rank, free module $M = F^{(\mathbb{Z}_{>0})}$, itself viewed as column vectors acted on the left by the matrices in R. Let $E_{i,j}$ be the standard matrix unit in R with 1 in the (i,j)-position and zeros elsewhere. Set $e_0 = E_{1,1} + E_{1,2} + E_{1,3} + \cdots$, and for $n \geq 1$ put $e_n = E_{n,n} - E_{n+1,n}$. It is easy to check that $e_ie_j = 0$ whenever $i < j$, and that $e_i^2 = e_i$. Define $v := \sum_{i=0}^{\infty} e_i$; this is a non-unit because $\mathrm{im}(v) \subsetneq M$, since the image does not contain $E_{1,1}M$. However, v is injective. It is well known that $J(R) = 0$.

Fix $e \in \mathrm{idem}(R)$, and fix $n \in \{0, 1, 2, \ldots, \omega\}$. Assume that $e_ie = 0$ for $i < n$ and $ee_i = 0$ for $i \geq n$ (so we can insert e into our sequence of idempotents and preserve the half-orthogonality). From the fact that $e_ie = 0$ for $1 \leq i < n$ we know that the first $n - 1$ rows of e are zero. In particular, if $n = \omega$, then $e = 0$, so we may as well assume that n is finite.

First consider what happens if $n \geq 1$. From the fact that $ee_i = 0$ for all $i \geq n$, we see that each of the rows of e become constant at the nth column. So $e = (x_{i,j})$ looks like

$$e = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots \\ * & \cdots & * & x_{n,n} & x_{n,n} & \cdots \\ * & \cdots & * & x_{n+1,n} & x_{n+1,n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

As e is also a column-finite matrix, it must have only finitely many non-zero rows; say the rows are zero after the mth row. Further, as $n \geq 1$ we have $e_0e = 0$, and so $\sum_{i=n}^{m} x_{i,n} = 0$. We then compute that e^2

is zero from the nth column on. As e is an idempotent, and e^2 is block lower-triangular, we must have $e = 0$.

Finally, assume $n = 0$. As above, e has only finitely many non-zero rows and each row is constant. From $ee_0 = 0$ we must have that each row starts with a zero, and so $e = 0$.

It is interesting to note that $w := \sum_{i=1}^{\infty} e_i$ is also injective, but the set $\{e_i\}_{i=1}^{\infty}$ of half-orthogonal idempotents can be expanded to a larger half-orthogonal set (e.g. by including e_0). □

Question 5.6. Suppose we are given a totally ordered set Γ and a collection of summable idempotents $\{e_i\}_{i\in\Gamma} \subseteq \mathrm{idem}(\mathrm{End}(M))$ for which $e_ie_j = 0$ whenever $i < j$. If this set is maximal (in the sense that there does not exist a nonzero idempotent which can be inserted somewhere into the set and preserve the half-orthogonality relations), then is $\sum_{i\in\Gamma} e_i$ an injective endomorphism on M?

Example 5.5 seems to prevent the orthogonalizing of half-orthogonal idempotents via multiplication by a unit, even in the simple case where the idempotents are well-ordered by ω. Yet, not all hope is lost. The problem at hand is a consequence of the lack of left-right symmetry in the choice of a left linear topology (rather than a right linear topology), and can be fixed by reversing the half-orthogonality relations.

Theorem 5.7. *Let R be a ring with a Σ-complete, left linear, Hausdorff topology. Let Γ be an ordinal, let $\{e_i\}_{i\in\Gamma} \subseteq \mathrm{idem}(R)$ be a summable family, and assume $e_ie_j = 0$ whenever $i > j$.*

Setting

$$u := 1 + \sum_{\substack{m\geq 2\\ i_1<i_2<\ldots<i_m}} (-1)^{m-1}(m-1)e_{i_1}e_{i_2}\cdots e_{i_m},$$

this is a well-defined unit of R. Further, letting $f_i = ue_i$ for each $i \in \Gamma$, we have that $\{f_i\}_{i\in\Gamma}$ consists of orthogonal idempotents with $f_i \sim_\ell e_i$ for each $i \in \Gamma$. Moreover, $e_0 = f_0$ and setting $f' = 1 - \sum_{i\in\Gamma} f_i$ we have $uf' = f'$. In particular, $\sum_{i\in\Gamma} e_i + f' = u^{-1}$.

Proof. We first need to see that the terms in the definition of u are summable. From the Σ-complete hypothesis, it suffices to show that those terms are Σ-Cauchy. In other words, letting I be an open, left ideal of R it suffices to show that all but finitely many of the terms in the sum lie inside I. (If we think of R as an endomorphism ring, we are using the fact that the finite topology is complete.) As $\{e_i\}_{i\in\Gamma}$ is summable, there exists a finite set $F_1 \subseteq \Gamma$ such that $e_i \notin I$ if and only if $i \in F_1$. Next, as $\{e_je_i\}_{i,j\in\Gamma}$ is summable there exists a finite set $F_2 \subseteq \{(j,i) \in \Gamma\times\Gamma \ : \ j < i\}$ such that $e_je_i \notin I$ if and only if $(j,i) \in F_2$. Since I is a left ideal, if $(j,i) \in F_2$, then $i \in F_1$. Repeating this process, we find finite sets $F_3, F_4, \ldots$ with similar properties. As ordinals satisfy the descending chain condition, there is no infinite sequence of choices $i_1 \in F_1, (i_2,i_1) \in F_2, (i_3,i_2,i_1) \in F_3, \ldots$. Since there is no such infinite sequence, and since each F_n is finite, by König's tree lemma there is an upper bound on the length of such sequences. Thus, only finitely many terms do not belong to I, as desired.

Proving that the equality $e_iue_j = \delta_{i,j}e_i$ holds is done just as in Proposition 4.4. Hence the elements $f_i := ue_i$ are orthogonal idempotents, and $f_i \sim_\ell e_i$. During these computations one also proves

$$f_i = \sum_{\substack{m\geq 1\\ i_1<\ldots<i_m=i}} (-1)^{m-1}e_{i_1}\cdots e_{i_m}.$$

(Following ideas of Corner as in [2], one can instead prove that

$$f_i = \lim_{F\subseteq\Gamma,\ |F|<\infty} \left[\prod_{j<i,\ j\in F}(1-e_j)\right] e_i,$$

where the product is interpreted as moving left to right as the indices increase. More formally, one recursively defines $\pi_i = 1 - \sum_{j<i} \pi_j e_j$, which can be thought of as "$\prod_{j<i}(1-e_j)$." Then set $f_i = \pi_i e_i$, and prove the necessary properties.)

That $e_0 = f_0$ is easy, and $uf' = f'$ is also just as before. We see that

$$1 = \sum_{i\in\Gamma} f_i + f' = \sum_{i\in\Gamma} ue_i + uf' = u\left(\sum_{i\in\Gamma} e_i + f'\right),$$

and hence u is right invertible. To prove that u is a unit takes more work than in Proposition 4.4. In this case, u is not unipotent (although we can think of it as locally unipotent when R is an endomorphism ring). Neither can we appeal to left-right symmetry considerations. We could show directly that $\sum_{i\in\Gamma} e_i + f'$ is a left inverse for u, but we pursue another option.

As u is right invertible, fix $v \in R$ such that $uv = 1$. Notice that $u(vu-1) = 0$. Thus, if we can prove u is not a left zero-divisor we must have $vu - 1 = 0$, and so $u \in U(R)$. Fix $x \in R$ such that $ux = 0$. Let I be an open left ideal (hence it is also closed); as R is Hausdorff it suffices to prove that $x \in I$. Let $F \subseteq \Gamma$ be the finite subset such that $e_i x \notin I$ for each $i \in F$. If $F = \emptyset$, then as

$$x + \sum_{\substack{m\geq 2\\ i_1<\ldots<i_m}} (-1)^{m-1}(m-1)e_{i_1}\cdots e_{i_m}x = ux = 0 \in I$$

we must also have $x \in I$. On the other hand, if $F \neq \emptyset$, then fix $j \in F$ maximal. We find

$$e_j x + \sum_{\substack{m\geq 2\\ j\leq i_1<\ldots<i_m}} (-1)^{m-1}(m-1)e_j e_{i_1}\cdots e_{i_m}x = e_j ux = 0 \in I,$$

and we then have $e_j x \in I$ after all, a contradiction. So $x \in I$ must hold, and the theorem is proved. □

The following corollary should be compared with [8, Theorem 9].

Corollary 5.8. *Let R be a ring with a Σ-complete, left linear, Hausdorff topology, and let $I \trianglelefteq R$ with $I \subseteq J(R)$ be a closed ideal. Let Γ be an ordinal, let $\{e_i\}_{i\in\Gamma} \subseteq \mathrm{idem}(R)$ be a summable family, and assume $e_ie_j \in I$ whenever $i > j$. If idempotents lift modulo I, then there exists a unit $w \in U(R)$ such that $\{we_i\}$ and $\{e_iw\}$ are idempotents in R, orthogonal modulo I. If the e_i are orthogonal modulo I, then we can drop the Σ-complete hypothesis and we can choose $w \equiv 1 \pmod{I}$.*

Proof. As I is a closed ideal, R/I can be endowed with the quotient topology. This topology is left linear and Hausdorff, but not necessarily Σ-complete. We will only use the Σ-complete hypothesis when working in R.

Let u be the explicit element constructed in the previous theorem (which still exists in the present context, by the same proof), and set $f' = 1 - \sum_{i\in\Gamma} ue_i$. By the previous theorem, we know f' is an idempotent modulo I and u is a unit modulo I. By the lifting hypothesis, we can find an idempotent $f'' \in f'R$ with $f'' \equiv f' \pmod{I}$. Also, as $I \subseteq J(R)$ we know that $u \in U(R)$. Again appealing to the previous theorem we know that $\sum_{i\in\Gamma} e_i + f'' \equiv u^{-1} \pmod{I}$. Hence, $v := \sum_{i\in\Gamma} e_i + f' \in U(R)$. By Lemma 5.4, putting $w = v^{-1}$ we are done. □

Note: If R is a ring with a left linear, Hausdorff topology, and every ideal without a non-zero idempotent is contained in the Jacobson radical, then $J(R)$ is a closed ideal in R. In particular, this holds true for exchange rings (see [2, Theorem 8.3] or [8, Lemma 11]). We will use this fact repeatedly in the remainder of the paper.

Corollary 5.9. *Let R be a ring with a Σ-complete, left linear, Hausdorff topology, and let $I \trianglelefteq R$ with $I \subseteq J(R)$ be a closed ideal. Let Γ be an ordinal, let $\{e_i\}_{i\in\Gamma} \subseteq \mathrm{idem}(R)$ be a summable family, and* *assume $e_ie_j \in I$ whenever $i > j$. If idempotents lift modulo I, then there exists an idempotent $e \in R$ such that $e_ie \in I$ for all $i \in \Gamma$ and $e + \sum_{i\in\Gamma} e_i \in U(R)$.*

Proof. Take $e = f''$, where f'' is the same element constructed in the previous corollary. □

Corollary 5.8 should be compared with [2, Lemma 9.8], where a small error is introduced when trying to remove the lifting hypothesis. Thus we ask:

Question 5.10. Can we drop the lifting hypothesis from the two corollaries above?

In many practical applications, the direction of half-orthogonality is forced upon us. Indeed, it turns out that in situations that arise naturally for exchange rings, the half-orthogonality relations are *opposite* those in Theorem 5.7 and Corollaries 5.8 and 5.9. Thus, we are still interested to know, in these situations, whether we can multiply by an element and turn the half-orthogonal family into a fully orthogonal family. The following example shows some of the interesting behavior that can occur.

Example 5.11. There exists an endomorphism ring R, and summable idempotents $\{e_i\}_{i=0}^{\infty}$ for which $e_ie_j = 0$ whenever $i < j$, where $\sum_{i=0}^{\infty} e_i$ is not a unit, but there still exists an element $u \in R$ such that $\{ue_i\}$ is a unisummable, orthogonal set of idempotents.

Construction. Just use the ring and idempotents constructed in Example 5.5. The matrix

$$u := \begin{pmatrix} 0 & -1 & -1 & -1 & \cdots \\ 0 & 0 & -1 & -1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

with -1 in all entries strictly above the diagonal, and 0's elsewhere, suffices. □

While in general the element u constructed in Proposition 4.4 isn't necessarily a well-defined element when generalized to an infinite collections of idempotents, there are many things we can say when it turns out that it does exist.

Proposition 5.12. *Let R be a ring with a Σ-complete, left linear, Hausdorff topology, let Γ be an ordinal, and let $\{e_i\}_{i\in\Gamma}$ be summable idempotents for which $e_ie_j = 0$ whenever $i < j$. Assume that*

$$u := 1 + \sum_{\substack{m\geq 2 \\ i_1<i_2<\ldots<i_m}} (-1)^{m-1}(m-1)e_{i_m}e_{i_{m-1}}\cdots e_{i_1}$$

is well-defined (meaning, the terms in the sum are summable).

We have $u \in U(R)$, and if we set $f_i = ue_i$, then $\{f_i\}_{i\in\Gamma}$ is a collection of orthogonal idempotents. Further, if we define $f' := 1 - \sum_{i\in\Gamma} f_i$, then $e_if' = 0$ for all $i \in \Gamma$ and $\sum_{i\in\Gamma} e_i + f' = u^{-1}$.

Proof. This follows from Proposition 4.4, *mutatis mutandis*, except the part about u being a unit. We still obtain that u is right invertible. Left invertibility follows by reasoning as in Theorem 5.7 (except we use a minimal subscript rather than a maximal one). □

In case $R = \mathrm{End}(M)$ is an endomorphism ring (using the finite topology), then the element u of Proposition 5.12 being well-defined is equivalent to the family $\{e_i\}_{i\in\Gamma}$ being locally right-T-nilpotent.

6. Understanding Harada modules

We end this paper with a foray into the subject of Harada modules, giving new and enlightening proofs of some known results, simultaneously providing new insights into the structure of their endomorphism rings. All modules in this section will be right k-modules for some ring k, with k not playing an important role, and endomorphisms written on the left. All mention of k will be suppressed.

Recall the following important definition.

Definition 6.1. A module M has the $\aleph$-*exchange property*, for some cardinal $\aleph$, if whenever there exist decompositions $A = M \oplus N = \bigoplus_{i\in\aleph} A_i$, then there are submodules $A_i' \subseteq A_i$ with $A = M \oplus \bigoplus_{i\in\aleph} A_i'$.

In other words, we can remove a piece of each of the summands of $\bigoplus_{i\in\aleph} A_i$ and "exchange" the sum of the missing parts with M. If this holds for all $\aleph$, we say that M has the *(full) exchange property.* It is known that 2-exchange implies n-exchange for all finite n, and thus 2-exchange is called *finite exchange.* It is an open question, going back fifty years to Crawley and Jónsson [3], whether finite exchange implies exchange (or even just countable exchange). There are many examples where it is known that finite exchange and exchange are equivalent, such as for square-free modules [12] and for direct sums of indecomposable modules [16]. We will focus on this latter situation, and to describe the main historical result we need to recall a few more definitions.

Definition 6.2. A decomposition $M = \bigoplus_{i\in\Gamma} M_i$ *complements summands* if for every direct summand $N \subseteq^{\oplus} M$ there exists a subset $\Omega \subseteq \Gamma$ such that $M = (\bigoplus_{i\in\Omega} M_i) \oplus N$.

Definition 6.3. A family of submodules $\mathfrak{X} = \{X_\ell\}_{\ell\in\Lambda}$ of a module M is called a *local summand* if the sum $\sum_{\ell\in F} X_\ell$ is both a direct sum and a direct summand of M, for each finite subset $F \subseteq \Gamma$. We say that *every local summand of M is a summand* if whenever $\mathfrak{X} = \{X_\ell\}_{\ell\in\Lambda}$ is a local summand of M, then the total sum $\sum_{\ell\in\Lambda} X_\ell$ is in fact a direct summand of M.

Definition 6.4. A family of modules $\{M_i\}_{i\in\Gamma}$ is *locally semi-T-nilpotent* if for every countable sequence of distinct indices $i_1, i_2, \ldots \in \Gamma$, every $m \in M_{i_1}$, and every sequence $f_1, f_2, \ldots$ of non-isomorphisms $f_n : M_{i_n} \to M_{i_{n+1}}$, then there exists an index $p \geq 1$ such that $f_p f_{p-1} \cdots f_1(m) = 0$.

With these notions in hand, we can now recall how to characterize when direct sums of indecomposables have the exchange property (see [7] for additional information).

Proposition 6.5. *Assume $M = \bigoplus_{i\in\Gamma} M_i$ where each M_i is indecomposable, and let $S = \mathrm{End}(M)$. The following are all equivalent:*

(1) *The module M has the exchange property.*
(2) *The module M has the finite exchange property.*
(3) *The family $\{M_i\}_{i\in\Gamma}$ is locally semi-T-nilpotent, and $\mathrm{End}(M_i)$ is a local ring for every $i \in \Gamma$.*
(4) *The decomposition complements summands, and $\mathrm{End}(M_i)$ is a local ring for every $i \in \Gamma$.*
(5) *Every local summand of M is a summand, and $\mathrm{End}(M_i)$ is a local ring for every $i \in \Gamma$.*
(6) *The ring $S/J(S)$ is von Neumann regular and idempotents lift modulo $J(S)$.*

A module M is called a *Harada module*, in honor of M. Harada's contributions to the proof of this proposition, if these conditions hold. Perhaps the most difficult implication in this proposition is (3) ⇒ (1), first proved in [16]. We will give another proof of the key step in this implication. We also give a new proof of the implication (2) ⇒ (3).

To begin, we recall [16, Proposition 3], with some additions found in [2], which tells us how to translate the exchange property on a module into statements about the endomorphism ring.

Proposition 6.6. *Given a cardinal $\aleph$, and a module M with $S = \mathrm{End}(M)$, the following are equivalent:*

(1) *The module M has the $\aleph$-exchange property.*
(2) *If $A = M \oplus N = \bigoplus_{\ell\in\Lambda} A_\ell$, where $A_\ell \cong M$ for all $\ell \in \Lambda$, $|\Lambda| \leq \aleph$ (and $N \cong \bigoplus_{|\Lambda|-1} M$), then there exist submodules $A_\ell' \subseteq A_\ell$ (such that $N \cap A_\ell \subseteq A_\ell'$) with*
$$A = M \oplus \bigoplus_{\ell\in\Lambda} A_\ell.$$
(3) *Given a unisummable family $\{x_\ell\}_{\ell\in\Lambda}$, with $|\Lambda| \leq \aleph$, there exist unisummable, orthogonal idempotents $e_\ell \in Sx_\ell$.*

In particular, M has the finite exchange property if and only if $S = \mathrm{End}(M)$ is an exchange ring, which was proven much earlier by Warfield. Remember that we can interpret summability in a more general sense if we are working in a ring R with a left linear, Hausdorff topology.

In trying to understand the endomorphism ring of a Harada module, we are naturally led to study the endomorphism ring of an arbitrary decomposable module $M = \bigoplus_{i\in\Gamma} M_i$. For convenience, we fix some notation, by putting $S = \mathrm{End}(M)$ and $S_i = \mathrm{End}(M_i)$ for each $i \in \Gamma$. Let $\pi_i \in S$ be the map which is the identity on M_i and zero on $\bigoplus_{j\in\Gamma\setminus\{i\}} M_j$. We can view S as the ring of $\Gamma \times \Gamma$ column-summable matrices, where $x = (x_{i,j}) \in S$ with $x_{i,j} = \pi_i x \pi_j$. We also identify $S_i = \pi_i S \pi_i \subseteq S$. We recall one of Zelmanowitz's characterizations of the Jacobson radical in S.

Proposition 6.7 (cf. [15, Theorem 1]). *For $M = \bigoplus_{i\in\Gamma} M_i$ with Γ infinite and $\alpha \in S = \mathrm{End}(M)$, then $\alpha \in J(S)$ if and only if for every $s \in S$ we have both*

(A) *$(\alpha s)_{i,i} \in J(S_i)$ for every $i \in \Gamma$, and*

(B) *for every sequence $i_1, i_2, \ldots$ of distinct elements of Γ and for every $m \in M$ there exists an integer $n \geq 1$ such that $(\alpha s)_{i_{n+1},i_n}(\alpha s)_{i_n,i_{n-1}} \cdots (\alpha s)_{i_2,i_1}(m) = 0$.*

The main ingredient in the original proof of the implication (3) ⇒ (1) for Proposition 6.5 is [16, Theorem 5(I)]. One of the hypotheses employed in Theorem 5(I) is that the collection $\{M_i\}_{i\in\Gamma}$ is locally semi-T-nilpotent, and when $i \neq j$ the modules M_i and M_j possess no nontrivial isomorphic summands. We now provide some alternate characterizations of these hypotheses. This simultaneously strengthens the main result of [4]. We note that one can use Proposition 6.7 to shorten the proof of this theorem, but we include elementary arguments when possible.

Theorem 6.8. *Let $M = \bigoplus_{i\in\Gamma} M_i$. The following are equivalent:*

(1) *Each M_i has the finite exchange property, the collection $\{M_i\}_{i\in\Gamma}$ is locally semi-T-nilpotent, and when $i \neq j$ the modules M_i and M_j possess no nontrivial isomorphic direct summands.*

(2) *The module M has the finite exchange property, and when $i \neq j$ the modules M_i and M_j possess no nontrivial isomorphic direct summands.*

(3) *The module M has the finite exchange property, and if $x \in S$, then $\pi_j x \pi_i \in J(S)$ whenever $i \neq j$.*

(4) *The module M has the finite exchange property, and the map $\varphi : S/J(S) \to \prod_{i\in\Gamma} S_i/J(S_i)$ given by the rule $x + J(S) \mapsto (\pi_i x \pi_i + J(S_i))_{i\in\Gamma}$ is a ring isomorphism.*

Proof. (1) ⇒ (4): Let $x, y \in S$. The first step is to show that $z := x_{i,j}y_{j,i} \in J(S_i)$ when $i \neq j$. Suppose, by way of contradiction, that $z \notin J(S_i)$. As S_i is an exchange ring, there exists some $r \in S_i$ with $e := rz = rx_{i,j}y_{j,i} \in S_i$ a nonzero idempotent. Thus e is isomorphic to the idempotent $f := y_{j,i}erx_{i,j} \in S_j$. This implies that M_i and M_j share a nontrivial isomorphic summand, yielding the needed contradiction.

Next, we prove that $K(S) := \{x \in S : x_{i,i} \in J(S_i) \text{ for all } i \in \Gamma\}$ equals $J(S)$. Towards that end, we need to prove that $K(S)$ is an ideal. Clearly $K(S)$ is closed under addition. Given $x \in K(S)$ and $y \in S$ we compute

$$(xy)_{i,i} = \sum_{j\in\Gamma} x_{i,j}y_{j,i}.$$

When $j = i$ we have $x_{i,i}y_{i,i} \in J(S_i)$ since $x_{i,i} \in J(S_i)$. When $j \neq i$, the previous paragraph tells us $x_{i,j}y_{j,i} \in J(S_i)$. As S_i is the endomorphism ring of a module with finite exchange, we know $J(S_i)$ is a closed ideal in the finite topology [7, Lemma 11]; hence, $(xy)_{i,i} \in J(S_i)$. This proves that $K(S)$ is closed under right multiplication from S, and a similar computation shows it is closed under left multiplication. Thus $K(S)$ is an ideal.

As $S/K(S) \cong \prod_{i\in\Gamma} S_i/J(S_i)$ is semiprimitive, we have $K(S) \supseteq J(S)$. To prove the other containment, first consider when Γ is finite. Fixing $x \in K(S)$, it suffices to show that $1 - xy$ is a unit for every

$y \in S$. We can reduce to the case where $x = x_{i,j} = \pi_i x \pi_j$, since if each entry of x lives in $J(S)$, then so does x (from the finiteness assumption on Γ). Thinking of elements of S as $\Gamma \times \Gamma$ matrices we calculate that $1 - xy = 1 - \sum_{k\in\Gamma} x_{i,j}y_{j,k}$ is a unit, since each diagonal entry is a unit and the only other nonzero entries are restricted to the ith row (so after a permutation of Γ, the matrix $1 - xy$ is upper-triangular with units down the diagonal).

In case Γ is infinite, we appeal to Proposition 6.7. We already proved that $(xy)_{i,i} \in J(S_i)$ for all $i \in \Gamma$ when $x \in K(S)$. The other condition of that proposition follows from the local semi-T-nilpotence hypothesis. Thus $K(S) = J(S)$, and the isomorphism in (4) follows.

Finally, we need to prove that M has the finite exchange property. Equivalently, we need to show that S is an exchange ring. We know $S/J(S)$ is a direct product of exchange rings, and hence an exchange ring. Appealing to [9, Proposition 1.5], it suffices to show that idempotents in $S/J(S)$ lift to S. This is easy; start by fixing an idempotent $\epsilon \in S/J(S)$ and so $\varphi(\epsilon) = (\epsilon_i)_{i\in\Gamma} \in \prod_{i\in\Gamma} S_i/J(S_i)$. We know that idempotents lift from $S_i/J(S_i)$ to S_i, since S_i is an exchange ring. Fix $e_i \in \text{idem}(S_i)$ lifting ϵ_i. If we set $e = \sum_{i\in\Gamma} e_i$, then this is an idempotent of S lifting ϵ, as desired.

(4) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (2): This implication is true even if we remove the exchange hypotheses from (2) and (3). We prove the contrapositive statement. Suppose we have distinct indices $i, j \in \Gamma$, with M_i and M_j possessing nonzero isomorphic summands. Write $M_i = A \oplus B$ and $M_j = C \oplus D$ with $A \cong C \neq 0$. Let $x : M_i \to M_j$ be the map which is zero on B and an isomorphism from A to C. Define $y : M_j \to M_i$ to be zero on D and to act as x^{-1} on C. We see that $yx \in S$ is a nonzero idempotent, and thus $1 - yx$ is not a unit. In particular, $x = \pi_j x \pi_i \notin J(S)$.

(2) $\Rightarrow$ (1): The finite exchange property passes to summands. Thus, it suffices to deduce that $\{M_i\}_{i\in\Gamma}$ is a locally semi-T-nilpotent family. If Γ is finite, then there is nothing to show, so we may assume Γ is infinite. Let $i_1, i_2, \ldots$ be a sequence of distinct indices from Γ. For ease of notation, we rename these indices as $1, 2, \ldots$. Without loss of generality, we assume $M = \bigoplus_{n=1}^{\infty} M_n$, and also assume $M_n \neq 0$ for each $n \geq 1$. Let $x_{n+1,n} : M_n \to M_{n+1}$ be a homomorphism. Each $x_{n+1,n}$ is not an isomorphism since M_n and M_{n+1} possess no nontrivial isomorphic summands.

We now claim that $x_{n+1,n} \in J(S)$ for each $n \geq 1$. If not, fix some n where it fails. As S is an exchange ring there exists an element $y \in S$ such that $0 \neq yx_{n+1,n} \in \text{idem}(S)$. But as $yx_{n+1,n} = yx_{n+1,n}\pi_n$ we see that $\pi_n y x_{n+1,n} = (\pi_n y \pi_{n+1})x_{n+1,n} \in S_n$ is also a nonzero idempotent. From the same proof as in the first paragraph of the implication (1) $\Rightarrow$ (4) above, we know that this leads to a contradiction. Thus $x_{n+1,n} \in J(S)$ for each $n \geq 1$.

As S is an exchange ring, we know $J(S)$ is a closed ideal. Thus $A = \sum_{n=1}^{\infty} x_{n+1,n} \in J(S)$, and in particular

$$1 - A = \begin{pmatrix} 1 & & & \\ -x_{2,1} & 1 & & \\ & -x_{3,2} & 1 & \\ & & \ddots & \ddots \end{pmatrix} \in U(S).$$

Calculating the (right) inverse of $1 - A$, we find

$$(1 - A)^{-1} = \begin{pmatrix} 1 & & & & \\ x_{2,1} & 1 & & & \\ x_{3,2}x_{2,1} & x_{3,2} & 1 & & \\ x_{4,3}x_{3,2}x_{2,1} & x_{4,3}x_{3,2} & x_{4,3} & 1 & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

As this is a column-summable matrix, we see that for each $m \in M_1$ there exists an index $p \geq 1$ such that $x_{p+1,p}x_{p,p-1} \cdots x_{2,1}(m) = 0$. This proves the local semi-T-nilpotence of $\{M_i\}_{i\in\Gamma}$. $\square$

We are now in a position to generalize [16, Theorem 5(I)], by interpreting that result as a statement about direct products. Recall that this is the key step in proving (3) ⇒ (1) in Proposition 6.5.

Theorem 6.9. *Let $M = \bigoplus_{i\in\Gamma} M_i$, where each M_i has the finite exchange property, the collection $\{M_i\}_{i\in\Gamma}$ is locally semi-T-nilpotent, and where M_i and M_j have no nontrivial isomorphic direct summands when $i \neq j$. Given a cardinal $\aleph$, the module M has $\aleph$-exchange if and only if each M_i has $\aleph$-exchange.*

Proof. As $\aleph$-exchange passes to direct summands, it suffices to do the "if" direction, so assume each M_i has $\aleph$-exchange. To show that M has $\aleph$-exchange, we will prove that condition (3) of Proposition 6.6 holds.

Accordingly, suppose $\{x_\ell\}_{\ell\in\Lambda}$ is a unisummable family in $S = \mathrm{End}(M)$, where $|\Lambda| \leq \aleph$. Theorem 6.8 tells us that $S/J(S) \cong \prod_{i\in\Gamma} S_i/J(S_i)$, and thus $x_\ell \equiv \sum_{i\in\Gamma} x_{i,\ell} \pmod{J(S)}$ where $x_{i,\ell} = \pi_i x_\ell \pi_i \in S_i$. (Note: Thinking of S as $\Gamma\times\Gamma$ column-summable matrices, we should perhaps write $x_{i,i,\ell}$ rather than $x_{i,\ell}$, since this element lives in the (i,i)-coordinate. However, throughout this proof we will only work with diagonal matrices in S, so this should cause no confusion.) Since $\sum_{\ell\in\Lambda} x_\ell = 1$ we have

$$\sum_{\ell\in\Lambda} x_{i,\ell} = \pi_i = 1_{S_i} \text{ for each } i \in \Gamma.$$

As M_i has $\aleph$-exchange for each $i \in \Gamma$, we see by Proposition 6.6 that there exist orthogonal, summable idempotents $e_{i,\ell} \in S_i x_{i,\ell}$ with $\sum_\ell e_{i,\ell} = 1_{S_i}$. For later use, for each $i \in \Gamma$ and $\ell \in \Lambda$ fix some $r_{i,\ell} \in S_i$, so that $e_{i,\ell} = r_{i,\ell} x_{i,\ell}$, and set $r_\ell := \sum_{i\in\Gamma} r_{i,\ell} \in S$.

For each $\ell \in \Lambda$ define $e_\ell := \sum_{i\in\Gamma} e_{i,\ell} \in S$. This is an idempotent, as it is the sum of orthogonal idempotents. Further, we compute

$$e_\ell = \sum_{i\in\Gamma} e_{i,\ell} = \sum_{i\in\Gamma} r_{i,\ell} x_{i,\ell} \equiv r_\ell x_\ell \pmod{J(S)}.$$

As $J(S)$ is enabling, we can fix idempotents $f_\ell \in S x_\ell$ with $f_\ell \equiv e_\ell \pmod{J(S)}$ for each $\ell \in \Lambda$.

We find that

$$v := \sum_{\ell\in\Lambda} f_\ell \equiv \sum_{\ell\in\Lambda} e_\ell = 1 \pmod{J(S)}$$

is a unit. Note that the family $\{f_\ell\}_{\ell\in\Lambda}$ is summable since $\{x_\ell\}_{\ell\in\Lambda}$ is summable. Setting $g_\ell = v^{-1} f_\ell \in S x_\ell$, by Lemma 5.4 we conclude that $\{g_\ell\}_{\ell\in\Lambda}$ is a unisummable collection of orthogonal idempotents. This demonstrates that property (3) of Proposition 6.6 holds. □

We now provide a new proof of the implication (2) ⇒ (3) in Proposition 6.5, which differs significantly from those found in the literature. We begin with an important lemma of independent interest, as it generalizes part of Proposition 4.1 from [13].

Lemma 6.10. *Assume $M = \bigoplus_{i\in\Gamma} M_i$, where $M_i \cong M_0$ for all $i \in \Gamma$, and Γ is infinite. Identify $S = \mathrm{End}(M)$ with the ring of column-summable matrices over $S_0 = \mathrm{End}(M_0)$. If M has the finite exchange property, then*

$$J(S) = \{s = (s_{i,j}) \in S \,:\, s_{i,j} \in J(S_0) \text{ for all } i,j \in \Gamma\}.$$

Furthermore, $J(S_0)$ is locally right T-nilpotent. In particular, if M_0 is finitely generated, then $J(S_0)$ is right T-nilpotent.

Proof. Let $K(S) = \{s = (s_{i,j}) \in S \,:\, s_{i,j} \in J(S_0) \text{ for all } i,j \in \Gamma\}$. The inclusion $J(S) \subseteq K(S)$ is true without assuming the finite exchange property. To prove this, fix $X = (x_{i,j}) \in J(S)$ and fix $i_0, j_0 \in \Gamma$; it suffices to show that $x_{i_0,j_0} \in J(S_0)$. After the appropriate matrix multiplication, we can send X to the new matrix $X' \in J(S)$ with x_{i_0,j_0} in the upper left-hand corner, and zeros elsewhere. Let $y \in S_0$, and let Y be the matrix with y in the upper left-hand corner, and zeros elsewhere. As $1 - X'Y$ is a unit since $X' \in J(S)$, we see that $1 - x_{i_0,j_0}y \in U(S_0)$. This holds true for each $y \in S_0$, therefore $x_{i_0,j_0} \in J(S_0)$, and hence $J(S) \subseteq K(S)$ as claimed.

For the reverse inclusion, let X be a matrix with one entry from $J(S_0)$ (in an arbitrary position) and all other entries zero. Given any $Y \in S$, we see that $1 - XY$ is a matrix with units on the main diagonal and the only other nonzero entries are restricted to a single row. Thus $1 - XY$ is a unit, and hence $X \in J(S)$. As $J(S)$ is a closed ideal in S (under the finite topology, again see [7, Lemma 8]), we must have $K(S) \subseteq J(S)$.

Given $x_1, x_2, \ldots \in J(S_0)$, consider the matrix

$$A = \begin{pmatrix} 0 & & & \\ x_1 & 0 & & \\ & x_2 & 0 & \\ & & \ddots & \ddots \end{pmatrix}.$$

Since $J(S) = K(S)$ we have $A \in J(S)$. Following the same argument as used in the proof of the implication (2) $\Rightarrow$ (1) in Theorem 6.8, we see that $\{x_1, x_2, \ldots\}$ is locally right T-nilpotent. (Alternatively, we can apply Proposition 6.7.) The last statement of the lemma is now clear. □

Proposition 6.11. *Assume $M = \bigoplus_{i\in\Gamma} M_i$ where each M_i is indecomposable. If M has the finite exchange property, then $\{M_i\}_{i\in\Gamma}$ is locally semi-T-nilpotent and* $\mathrm{End}(M_i)$ *is a local ring for each $i \in \Gamma$.*

Proof. Assume M has the finite exchange property. Each M_i also then has the finite exchange property. As M_i is indecomposable, it then has a local endomorphism ring [14, Proposition 1]. All that remains is proving local semi-T-nilpotence.

Initially consider the case when all of the M_i are isomorphic, say $M_i \cong M_0$ for all $i \in \Gamma$. If $|\Gamma| < \infty$, then local semi-T-nilpotence automatically holds. We thus assume Γ is infinite; in this case the local semi-T-nilpotence of $\{M_i\}_{i\in\Gamma}$ is equivalent to the local right T-nilpotence of $J(S_0)$, and that follows from the previous lemma.

In the general case, write $M = \bigoplus_{j\in\Omega} N_j$ where $\{N_j\}_{j\in\Omega}$ is the set of homogeneous components in the decomposition $M = \bigoplus_{i\in\Gamma} M_i$. (A homogeneous component is just the sum of all summands of a given isomorphism type.) By the Krull-Schmidt Theorem, the N_j share no isomorphic summands. Thus, by Theorem 6.8 we know that the set $\{N_j\}_{j\in\Omega}$ is semi-T-nilpotent.

Fix a sequence of distinct indices $i_1, i_2, \ldots \in \Gamma$, fix a set of non-isomorphisms $\varphi_n : M_{i_n} \to M_{i_{n+1}}$, and fix $m \in M_{i_1}$. There are two main possibilities. First, infinitely many of the M_{i_n}'s could belong to the same homogeneous component. In that case we may assume that each φ_n is an endomorphism of a fixed homogeneous component (after composing maps if necessary), and the homogeneous component is composed of infinitely many direct summands. Notice that each of the maps φ_n (even after composing) is still a non-isomorphism, and in fact belongs to the Jacobson radical, since any composite map $N_i \to N_j \to N_i$ (where $i \neq j$) belongs to $J(\mathrm{End}(N_i))$, and any non-isomorphism $\varphi : M_{i_n} \to M_{i_{n+1}}$ of isomorphic strongly indecomposable modules $M_{i_n} \cong M_{i_{n+1}}$ also belongs to the Jacobson radical. As the Jacobson radical of $\mathrm{End}(N_i)$ is locally right T-nilpotent by the previous lemma, there is some $p \geq 1$ such that $\varphi_p\varphi_{p-1}\cdots\varphi_1(m) = 0$. The second possibility is that infinitely many M_{i_n}'s belong to distinct homogeneous components. In that case, local semi-T-nilpotence is exhibited by the fact that the N's are locally semi-T-nilpotent. □

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