HOMOGENEOUS NILRADICALS OVER SEMIGROUP GRADED RINGS

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Abstract. In this paper we study the homogeneity of radicals defined by nilpotence or primality conditions, in rings graded by a semigroup S. When S is a unique product semigroup, we show that the right (and left) strongly prime and uniformly strongly prime radicals are homogeneous, and an even stronger result holds for the generalized nilradical. We further prove that rings graded by torsion-free, nilpotent groups have homogeneous upper nilradical. We conclude by showing that non-semiprime rings graded by a large class of semigroups must always contain nonzero homogeneous nilpotent ideals.

1. Introduction

To understand the structure of a ring it is extremely useful to consider many different radicals of the ring. A natural problem in ring theory is to determine which radicals of a graded ring are generated by homogeneous elements, and moreover describe the elements of such radicals under (graded) extensions. For instance, the famous (and currently open) Köthe Conjecture posits that any nil one-sided ideal is contained in a nil two-sided ideal. Surprisingly, this is equivalent to showing $R[x]$ is Jacobson radical if (and only if) R is a nil ring.

Many homogeneity results for classical radicals are known, with arguably one of the earliest and most influential results being that of Bergman [1], who proved that the Jacobson radical of any $\mathbb{Z}$-graded ring is homogeneous. More generally, it has been proven in [10] that if G is a group which is residually p-finite for two distinct primes p and R is a G-graded ring, then $J(R)$ is homogeneous. Since this large class of groups contains all free groups and finitely generated torsion-free nilpotent groups, any ring graded by such a group must have a homogeneous Jacobson radical.

In this paper we will work more broadly with semigroup graded rings (see Kelarev's book [14] as a resource for information on graded rings), and a very natural class of semigroups in this context are the so-called *unique product semigroups*, or u.p.-semigroups for short. We will recall the definition and basic properties of such semigroups in Section 2, and give an interesting generalization of the unique product property a little later in Section 3. By two well-known theorems of Jespers, Krempa, and Puczylowski found in [9], if S is a u.p.-semigroup, then any S-graded ring has homogeneous prime radical and Levitzki (or locally nilpotent) radical. It is still open whether this result is true for the Jacobson radical and upper nilradical, but some partial results are known (see [8] and [11] for a collection of such results). However, there are two known necessary conditions; the semigroup must be cancellative and torsion free (see [18]).

For commutative semigroups, the situation is more understood. The following proposition summarizes information that can be found in [9] and [18].

Proposition A. *Let S be a commutative semigroup. The prime radical* (*respectively, Levitzki radical, Jacobson radical, upper nilradical, bounded nilradical, Wedderburn radical*) *is homogeneous in every S-graded ring if and only if S is cancellative and torsion-free.*

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This proposition may give the impression that cancellativity and torsion-freeness are always necessary (at least when considering "nice" radicals). Indeed, there are ranges of radicals $\mathfrak{F}$ where these two properties hold in any semigroup S such that $\mathfrak{F}(R)$ is homogeneous whenever R is S-graded (see [18, Theorem 6 and Proposition 11]). However, we will see that for those radicals which fall outside these ranges, those properties are not necessary.

In Section 3 we work with the right strongly prime radical $s_r(R)$, the uniformly strongly prime radical $u(R)$, and the generalized nilradical $N_g(R)$ for a ring R. (The definitions and basic properties of these radicals are recalled in that section, but the reader may consult [4] for additional information. In particular, there is a nice diagram showing the relationship between these radicals and those discussed above, on page 293 of [4]. We will cite the relevant proofs when we need to use any of these containments.) The first main result of this paper is the following:

Theorem B. *If S is a u.p.-semigroup and R is an S-graded ring, then $s_r(R)$, $u(R)$, and $N_g(R)$ are homogeneous.*

Theorem B incorporates information from Theorems 3.7, 3.10, and 3.12 below. In fact, we obtain a slight improvement on the result for the generalized nilradical. Moreover, as a corollary, we find that we can add $s_r(R)$ and $u(R)$ to the list of radicals given in Proposition A.

We next turn to the context of *group* (rather than semigroup) graded rings. In Section 4 we study the upper nilradical of group graded rings. Much less is known regarding the homogeneity of the upper nilradical of such a graded ring in comparison with the other standard radicals. However, by bootstrapping an argument due to Smoktunowicz from [26], we are able to prove the second main result of the paper, which will appear as Theorem 4.1, and which we restate here.

Theorem C. *If G is a torsion-free, nilpotent group and R is a G-graded ring, then* $\mathrm{Nil}^*(R)$ *is homogeneous.*

As a consequence of this theorem, we are able to generalize [18, Theorem 13] for group graded rings.

We end the paper in Section 5 by also proving homogeneity results for the Wedderburn radical of a ring. Our final main result, appearing as Theorem 5.2 is the following:

Theorem D. *Let S be a semigroup and let R be any S-graded ring. If the Wedderburn radical has no nonzero homogeneous elements, then the prime radical also has no nonzero homogeneous elements.*

Throughout this paper we have the following conventions. We let S denote a semigroup and let G denote a group. Rings will be associative, but not necessarily be commutative or have multiplicative identity, and we always reserve the letter R to denote a ring. If R is an S-graded ring and $s \in S$, we will write R_s for the s-component of R; so that $R = \bigoplus_{s\in S} R_s$. It is easy to show that when S is a cancellative semigroup and R is a nonzero unital ring which is S-graded, then S is a monoid (say with identity e) and the multiplicative identity $1 \in R$ is homogeneous and belongs to the identity component R_e. (The proof relies on the fact that if $st = t$ for some $s, t \in S$, with S a cancellative semigroup, then s is a two-sided identity for S.)

We write $I \leq R$ to mean that I is a two-sided ideal of R. Given any ideal I of a graded ring R, we let I_h denote the subideal generated by the homogeneous elements in I, or in other words I_h is the largest homogeneous ideal of R contained in I.

For each element r in an S-graded ring R, let $S_r = \mathrm{supp}_S(r) \subseteq S$ denote the support of r. Given $r \in R$ and $s \in S_r$, we write r_s for the s-component of r. Thus $r = \sum_{s\in S_r} r_s$ with $r_s \in R_s$.

2. Semigroups with unique products

A common technique when showing homogeneity for ideals is working with minimal length elements. If $I \leq R$ (or more generally, if I is just a subset of R), we say that $0 \neq a \in I$ is *of minimal length* (in I)

if $|S_a|$ is minimal in $\{|S_b| \ : \ 0 \neq b \in I\}$. The following lemma collects some of the most fundamental information about elements of minimal length, and slightly generalizes [9, Lemma 2.1].

Lemma 2.1. *Let S be a right cancellative semigroup, let R be an S-graded ring, let I be a right ideal in R, and let $b \in R$ be homogeneous.*

If $a \in I$ is of minimal length, then the following are equivalent:

(1) $ab = 0$.
(2) $a_p b = 0$ *for each* $p \in S_a$.
(3) $a_p b = 0$ *for some* $p \in S_a$.

Proof. (1) $\Rightarrow$ (2): Let $\mathrm{supp}(b) = \{q\}$. By right cancellativity the elements pq, as p runs over S_a, are distinct. Thus $a_p b = 0$ for each $p \in S_a$.

(2) $\Rightarrow$ (3): Tautological weakening.

(3) $\Rightarrow$ (1): Assuming $a_p b = 0$ for some $p \in S_a$, then $ab \in I$ has smaller length than $a \in I$, so by minimality $ab = 0$. □

Note that the lemma is true in much greater generality, as we never used the fact that R is an associative ring. We could have taken S to be a right cancellative magma, and R to be an S-graded (possibly non-associative, non-unital, and even non-left-distributive) ring. Further, not only is there a natural left-right dual version of this lemma, there is also a two-sided version which we record as the following corollary.

Corollary 2.2. *Let S be a left and right cancellative semigroup, let R be an S-graded ring, let I be an ideal of R, and let $b, c \in R$ be homogeneous. If $a \in I$ is of minimal length, then the following are equivalent:*

(1) $bac = 0$.
(2) $ba_p c = 0$ *for each* $p \in S_a$.
(3) $ba_p c = 0$ *for some* $p \in S_a$.

Proof. Since I is a left ideal, we have $ba \in I$. From left cancellativity, the homogeneous components of ba are ba_p for each $p \in S_a$. Thus either all of these components are zero, or $ba \in I$ has minimal length. In the first case, all three conditions hold. In the second case, we can replace a with ba, and the corollary follows by applying Lemma 2.1. □

We would like to remove the homogeneity assumption on b in Lemma 2.1. It is not difficult to show that such a generalization requires a stronger assumption on the semigroup S. One option is to use the following condition.

Definition 2.3. A semigroup S is called a *u.p.-semigroup* (short for unique product semigroup) if for any finite nonempty subsets $A, B \subseteq S$, there is an element $s \in AB$ that has a unique representation in the form $s = ab$ for some $a \in A$ and $b \in B$.

The u.p.-(semi)groups show up in many different contexts, and generalize the concept of ordered groups. For instance, it is easy to show that if R is a domain and S is a u.p.-semigroup, then $R[S]$ has no nonzero zero-divisors. (Thus, this is a first approximation to a solution of Kaplansky's zero-divisor conjecture for group rings.) Quite recently it has been claimed that the universal group of the group of symmetries is a u.p.-group [2], and other applications of u.p.-groups continue to appear in geometric group theory.

The unique product condition is useful because it replaces the usual "leading term" arguments normally used when dealing with ordered structures. The u.p.-semigroups are cancellative and torsion-free (where by *torsion-free* we mean that if $s, t \in S$ are distinct and commute, then $s^n \neq t^n$ for all $n \geq 1$), and the converse holds for commutative semigroups. In fact, in the commutative setting, the u.p.-semigroup hypothesis is equivalent to the existence of a total strict semigroup ordering. We will make use of these facts later, so we record them here, along with a short proof.

Proposition 2.4. *Let S be a semigroup, and consider the following conditions:*

(1) *S is totally, strictly orderable.*
(2) *S is a u.p.-semigroup.*
(3) *S is cancellative and torsion-free.*

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold and are irreversible in general, but the converses hold if S is commutative.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are straightforward, although see Proposition 3.2 below for a generalization of the proof that u.p.-semigroups are torsion-free. The irreversibility of the first implication is shown in [17], and the second in [25] (even for groups). In the commutative case, we can embed S in its fraction group, and (3) $\Rightarrow$ (1) is then a classical theorem of Levi (see [15, Theorem 6.31] for a proof). □

For more information on u.p.-semigroups we direct the reader to [21]. Many authors focus on u.p.-monoids (rather than semigroups) but this causes no real problems because adjoining an identity element to any u.p.-semigroup S without an identity results in a u.p.-monoid M by [13, Lemma 4.2(i)]; and of course any S-graded ring is an M-graded ring with zero identity component. We are now able to generalize Lemma 2.1.

Lemma 2.5. *Let S be a u.p.-semigroup, let R be an S-graded ring, let I be a right ideal in R, and let $b \in R$ be arbitrary. If $a \in I$ is of minimal length, then the following are equivalent:*

(1) *$ab = 0$.*
(2) *$a_p b_q = 0$ for each $p \in S_a$ and each $q \in S_b$.*

Proof. (2) $\Rightarrow$ (1) is easy, so it remains to show that (1) $\Rightarrow$ (2). We induct on $|S_b|$. If $|S_b| = 0$, then both conditions automatically hold, while if $|S_b| = 1$ the implication holds by Lemma 2.1, since u.p.-semigroups are cancellative. Thus, we may assume $|S_b| \geq 2$ and that the implication holds true for elements with smaller support.

Assume $ab = 0$. Applying the unique product property to the two sets S_a and S_b, it follows that there exist $p_0 \in S_a$ and $q_0 \in S_b$ with $a_{p_0} b_{q_0} = 0$. From Lemma 2.1 we see that $ab_{q_0} = 0$, and thus $a(b - b_{q_0}) = 0$. Now the induction hypothesis applies since $|S_{b-b_{q_0}}| < |S_b|$, and hence condition (2) follows. □

It seems quite difficult to obtain this same result with a sandwiched between two non-homogeneous elements. By assuming even more restrictive semigroup properties, these issues can be circumvented.

Lemma 2.6. *Let S be a cancellative, torsion-free, commutative semigroup, let R be an S-graded ring, let I be an ideal of R, and let $b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_k \in R$ be arbitrary elements (where $k \geq 1$). If $a \in I$ is of minimal length and*

$$b_1ac_1 + b_2ac_2 + \cdots + b_kac_k = 0,$$

then

$$b_1a_pc_1 + b_2a_pc_2 + \cdots + b_ka_pc_k = 0 \text{ for each } p \in S_a.$$

Proof. For each $q \in S$, consider the element

$$x(q) := \sum_{i=1}^{k} \sum_{\substack{r,s \in S \\ rs=q}} (b_i)_r a (c_i)_s \in I.$$

As S is commutative and cancellative, we see that the homogeneous components of $x(q)$ are exactly

$$x(q)_{pq} = \sum_{i=1}^{k} \sum_{\substack{r,s\in S\\ rs=q}} (b_i)_r a_p (c_i)_s$$

for each $p \in S_a$, and so $x(q)$ is an element of minimal length in I or zero. In particular, if $x(q)_{pq} = 0$ for some $p \in S_a$, then $x(q)_{pq} = 0$ for all p.

We can fix a strict, total semigroup ordering $<$ on S by Proposition 2.4. Let $q_1 < q_2 < \ldots < q_n$ be the elements in $\bigcup_{i=1}^{k} S_{b_i} \cdot S_{c_i}$. Similarly, let $p_1 < p_2 < \ldots < p_m$ be the elements in S_a. We have

$$\sum_{j=1}^{n} x(q_j) = \sum_{i=1}^{k} b_i a c_i = 0. \tag{2.7}$$

The p_1q_1 component of this sum is just $x(q_1)_{p_1q_1}$, and so equals zero. Therefore $x(q_1)_{pq_1} = 0$ for all $p \in S$. This implies $x(q_1) = 0$, and so we can remove that term from the sum (2.7) to get $\sum_{j=2}^{n} x(q_j) = 0$. Repeating this process recursively, we obtain $x(q_j)_{pq_j} = 0$ for each $p \in S_a$ and each j. Hence,

$$\sum_{i=1}^{k} b_i a_p c_i = \sum_{i=1}^{k} \sum_{q\in S} \sum_{\substack{r,s\in S\\ rs=q}} (b_i)_r a_p (c_i)_s = \sum_{q\in S} x(q)_{pq} = \sum_{j=1}^{n} x(q_j)_{pq_j} = 0. \qquad \square$$

Remark 2.8. (1) By passing to the monoid extension of S if it is not already a monoid (noting that it remains cancellative, torsion-free, and commutative), the previous result remains true, with only trivial changes to the proof, if we remove any of the b_i or c_i from the statement of the lemma. Thus, for instance, if $b_1a + ac_2 + b_3ac_3 + a = 0$ then $b_1a_p + a_pc_2 + b_3a_pc_3 + a_p = 0$.

(2) This lemma can also easily be adapted to give an alternate proof that the upper nilradical is homogeneous in an S-graded ring, where S is cancellative, torsion-free, and commutative.

We now return to the u.p.-semigroup hypothesis. While we've been unable to prove an analog of Lemma 2.6 in this case, there still is a natural way to combine Lemmas 2.1 and 2.5.

Proposition 2.9. *Let S be a u.p.-semigroup, let R be an S-graded ring, let I be an ideal of R, and let $b_0, b_1, \ldots, b_n \in R$ be homogeneous (for some $n \geq 1$). If $a_1, \ldots, a_n \in I$ are of minimal length, then the following are equivalent:*

1. $b_0a_1b_1a_2b_2 \cdots b_{n-1}a_nb_n = 0$.
2. $b_0(a_1)_{p_1}b_1(a_2)_{p_2}b_2 \cdots b_{n-1}(a_n)_{p_n}b_n = 0$ *for each* $p_i \in S_{a_i}$ *(for every* $i \geq 1$*).*
3. $b_0(a_1)_{p_1}b_1(a_2)_{p_2}b_2 \cdots b_{n-1}(a_n)_{p_n}b_n = 0$ *for some* $p_i \in S_{a_i}$ *(for every* $i \geq 1$*).*

Proof. Clearly (2) implies both (1) and (3). Further, (3) $\Rightarrow$ (2) by recursively applying the same implication from Corollary 2.2. So it just remains to show (1) $\Rightarrow$ (2). We work by induction on n, with the case $n = 1$ done by Corollary 2.2. So now assume $n \geq 2$, and that the implication holds for smaller values of n.

Note that $b_0a_1 \in I$ is of minimal length or zero. Letting $c := b_1a_2b_2 \cdots b_{n-1}a_nb_n$, then by Lemma 2.5 we have $b_0(a_1)_{p_1}c_s$ for all $p_1 \in S_{a_1}$ and all $s \in S_c$. In particular, $b_0(a_1)_{p_1}b_1a_2b_2 \cdots b_{n-1}a_nb_n = 0$ for each $p_1 \in S_{a_1}$. The induction hypothesis now applies to give us the condition in (2). $\square$

Remark 2.10. The proposition is also true if we remove any subcollection of the b_i's from the product (whether or not R is unital). In particular, if $a^n = 0$ with $a \in I$ of minimal length, then $a_p^n = 0$ for each $p \in S_a$.

The goal in the next section will be to use these results to prove that many radical properties are homogeneous in semigroup graded rings, as long as a sufficiently strong semigroup hypothesis is posited.

3. Prime and nil radicals

It usually comes as a surprise to those learning noncommutative ring theory that the notion of a prime ideal differs from that given in the commutative case. However, such ideals still have a place in noncommutative rings. Following the literature, an ideal $I \leq R$ is *completely prime* if R/I is a domain; equivalently, for all $a, b \in R$, if $ab \in I$ then $a \in I$ or $b \in I$.

In analogy with the prime radical, the *generalized nilradical*, written $N_g(R)$, is defined to be the intersection of all completely prime ideals in R. A celebrated theorem of Andrunakievich and Ryabukhin (see Theorem 12.7 and the paragraphs following in [15]) is that R is reduced if and only if $N_g(R) = 0$. Thus, another way to think about the generalized nilradical is that it is the smallest ideal $I \leq R$ for which R/I is reduced. This means that for some nonzero unital rings we may have $N_g(R) = R$, such as for matrix rings over fields, which is one reason that this radical has not enjoyed much popularity.

Before proving our main theorem about the generalized nilradical, we need to introduce a new type of semigroup. We say that a semigroup S is a *one set unique product* semigroup (or, more concisely, o.s.u.p.-semigroup) if for any finite non-empty set $A \subseteq S$, there is an element $s \in A^2$ which can be written uniquely as a product $s = s_1 s_2$ with $s_1, s_2 \in A$. By [21, Remark 7 on page 122], S is a u.p.-semigroup if and only if S is cancellative and an o.s.u.p.-semigroup. (The proof is based on ideas of Strojnowski [27], in the group case.) Without the cancellativity condition, there is no reverse implication as the following example demonstrates.

Example 3.1. *There exists a commutative monoid which is an o.s.u.p.-semigroup and which is not cancellative.*

Construction. Let S be the commutative monoid generated the letters s, t, u, subject to the relation $su = st$. Thus, an arbitrary element $x \in S$ is of one of two reduced forms: $x = s^m t^n$ for some $m \geq 1$ and $n \geq 0$, or $x = t^m u^n$ for some $m, n \geq 0$. Since the single relation preserves lengths of words and the number of copies of s in a word, these are also well-defined notions for arbitrary elements of S.

Clearly, S is not cancellative, so it only remains to check the o.s.u.p. condition. Let $A \subseteq S$ be finite and non-empty. Let A' be the set of elements of A with maximal length, and subject to that constraint they also have a maximal number of s's in them. No element of $A \cdot (A \setminus A') \cup (A \setminus A') \cdot A$ can equal an element from $A'A'$. Thus, after replacing A by A', we may as well assume that each element of A has the same length, and the same number of s's.

If that number of s's is positive, then A is a singleton set, and thus clearly has a unique product. If that number of s's is zero, then A is a subset of $\{t^k, t^{k-1}u, t^{k-2}u^2, \ldots, tu^{k-1}, u^k\}$ for some $k \geq 0$. Now, taking $x \in A$ to be the element with the maximal number of t's, we see that x^2 is a unique product. □

The fact that u.p.-semigroups are torsion-free is also true for o.s.u.p.-semigroups.

Proposition 3.2. *If S is an o.s.u.p.-semigroup, then S is torsion-free.*

Proof. Let $s, t \in S$ with $st = ts$ and $s \neq t$. Working contrapositively, assume that $s^k = t^k$ for some (minimal) $k \geq 2$. Consider the finite set $A = \{s^{k-1}, s^{k-2}t, \ldots, st^{k-2}, t^{k-1}\}$. While the elements we have listed are not necessarily distinct, the two elements at the ends are different, by minimality on k.

If $a, b \in A$ are distinct, then $ab = ba$ (since s and t commute) so that product is not unique. Thus, any possible unique product from A^2 would be of the form $x = (s^i t^j)(s^i t^j)$ with $i + j = k - 1$.

Case 1: If $i = j = (k-1)/2$, then $x = s^{k-1}t^{k-1} = t^{k-1}s^{k-1}$, is a product from A^2 in at least two ways, since $s^{k-1} \neq t^{k-1}$.

Case 2: If $i > j$, then $2i \geq k$ and so $x = s^{2i}t^{2j} = s^{k-1}(s^{2i-k+1}t^{2j}) = t^{k-1}(s^{2i-k}t^{2j+1})$ is again a product from A^2 in at least two ways.

Case 3: If $i < j$, this is similar to case 2.

Thus, in every case A^2 has no unique product. □

Surprisingly, the converse of the previous proposition holds when S is commutative. Any proof will require methods somewhat orthogonal to those commonly used to verify the equivalence (in the commutative setting) of conditions (2) and (3) in Lemma 2.6, since classically those schemes pass through the orderability condition (1), which is unavailable without cancellativity. Thus, the proof below is somewhat involved.

Proposition 3.3. *If S is a torsion-free, commutative semigroup, then S is an o.s.u.p.-semigroup.*

Proof. We work contrapositively, so assume $A \subseteq S$ is a finite, non-empty set, with no unique product in A^2. To begin, we make a few simplifications. First, we may assume $|A|$ is minimal, subject to the conditions already assumed. Write $A = \{a_1, a_2, \ldots, a_n\}$, and note that $n \geq 2$.

Given $1 \leq i \leq n$, there exist $1 \leq i(1) \leq i(2) \leq n$ (depending on i) with $a_i^2 = a_{i(1)}a_{i(2)}$ and $(i(1), i(2)) \neq (i, i)$. If $i(1) = i(2)$, then we have $a_i^2 = a_{i(1)}^2$ which demonstrates that S is not torsion-free. So, our second simplification is that we may assume $i(1) \neq i(2)$, and we fix such a choice for each integer i in the range $1 \leq i \leq n$.

Let T be the subsemigroup of S generated by the elements in A; it suffices to show that T is not torsion-free. Each element $t \in T$ can be written (in possibly many ways) as a product $\prod_{i=1}^{n} a_i^{m_i}$ where each $m_i \geq 0$ and not all exponents are zero. Let us say that a *valid reduction* associated to $i > 1$ is when we replace an instance of a_i^2 with $a_{i(1)}a_{i(2)}$, in such a product. Note that a valid reduction never decreases the number of a_1's in a product representation (since we only use $i > 1$), and it also never changes the total length of the product (where the length is $\sum_{i=1}^{n} m_i$).

We claim that any product representation will, after enough valid reductions, have no further valid reductions available. To see this, suppose by way of contradiction (using König's tree lemma, see [3] for a historical overview of the lemma) that there is an infinite sequence of valid reductions available for some product. After passing to the tail of the sequence, we may assume no additional a_1's arise. Since the sequence is infinite, there exists some j_1 where we use the valid reduction $a_{j_1}^2 = a_{j_1(1)}a_{j_1(2)}$ infinitely often. Putting $j_2 = j_1(1)$ and $j_3 = j_1(2)$, we thus see that the exponents on a_{j_2} and a_{j_3} are increased infinitely often; but as the total length never changes, those exponents must also be lowered infinitely often. Hence in our sequence of valid reductions, we must use the valid reductions associated to j_2 and j_3 infinitely often. We can recursively repeat this process, for instance at the next step putting $j_4 = j_2(1)$, $j_5 = j_2(2)$, $j_6 = j_3(1)$, and $j_7 = j_3(2)$, and at each stage all of these new j's are involved in infinitely many valid reductions.

Now let $A' = \{a_{j_p} \ : \ p \geq 1\}$. We have $a_1 \notin A'$ since a_1 stopped arising in the sequence of valid reductions, and therefore A' is a proper, nonempty subset of A. Moreover, it is easy to see that A' has no unique product, as follows. If $a_j \in A'$, then the "squared" element $a_j^2 \in A'^2$ is not a unique product since it equals $a_{j(1)}a_{j(2)}$, and $a_{j(1)}, a_{j(2)} \in A'$ by the recursion argument of the previous paragraph. On the other hand, any non-"squared" element of A'^2 is automatically a non-unique product since S is commutative, and hence $a_j a_k = a_k a_j$ are two different product representations. This contradicts the minimality hypothesis on $|A|$, and thus we have proven there are only finitely many valid reductions that can be made to any product.

Given a product $\prod_{i=1}^{n} a_i^{m_i}$, let $\prod_{i=1}^{n} a_i^{m_i'}$ be any product representation arising from the first product using a maximal sequence of valid reductions. (It isn't hard to show that the exponents m_i' are uniquely determined, but we won't need this fact.) In particular, note that $m_i' \in \{0, 1\}$ for $i > 1$. We call $r := \prod_{i=2}^{n} a_i^{m_i'}$ the *remainder* of the sequence of valid reductions. There are only finitely many possible remainders since $0 \leq m_i' \leq 1$ for $2 \leq i \leq n$.

Now, consider the sequence of products, $a_2, a_2^2, a_2^3, \ldots$. Infinitely many of these products must share a remainder, and so in particular we have

$$a_2^p = a_1^{p'} r \qquad \text{and} \qquad a_2^{p+q} = a_1^{p'+q} r$$

for some $p, q \geq 1$, and some remainder r. Thus

$$a_1^q a_2^p = a_1^q(a_1^{p'} r) = a_1^{p'+q} r = a_2^{p+q}. \tag{3.4}$$

By symmetry, repeating the entire argument to this point with the roles of a_1 and a_2 interchanged, we must have

$$a_1^r a_2^s = a_1^{r+s} \tag{3.5}$$

for some $r, s \geq 1$. Using (3.5) repeatedly, we have for each integer $k \geq 1$ that

$$a_1^r a_2^{ks} = (a_1^r a_2^s) a_2^{(k-1)s} = a_1^{r+s} a_2^{(k-1)s} = \cdots = a_1^{r+ks}.$$

So, we may as well assume that $s > p$. Multiplying (3.5) on both sides by an appropriate power of a_1, we may also assume q divides r. But then applying (3.4) repeatedly to (3.5), we may as well assume that $r = 0$. Hence $a_1^s = a_2^s$, demonstrating that S is not torsion-free. □

The following lemma was motivated by the statement of [9, Lemma 1.1], and allows us to connect the o.s.u.p. condition to completely prime ideals.

Lemma 3.6. *Let S be an o.s.u.p.-semigroup, and let R be an S-graded, non-reduced ring. If P is a completely prime ideal, then P contains a nonzero homogeneous element.*

Proof. As R is not reduced, fix $0 \neq a \in R$ with $a^2 = 0$. From the o.s.u.p.-semigroup hypothesis, we can fix $s, t \in S_a$ such that $a_s a_t = 0$. Since P is completely prime, either $a_s \in P$ or $a_t \in P$, which yields the conclusion. □

We are now ready to prove the homogeneity of the generalized nilradical, when graded by o.s.u.p.-semigroups.

Theorem 3.7. *If S is an o.s.u.p.-semigroup and R be an S-graded ring, then $N_g(R)$ is homogeneous.*

Proof. Let P be a completely prime ideal of R. We wish to show that the homogeneous part of P, denoted P_h, is an intersection of completely prime ideals. Equivalently, by the result of Andrunakievich and Ryabukhin cited above, we wish to show that R/P_h is reduced.

Assume to the contrary that R/P_h is not reduced. This is still naturally an S-graded ring, since P_h is homogeneous. Applying Lemma 3.6 to this factor ring, noting that P/P_h is still completely prime, we must have a nonzero homogeneous element in P/P_h which directly contradicts the definition of P_h.

Therefore, $P_h = \bigcap Q$ is an intersection of completely prime ideals Q, for each completely prime ideal P. Hence

$$N_g(R) = \bigcap_{P \text{ comp. prime}} P \supseteq \bigcap_{P \text{ comp. prime}} P_h \supseteq \bigcap_{P \text{ comp. prime}} \bigcap_{\substack{Q \supseteq P_h \\ Q \text{ comp. prime}}} Q \supseteq N_g(R)$$

where the last inclusion holds since $N_g(R)$ is the intersection of *every* completely prime ideal. Therefore, equality holds throughout and so $N_g(R) = \bigcap P_h$ is homogeneous. □

We mentioned in the Introduction that for many radicals $\mathfrak{F}$, if $\mathfrak{F}(R)$ is homogeneous for all S-graded rings R, then S must be cancellative. The most general result along those lines appears to be Theorem 6 in [18]. Looking at the hypotheses of that theorem, we see that the only condition which fails to hold for the generalized nilradical is *precisely* the defining characteristic for $R/N_g(R)$; that it be non-reduced!

There is a similar range of radicals where S is required to be torsion-free; and the generalized nilradical fits in this range. As a consequence, we can characterize precisely what happens in the case of commutative gradings.

Corollary 3.8. *Let S be a commutative semigroup. The generalized nilradical is homogeneous for all S-graded rings if and only if S is torsion-free.*

Proof. The forward direction is a consequence of [18, Proposition 11], and the backwards direction follows immediately from Theorem 3.7 in conjunction with Proposition 3.3. □

There are other natural generalizations of the prime radical. For instance, Handelman and Lawrence [6] and others have made the following definitions. An element $x \in R$ has a *right insulator* if there is a finite subset $F \subseteq R$ (possibly depending on x) such that $\operatorname{ann}_r(xF) = 0$; similarly, a finite subset $F \subseteq R$ is said to be a *right insulator* if it has zero right annihilator. A ring is *right strongly prime* if each nonzero element has a right insulator. Equivalently, every nonzero ideal contains an insulator [23, Proposition 1]. An ideal $I \leq R$ is said to be *right strongly prime* if R/I is right strongly prime as a ring. (Note that R is always right strongly prime *as an ideal in itself*, but may not be right strongly prime as a ring. However, this small ambiguity in terminology should cause no problems.)

We define the *right strongly prime radical* of a ring as the intersection of all right strongly prime ideals of the ring; see [5] for more information. We will denote this radical in this paper by $s_r(R)$. The right strongly prime radical contains the Levitzki radical, first proved in [12, Theorem 3.3], but also see [4, Proposition 4.11.16].

Since we will consistently work on the right and not the left, hereafter we will drop the word "right" in reference to insulators and strongly prime rings, ideals, and radicals. However, the reader should be aware that each of the results we prove will have a "left" analog.

Proposition 3.9. *If S is a u.p.-semigroup and R is an S-graded ring, then the homogeneous component of any strongly prime ideal is strongly prime.*

Proof. Let P be any strongly prime ideal of R. Passing to R/P_h, we may as well assume $P_h = 0$ and show that R is strongly prime. If $R = 0$ we are done, so we may assume $R \neq 0$. Letting I be a nonzero ideal, it suffices to show that I contains an insulator. Fix $0 \neq x \in I$ having minimal length and fix some $s \in S_x$. Note that $x_s \notin P$ since $P_h = 0$. As R/P is strongly prime, fix an insulator $\overline{F}$ for $x_s + P$. We may as well assume $\overline{F}$ is the image of a finite, homogeneous set $F \subseteq R$, after replacing the elements of F by their homogeneous components if necessary. Define $G = FxF$ which is a finite subset of I.

We now claim that G is an insulator in I. Indeed, suppose $Gy = 0$ for some $y \in R$; in other words, for all $f, f' \in F$ we have $fxf'y = 0$. By Lemma 2.5, we have $fx_sf'y_q = 0$ for all $q \in S_y$, and thus $x_sFx_sFy_q = 0$. From the definition of F, this implies $x_sFy_q \subseteq P$. But x_sFy_q consists of homogeneous elements, and hence $x_sFy_q \subseteq P_h = 0$. Applying the definition of F once more, we get $y_q \in P_h = 0$. As this holds for all $q \in S_y$, we must have $y = 0$ as desired. □

The following theorem is a generalization of [19, Corollary 2], where in that paper S is taken to be an ordered group.

Theorem 3.10. *If S is a u.p.-semigroup and R is an S-graded ring, then $s_r(R)$ is homogeneous.*

Proof. Let $\mathcal{S}$ be the set of strongly prime ideals of R, and let $\mathcal{T}$ be the subset of those ideals which are homogeneous. On one hand we have $\mathcal{T} \subseteq \mathcal{S}$, and so

$$s_r(R) = \bigcap_{I \in \mathcal{S}} I \subseteq \bigcap_{I \in \mathcal{T}} I.$$

On the other hand,

$$\bigcap_{I \in \mathcal{S}} I \supseteq \bigcap_{I \in \mathcal{S}} I_h = \bigcap_{I \in \mathcal{T}} I$$

where the last equality comes from Proposition 3.9. Putting these containments together proves that $s_r(R)$ is homogeneous, being an intersection of homogeneous ideals. □

We finish this section by studying one more radical, which generalizes the (right) strongly prime radical. Following, but slightly modifying terminology introduced by Olson [22], let us say that a finite subset $F \subseteq R$ is a *uniform insulator* if $xFy = 0$ entails $x = 0$ or $y = 0$ for any $x, y \in R$. A ring is *uniformly strongly prime* if it contains a uniform insulator, and an ideal $I \leq R$ is *uniformly strongly prime* if R/I is a uniformly strongly prime ring. Finally the *uniformly strongly prime radical* of a ring R is the intersection of all uniformly strongly prime ideals.

The paper [22] contains many more facts about this radical, including its relationship to the strongly prime radical and other well-known radicals. Our current goal is to prove an analog of Proposition 3.9 for uniformly strongly prime ideal.

Proposition 3.11. *If S is a u.p.-semigroup and R is an S-graded ring, then the homogeneous component of any uniformly strongly prime ideal is uniformly strongly prime.*

Proof. Let I be any uniformly strongly prime ideal of R. Passing to R/I_h, we may as well assume $I_h = 0$ and show that R is uniformly strongly prime. As R/I is uniformly strongly prime, fix a uniform insulator $\overline{F}$. We may as well assume $\overline{F}$ is the image of a finite, homogeneous set $F \subseteq R$.

Consider, for a moment, what happens if $aFb = 0$ for some $a, b \in R$ with b nonzero and homogeneous. Since S is right cancellative, we must have $a_s Fb = 0$ for all $s \in S$ by the proof of (1)⇒(2) in Lemma 2.1. Working modulo I we have $(a_s + I)\overline{F}(b + I) = 0$. As $\overline{F}$ is a uniform insulator, this implies $a_s \in I$ or $b \in I$. However $I_h = 0$ and $b \neq 0$, so $a_s = 0$ for all $s \in S$, and in particular $a = 0$. Of course, a similar result holds if we instead assume a is nonzero and homogeneous. Thus, if $a, b \in R$ are nonzero and $aFb = 0$, then neither a nor b is homogeneous. We will use this fact repeatedly below.

If $R = 0$ we are done, so assume $R \neq 0$. Fix nonzero homogeneous elements $x, y \in R$ such that $F' = \{f \in F \,:\, xfy = 0\}$ has as large cardinality as possible. By the argument above we know that F' is a proper subset of F. Let $G := FxFyF$. We claim that this set is our uniform insulator. To prove the claim suppose, by way of contradiction, that $uFxFyFv = 0$ for some *nonzero* $u, v \in R$.

By the argument from two paragraphs previous, we have $uFx, yFv \neq 0$. Fix $f_1, f_2 \in F$ with $uf_1x \neq 0$ and $yf_2v \neq 0$. Further, fix $f_3 \in F \setminus F'$ (recalling that F' is a proper subset of F). As $uFxFyFv = 0$, we have $uf_1xf_3yf_2v = 0$. Therefore, since S is a u.p.-semigroup there exist $s \in S$ and $t \in S$ such that $u_s f_1 x \neq 0$, $yf_2v_t \neq 0$, and $u_s f_1 x f_3 y f_2 v_t = 0$. This contradicts the maximality hypothesis on x, y since $u_s f_1 x(F' \cup \{f_3\})yf_2v_t = 0$. □

Theorem 3.12. *If S is a u.p.-semigroup and R is an S-graded ring, then $u(R)$ is homogeneous.*

Proof. This follows from Proposition 3.11 in exactly the same way Theorem 3.10 followed from Proposition 3.9. □

As a corollary to Theorems 3.10 and 3.12, we have the following.

Corollary 3.13. *Let S be a commutative semigroup. The ideal $s_r(R)$ (respectively $u(R)$) is homogeneous in every S-graded ring R if and only if S is cancellative and torsion-free.*

Proof. As a cancellative, torsion-free, and commutative semigroup is a u.p.-semigroup by Proposition 2.4, sufficiency follows by Theorems 3.10 and 3.12. Necessity comes from [18, Corollary 7(3), Proposition 11] and [4, Proposition 4.11.21]. □

4. The upper nilradical of group and semigroup graded rings

We now turn our attention to the upper nilradical of graded rings. Only as recently as 2014 was it proven that this radical is homogeneous for $\mathbb{Z}$-graded rings [26]. We now generalize Smoktunowicz's result.

Theorem 4.1. *If G is a torsion-free, nilpotent group and R is a G-graded ring, then* $\mathrm{Nil}^*(R)$ *is homogeneous.*

Proof. We work by induction on the index of nilpotence k of the group G. If $k = 0$ then $G = 1$, and the result is trivially true. If $k = 1$ then G is abelian, and this case follows from [18, Theorem 13]. So we may assume $k \geq 2$ and that the result holds for smaller indices. Also, if $I \leq R$ is the largest homogeneous ideal contained in the upper nilradical, after passing to R/I we may as well assume that $I = 0$ (so that R has no nonzero nil homogeneous ideal).

Assume by way of contradiction $\mathrm{Nil}^*(R)$ is not homogeneous. We can fix $a \in \mathrm{Nil}^*(R)$ of minimal length $\ell \geq 2$. Note that $G/Z(G)$ has index $k-1$ and is torsion-free by [24, Lemma 11.1.3], and R is a $G/Z(G)$-graded ring. Thus $\mathrm{Nil}^*(R)$ is homogeneous in this grading by the induction hypothesis, which immediately implies (by minimality of ℓ) that the components of a all lie in the same coset modulo $Z(G)$. Thus,

$$sgt = tgs \text{ for all } s, t \in \mathrm{supp}_G(a) \text{ and all } g \in G. \tag{4.2}$$

Using this key fact, we now repeat the argument given by Smoktunowicz in [26].

Fix $s \in \mathrm{supp}_G(a)$. If $r \in R$ is homogeneous, then by (4.2) we have that $ara_s - a_s ra \in \mathrm{Nil}^*(R)$ has less than ℓ nonzero homogeneous components, and thus must be zero. Therefore $ara_s = a_s ra$ for all homogeneous $r \in R$, and hence *a fortiori* we have

$$ara_s = a_s ra \text{ for all } r \in R. \tag{4.3}$$

Since $a_s \notin \mathrm{Nil}^*(R)$, there exist elements $p_i, q_i \in R$ such that $\alpha := \sum_{i=1}^n p_i a_s q_i$ is not nilpotent. As $a \in \mathrm{Nil}^*(R)$ we may fix $m \geq 1$ such that $(\sum_{i=1}^n p_i a q_i)^m = 0$. We then find, using (4.3), that

$$(\alpha^m RaR)^m \subseteq \alpha^m (RaR)^m = \left(\sum_{i=1}^n p_i a_s q_i\right)^m (RaR)^m \subseteq \left(\sum_{i=1}^n p_i a q_i\right)^m (Ra_sR)^m = 0.$$

Hence, if P is a minimal prime ideal then either $\alpha^m \in P$ or $a \in P$. However, the group G is a totally orderable semigroup by [24, Lemma 13.1.6], and hence by [9, Theorem 1.2] we know P is homogeneous. Thus, if $a \in P$ then $a_s \in P$, and consequently $\alpha^m \in P$. Therefore, in any case α^m is in all minimal prime ideals, so α^m belongs to the prime radical and is therefore nilpotent, contradicting the fact α is not nilpotent. □

Corollary 4.4. *If S is a subsemigroup of a torsion-free, nilpotent group G and R is an S-graded ring, then* $\mathrm{Nil}^*(R)$ *is homogeneous.*

Proof. We can give R a G-grading by setting $R_g = 0$ whenever $g \in G \setminus S$. The result now follows from Theorem 4.1. □

According to Theorems 7.3 and 10.6 and Proposition 10.9 from [21], the semigroups described in Corollary 4.4 are exactly the cancellative, torsion-free, weakly nilpotent semigroups, and they are u.p.-semigroups. Thus, we end this section with the following:

Question 4.5. If S is a u.p.-semigroup and R is an S-graded ring, is $\mathrm{Nil}^*(R)$ homogeneous?

Of course, if $\mathrm{Nil}^*(R)$ is locally nilpotent, then the answer is yes since the Levitzki radical will be homogeneous in that case.

5. The Wedderburn radical

The Wedderburn radical of a ring R, denoted $W(R)$, is the sum of all nilpotent ideals in R, or equivalently is the set of elements which generate nilpotent ideals. This is not a Kurosh-Amitsur radical as $W(R/W(R))$ may be nonzero, and so we define by transfinite recursion on ordinals α, the following "higher Wedderburn radicals":

$$W_\alpha(R) = \begin{cases} \{r \in R \ : \ r + W_\beta(R) \in W(R/W_\beta(R))\} & \text{if } \alpha \text{ is the successor of } \beta \\ \bigcup_{\beta < \alpha} W_\beta(R) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

This chain of ideals stabilizes to the prime radical $P(R)$, by a theorem of Levitzki (see [16, Exercise 10.11] for a proof).

How do the homogeneous components of these higher Wedderburn radical behave? We can naively define the "higher homogeneous Wedderburn radicals" for a graded ring R in a similar way, by the recursive formula

$$W_{h,\alpha}(R) = \begin{cases} \{r \in R \;:\; r + W_{h,\beta}(R) \in W(R/W_{h,\beta}(R))_h\} & \text{if } \alpha \text{ is the successor of } \beta \\ \bigcup_{\beta<\alpha} W_{h,\beta}(R) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Do we have the equality $W_\alpha(R)_h = W_{h,\alpha}(R)$ for every ordinal α? Surprisingly, the answer is yes.

Proposition 5.1. *Let S be a semigroup and let R be an S-graded ring. For every ordinal α, we have $W_\alpha(R)_h = W_{h,\alpha}(R)$.*

Proof. We work by transfinite induction, and thus we may assume the statement is true for all ordinals smaller than α. First, consider the case when α is a limit ordinal (including the case when $\alpha = 0$). We then have

$$W_{h,\alpha}(R) = \bigcup_{\beta<\alpha} W_{h,\beta}(R) = \bigcup_{\beta<\alpha} W_\beta(R)_h = \left(\bigcup_{\beta<\alpha} W_\beta(R)\right)_h = W_\alpha(R)_h$$

where the second equality holds from the induction hypothesis.

Second, consider the case when α is a successor of β. To prove that $W_{h,\alpha}(R) = W_\alpha(R)_h$ it suffices to show that the set of homogeneous elements in both ideals are equal, since both ideals are homogeneous. We first show the containment $W_{h,\alpha}(R) \subseteq W_\alpha(R)_h$. Fix a homogeneous $r \in W_{h,\alpha}(R)$. By definition, r generates a nilpotent ideal modulo $W_{h,\beta}(R) = W_\beta(R)_h$, hence also modulo the larger ideal $W_\beta(R)$. Therefore $r \in W_\alpha(R)$.

Conversely, fix a homogeneous element $r \in W_\alpha(R)$. By definition of the higher Wedderburn radicals, the ideal generated by $r + W_\beta(R)$ is nilpotent in the factor ring $R/W_\beta(R)$, say with nilpotence index $k \geq 0$. This means $(r)^k \subseteq W_\beta(R)$. As $(r)^k$ is a homogeneous ideal, $(r)^k \subseteq W_\beta(R)_h = W_{h,\beta}(R)$. Thus $r + W_{h,\beta}(R)$ generates a nilpotent ideal in the factor ring $R/W_{h,\beta}(R)$, and so $r \in W_{h,\alpha}(R)$ by definition of the higher homogeneous Wedderburn radicals, giving us the needed inclusion. □

For rings R graded by totally, strictly ordered semigroups, the Wedderburn radical is homogeneous [18, Proposition 18]. The difficulty in extending this result further arises precisely from the fact that this ideal may "grow" after factoring out its homogeneous component. However, we can use the previous proposition to prove that whenever the homogeneous component of $W(R)$ is zero, then the homogeneous component of $P(R)$ is zero. Equivalently, if $P(R)$ contains nonzero homogeneous elements, then there exists a nonzero homogeneous nilpotent ideal.

Theorem 5.2. *Let S be a semigroup and let R be an S-graded ring. If $W(R)_h = 0$, then $P(R)_h = 0$.*

Proof. As $P(R)$ is the limit of the higher Wedderburn radicals, Proposition 5.1 implies that $P(R)_h$ is the limit of the higher homogeneous Wedderburn radicals. If $W(R)_h = 0$, then these higher radicals stabilized at the zeroth stage, and so $P(R)_h = 0$. □

The anonymous referee has kindly pointed out that this theorem can be proven in a way that avoids Proposition 5.1 as follows.

Alternate proof. A sequence of elements $a_0, a_1, a_2, \ldots \in R$ where $a_{n+1} \in a_n R a_n$ (for each $n \geq 0$) is called an m-sequence. An element $a \in R$ is called strongly nilpotent if each m-sequence starting with $a_0 = a$ is eventually zero. It is a well-known theorem of Levitzki that the prime radical is precisely the set of strongly nilpotent elements (see [16, Exercise 10.17] for a proof).

Fix a_0 to be a nonzero homogeneous element of $P(R)$. For each integer $n \geq 0$, as long as possible recursively fix a homogeneous element $r_n \in R$ such that $a_{n+1} := a_n r_n a_n \neq 0$. Since a_0 is strongly nilpotent, this process must terminate. Thus, for some integer $m \geq 0$ we have $a_m \neq 0$, but $a_m r_m a_m = 0$ for every homogeneous element $r_m \in R$. In this case, a_m is a nonzero homogeneous element which generates a nilpotent ideal (of index at most 3) in R, and hence belongs to $W(R)$. □

We have the following immediate corollaries.

Corollary 5.3. *Let S be a semigroup such that for any S-graded ring the prime radical is homogeneous (for instance, this holds if S is a u.p.-semigroup), and let R be an S-graded ring. If the prime radical is nonzero, then there is a nonzero homogeneous nilpotent ideal.*

Corollary 5.4. *Let S be a semigroup such that any S-graded ring has a homogeneous prime radical, let R be an S-graded ring, and let $I \leq R$ be any ideal with $W(R) \subseteq I \subseteq P(R)$ (for instance, I could be the bounded nilradical, see* [20]*). If I contains no nonzero homogeneous elements, then $I = 0$.*

If R is a graded ring and $P(R)$ is homogeneous, it does not need to be true that $W(R)$ is homogeneous. However, in that case we must have $W(R) \subseteq W_{h,\alpha}(R)$ for some α. It is thus natural to ask whether there is a universal upper-bound on α. The following example negates such a possibility.

Example 5.5. *There exists a semigroup S, such that for each ordinal α there exists an S-graded ring R for which $P(R)$ is homogeneous, but $W(R) \nsubseteq W_{h,\alpha}(R)$.*

Sketch of construction. We follow the construction given in [7, Theorem 4.8], with a few minor changes. Put $S = \langle s, t \ : \ s^2 = st = ts = t^2 \rangle$, and let α be any ordinal. Let F be a field and set

$$R = F\langle b_{n,\beta},\ c_{n,\alpha}\ (n \in \mathbb{Z}_{>0}, \beta \leq \alpha \text{ is an ordinal})\rangle$$

where the letters are subject to the following two types of relations we will describe. Let $R_{\geq\gamma}$ be the subring of R generated by the letters $b_{n,\beta}$ for β in the range $\gamma \leq \beta \leq \alpha$, and by $c_{n,\alpha}$ (for all $n \geq 1$). Our relations are

- Any monomial in $R_{\geq\beta}$ containing $b_{n,\beta}$ and of total degree $> n$ is zero.
- Given any letter x in R, we put

$$c_{n,\alpha}x = b_{n,\alpha}x \quad \text{and} \quad xc_{n,\alpha} = xb_{n,\alpha}.$$

The first set of relations guarantee that $b_{n,\alpha} \notin W_\alpha(R)$, by essentially the same proof as in [7]. However, it is easy to see that each of the letters defining R belongs to the prime radical. Moreover, we claim $r := b_{n,\alpha} - c_{n,\alpha} \in W(R)$. Indeed, $rRr = 0$ from the second set of relations.

We make R an S-graded ring by letting each of the b letters have grade s, while the c letters have grade t. The relations respect this grading. □

While this example gives the most general picture, for certain semigroups S we can get an absolute bound on α.

Proposition 5.6. *If S is a u.p.-semigroup and R is an S-graded ring, then $W(R) \subseteq W_{h,\omega}(R)$.*

Proof. We will prove, by induction, that if $x \in W(R)$ has length ℓ, then $x \in W_{h,\ell}(R)$. The case when $\ell = 1$ is obviously true, and so we assume $\ell \geq 2$ and that the claim is true for smaller lengths. If $x \in W(R) \cap W_{h,\ell-1}(R)$ we are done, so we may assume $\overline{x} \in W(R)/(W(R) \cap W_{h,\ell-1}(R))$ is nonzero. Let $k \geq 2$ be the index of nilpotence for the ideal generated by $\overline{x}$.

From the inductive hypothesis, $\overline{x} \in W(R)/(W(R) \cap W_{h,\ell-1}(R))$ has minimal length, so we may apply Proposition 2.9 and Remark 2.10, which tell us that $\overline{x}_s$ generates a nilpotent ideal of index $\leq k$, for each $s \in S_x$. Therefore $x_s \in W_{h,\ell}(R)$, since $W(R) \cap W_{h,\ell-1}(R) \subseteq W_{h,\ell-1}(R)$, using the definition of the higher homogeneous Wedderburn radicals. But as $s \in S_x$ is arbitrary, we have $x \in W_{h,\ell}(R)$. This completes our induction. □

An easy modification of this argument shows (still assuming S is a u.p.-semigroup) that for each ordinal α, if $x \in W_\alpha(R)$ has length ℓ, then $x \in W_{h,\alpha+\ell-1}(R)$. As a consequence, when α is a limit ordinal we have that $W_\alpha(R)$ is homogeneous. We were unable to show whether or not the hypothesis that α be a limit ordinal is necessary, so we ask:

Question 5.7. If S is a u.p.-semigroup and R is an S-graded ring, is $W(R)$ homogeneous?

As mentioned previously, the answer is positive if S is a strictly, totally ordered semigroup. The argument used in that case needs a semigroup property which is stronger than the unique product hypothesis.

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