INNER INVERSES AND INNER ANNIHILATORS IN RINGS

T. Y. LAM AND PACE P. NIELSEN

Abstract. For any ring element $\alpha \in R$, we study the group of inner annihilators $\mathrm{IAnn}\,(\alpha) = \{p \in R : \alpha\, p\, \alpha = 0\}$ and the set $I(\alpha)$ of inner inverses of α. For any Jacobson pair $\alpha = 1 - ab$ and $\beta = 1 - ba$, the groups $A = \mathrm{IAnn}\,(\alpha)$ and $B = \mathrm{IAnn}\,(\beta)$ are shown to be equipotent, and $A \oplus C$ is shown to be group isomorphic to $B \oplus C$ where $C = \mathrm{Ann}_\ell(\alpha) \oplus \mathrm{Ann}_r(\alpha)$. In the case where α is (von Neumann) regular, we show further that $A \cong B$ as groups. For any Jacobson pair $\{\alpha, \beta\}$, a "new Jacobson map" $\Phi : I(\alpha) \to I(\beta)$ is constructed that is a semigroup homomorphism with respect to the von Neumann product, and preserves units, reflexive inverses and commuting inner inverses. In particular, for any abelian ring R, Φ is a semigroup isomorphism between $I(\alpha)$ and $I(\beta)$. As a byproduct of our methods, we also show that a ring R satisfies internal cancellation iff every Jacobson pair of regular elements are equivalent over R. In particular, the latter property holds for many rings, including semilocal rings, unit-regular rings, strongly π-regular rings, and finite von Neumann algebras.

1. Introduction

Two elements α, β in a ring R are said to form a *Jacobson pair* if there exist elements $a, b \in R$ such that $\alpha = 1 - ab$ and $\beta = 1 - ba$. For such a pair, "Jacobson's Lemma" is the statement that if α is invertible with inverse s, then β is invertible with inverse $1 + bsa$. This leads easily to a proof of the similar fact that if α is regular (in the sense of von Neumann) with inner inverse s, then β is also regular with inner inverse $1 + bsa$. However, the standard "Jacobson map" $s \mapsto 1 + bsa$ from the inner inverses of α to those of β is usually neither injective nor surjective, fails to take units to units (if there are unit inner inverses), or group inverses to group inverses (if those exist).

In this paper, we define for every $\alpha \in R$ the set $\mathrm{IAnn}\,(\alpha) = \{p \in R \,:\, \alpha p \alpha = 0\}$ (consisting of the "inner annihilators" of α). In case α is regular with an inner inverse s, the set of *all* inner inverses of α, denoted by $I(\alpha)$, is given by $s + \mathrm{IAnn}\,(\alpha)$. So we expect $\mathrm{IAnn}\,(\alpha)$ to be a useful object to study in general, even when α is not regular. A search of the literature did not turn up very many results on $\mathrm{IAnn}\,(\alpha)$ and $I(\alpha)$, although some work has been done by Hartwig and Luh [7] on the set of *unit* inner inverses of unit-regular elements in a ring R. It is hoped that, by contributing a number of new results on $\mathrm{IAnn}\,(\alpha)$ and $I(\alpha)$, the present paper will draw interest on these objects from the ring theory community, and from algebraists in general.

In §2, we initiate the study of the abelian group $\mathrm{IAnn}\,(\alpha)$ in a ring R. Various motivating examples are given, with and without assuming that α is regular. In §3, we prove several theorems on ring elements with only a finite number of inner inverses or unit inner inverses that did not seem to have been noted before in the literature. In the second half of §3, we also formally introduce the two Hartwig-Luh self-maps φ and φ' on $I(\alpha)$ for any regular element $\alpha \in R$. These maps will play a substantial role in the subsequent study (in §§6–7) of mappings from $I(\alpha)$ to $I(\beta)$ for any Jacobson pair $\{\alpha, \beta\}$. There has been considerable recent interest on Jacobson pairs, mainly in connection with the study of generalized inverses (e.g. Drazin inverses) in rings; see, e.g. [2], [4], [17], [18], and [21]. Indeed, it was

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this series of papers which aroused our own interest in the structure of the sets $\mathrm{IAnn}(\alpha)$ and $I(\alpha)$, and their relationship with $\mathrm{IAnn}(\beta)$ and $I(\beta)$ in case α and β form a Jacobson pair.

In §4, we make our first analysis of $\mathrm{IAnn}(\alpha)$. By considering its subgroups $\mathrm{Ann}_\ell(\alpha)$ and $\mathrm{Ann}_r(\alpha)$ along with their sum and intersection, we prove the following result over any ring.

Theorem A. *For any Jacobson pair $\{\alpha, \beta\}$, the groups $\mathrm{IAnn}(\alpha)$ and $\mathrm{IAnn}(\beta)$ are always equipotent. Moreover, these two groups become isomorphic after "adding" the direct sum $\mathrm{Ann}_\ell(\alpha) \oplus \mathrm{Ann}_r(\alpha)$.*

In §5, we address the case of *regular* Jacobson pairs $\{\alpha, \beta\}$ (where α, and hence β, is regular). In this more tractable case, we prove the following sharper version of Theorem A, again over any ring.

Theorem B. *If $\{\alpha, \beta\}$ is a regular Jacobson pair, then $\mathrm{IAnn}(\alpha) \cong \mathrm{IAnn}(\beta)$ as groups.*

While we are able to prove the existence of a group isomorphism in the setting of Theorem B, it seems difficult to construct an explicit *canonical* isomorphism from $\mathrm{IAnn}(\alpha)$ to $\mathrm{IAnn}(\beta)$. In the hope of solving this problem on the level of inner inverses (instead of inner annihilators), we turn our efforts to the alternative problem of finding "new Jacobson maps" $I(\alpha) \to I(\beta)$ that have better properties than the original Jacobson map (which sends any $s \in I(\alpha)$ to $1 + bsa \in I(\beta)$). Toward this goal, we prove the following main result in §§4–7.

Theorem C. *For any regular Jacobson pair $\{\alpha, \beta\}$ in a ring, the new Jacobson map $\Phi : I(\alpha) \to I(\beta)$ defined by*

$$(\dagger) \qquad \Phi(s) = 1 + b\,s\,a - b\,(1 - s\,\alpha)\,(1 - \alpha\,s)\,a \quad (\textit{for any } s \in I(\alpha))$$

has the following properties:

1. *Φ preserves and reflects units; that is, for $s \in I(\alpha)$, s is a unit iff $\Phi(s)$ is a unit.*
2. *Φ preserves and reflects regular elements and unit-regular elements.*
3. *Φ is a semigroup homomorphism with respect to the von Neumann products in $I(\alpha)$ and $I(\beta)$. (In $I(\alpha)$, the von Neumann product is defined by $x * y = x\alpha y$, and similarly for $I(\beta)$.)*
4. *Φ preserves and reflects reflexive inverses. (A reflexive inverse of α is an element $s \in I(\alpha)$ such that $s = s\,\alpha\,s$, and similarly for β.)*
5. *Φ restricts to a bijection between the "commuting inner inverses" of α and those of β. Furthermore, the corresponding map from the commuting inner inverses of β to those of α is the inverse map.*

As far as we know, the expression on the right side of (†) made its first appearance in the work of Castro-González, Mendes-Araújo and Patrício; see [2, Theorem 3.4]. (In the paper, they proved half of part (4) above, namely, that Φ preserves reflexive inverses.) From our viewpoint, the mapping (†) came into being through a careful analysis of the important (but essentially classical) observation that, upon suspension, *any Jacobson pair α, β over R become "equivalent" in the 2×2 matrix ring over R.*[1] The full details of this analysis are given in §§4–6. The proof of Theorem C is completed in §7 where we present the computations needed for checking the second and the third parts of the theorem.

A consequence of the methods used in this paper is the following surprising new characterization of the IC rings (rings satisfying "internal cancellation") introduced by Khurana and Lam in [8].

[1] Throughout this paper, the word "equivalence" will be used in the classical sense of linear algebra and matrix theory: two elements x and y in a ring S are said to be *equivalent* if there exist two units u, v in S such that $y = uxv \in S$.

Theorem D. *A ring R is IC iff every regular Jacobson pair of elements $1-ab$ and $1-ba$ in R are equivalent. In particular, a regular ring R is unit-regular iff every Jacobson pair of elements in R are equivalent.*

For any Jacobson pair $\alpha = 1-ab$ and $\beta = 1-ba$, the two basic commutation rules $\alpha\, a = a\, \beta$ and $b\, \alpha = \beta\, b$ are crucial for our work, *and will be used freely without further mention in the rest of this paper.* An important part for much of the work in §§4–7 will be not only to exploit these commutation relations, but (as in [14]) also to extend them to other pairs of elements arising from $\{\alpha, \beta\}$. For instance, a key step in proving Theorem C is to realize that the map Φ defined in (†) is related to the Hartwig-Luh self-maps φ and φ' on $I(\alpha)$ by the fact that, *for any $s \in I(\alpha)$, $\{\varphi(s), \Phi(s)\}$ and $\{\varphi'(s), \Phi(s)\}$ are both Jacobson pairs.* This leads to the "generalized commutation rules" $\Phi(s)\, b = b\, \varphi(s)$ and $a\, \Phi(s) = \varphi'(s)\, a$, which are essential to the proof of Theorem C.

Combining items (4) and (5) in Theorem C (and using an observation from [2] and [14]), we also arrive at a purely conceptual derivation of the following known result in the literature.

Corollary E. *For any Jacobson pair $\{\alpha, \beta\}$ in R, if α is group invertible or Drazin invertible, then so is β. In the group invertible case* (*where α is strongly regular*)*, the new Jacobson map in Theorem B maps the group inverse of α to that of β.*

The point about this conceptual derivation is that the known group inverse formula for β is now seen to be just a special case of the general formula (†) for the new Jacobson map Φ, and that the original Jacobson map would have failed to do the job through the omission of the second crucial term on the right side of the equation (†).

Common notations, terminology, and definitions in ring theory follow closely those in [6] and [10]. Specifically, $\mathrm{U}(R)$ denotes the group of units in a ring R, and $\mathbb{M}_n(R)$ denotes the $n \times n$ matrix ring over R with matrix units $\{e_{ij}\}$. For this paper, it will also be convenient for us to use consistently the following shorthand notations:

$$\begin{aligned}
\mathrm{reg}\,(R) &= \{x \in R : \exists\, y \in R,\ xyx = x\} \quad \text{(regular elements)},\\
\mathrm{ureg}\,(R) &= \{x \in R : \exists\, y \in \mathrm{U}(R),\ xyx = x\} \quad \text{(unit-regular elements)},\\
\mathrm{sreg}\,(R) &= \{x \in R : x \in x^2R \cap R\,x^2\} \quad \text{(strongly regular (or group invertible) elements)},\\
\mathrm{sreg}_\pi(R) &= \{x \in R : \exists\, n \geq 1,\ x^n \in x^{n+1}R \cap R\,x^{n+1}\} \quad \text{(strongly } \pi\text{-regular elements)}.
\end{aligned}$$

2. Inner Annihilators: Definitions and Examples

In this beginning section, we introduce the first basic object of our study; namely, the group of inner annihilators of an element in a ring R. To motivate our discussions, we'll give a number of basic examples from ring theory for this notion.

For any element $\alpha \in R$, we define

$$\mathrm{IAnn}\,(\alpha) = \{p \in R \ : \ \alpha\, p\, \alpha = 0\}.$$

This is an abelian group which we call the group of *inner annihilators* of α. This group contains the left annihilator ideal $\mathrm{Ann}_\ell(\alpha)$ and the right annihilator ideal $\mathrm{Ann}_r(\alpha)$ (and hence also their sum). We say that $\mathrm{IAnn}\,(\alpha)$ is *decomposable* if $\mathrm{IAnn}\,(\alpha) = \mathrm{Ann}_\ell(\alpha) + \mathrm{Ann}_r(\alpha)$.

For later reference, we begin by stating some basic formulas for the three annihilators for a pair of equivalent elements in a ring R. (The routine proofs for these facts will be left to the reader.)

Lemma 2.1. *Given a ring R and elements $\alpha \in R$, $u, v \in \mathrm{U}(R)$, the following hold:*

1. $\mathrm{IAnn}\,(u\alpha v) = v^{-1}\,\mathrm{IAnn}(\alpha)\, u^{-1}$,

2. $\mathrm{Ann}_\ell(u\alpha v) = \mathrm{Ann}_\ell(\alpha)\, u^{-1}$,
3. $\mathrm{Ann}_r(u\alpha v) = v^{-1}\mathrm{Ann}_r(\alpha)$.

The following simple examples will show to some extent how $\mathrm{IAnn}\,(\alpha)$ behaves in rings.

Example 2.2. If R is a reduced ring then $\mathrm{IAnn}\,(\alpha) = \mathrm{Ann}_\ell(\alpha) = \mathrm{Ann}_r(\alpha)$. In particular, the inner annihilator group here is decomposable, and it is an ideal.

Example 2.3. If α is *central* (for example if R is a commutative ring), obviously $\mathrm{IAnn}\,(\alpha) = \mathrm{Ann}\,(\alpha^2)$ is an ideal, but it is usually larger than $\mathrm{Ann}\,(\alpha)$. This leads easily to some examples where $\mathrm{IAnn}\,(\alpha)$ is not decomposable.

Example 2.4. More generally, if $\alpha \in R$ is a "duo element" (that is, if $\alpha R = R\,\alpha$), then $\mathrm{IAnn}\,(\alpha)$ is an ideal in R, in view of the fact that $\alpha\,(R\,p\,R)\,\alpha = R\,(\alpha\,p\,\alpha)\,R$ for any $p \in R$.

Example 2.5. Suppose R is a *reversible ring,* meaning that whenever $xy = 0$ then $yx = 0$. Here, we have $\mathrm{IAnn}\,(\alpha) = \mathrm{Ann}_\ell(\alpha^2) = \mathrm{Ann}_r(\alpha^2)$. If we assume instead the weaker condition that R is a semicommutative ring, meaning that $xy = 0$ implies $xR\,y = 0$, then $\mathrm{IAnn}\,(\alpha)$ is obviously an ideal, but we may no longer have the above equalities. Indeed, this class of rings will lead to some easy noncommutative examples of inner annihilators that are not decomposable. For instance, take F to be a field and consider the F-algebra

$$R = F\,\langle\, x, y\, :\, \text{monomials of degree three} \equiv\, 0\,\rangle.$$

This ring is easily seen to be semicommutative, using a degree argument. The element y is an inner annihilator of x, but $\mathrm{Ann}_\ell(x) = \mathrm{Ann}_r(x)$ is the F-span of $\{x^2, xy, y^2\}$ (which does not contain y). Thus, $\mathrm{IAnn}\,(x)$ is not decomposable in this case.

Example 2.6. Assume $\alpha \in \mathrm{reg}\,(R)$, say $\alpha = \alpha s\alpha$. Write $e = \alpha s$, $f = s\alpha$ and write $e' = 1 - e$, $f' = 1 - f$ for their complementary idempotents. (These notations will be used rather consistently in the rest of this paper.) Since $\alpha\,R = e\,R$ and $R\,\alpha = Rf$, we have

$$P_\ell := \mathrm{Ann}_\ell(\alpha) = R\,e', \quad \text{and} \quad P_r := \mathrm{Ann}_r(\alpha) = f'R.$$

We note further the following nice information about the sum and the intersection of P_ℓ and P_r.

(A) $P_\ell + P_r = \mathrm{IAnn}\,(\alpha)$; that is, $\mathrm{IAnn}\,(\alpha)$ *is always decomposable.* This is a well known result. Indeed, every $p \in \mathrm{IAnn}\,(\alpha)$ can be decomposed into $p\,e' + p\,e$, where $p\,e' \in R\,e' = \mathrm{Ann}_\ell(\alpha)$, while $\alpha \cdot (p\,e) = (\alpha p\alpha)\,s = 0$ shows that $p\,e \in \mathrm{Ann}_r(\alpha)$. (However, $\mathrm{IAnn}\,(\alpha)$ may not be an ideal; for instance, see Example 2.8 below.)

(B) $P_\ell \cap P_r = \{\, x - x\alpha x : x \in I(\alpha)\}$, where (as in the Introduction) $I(\alpha)$ denotes the set of inner inverses of α. Here, the inclusion "$\supseteq$" is obvious. To check "$\subseteq$", let $p \in P_\ell \cap P_r$. Fix any reflexive inverse s_0 for α (e.g. $s_0 = s\,\alpha\,s$), and let $x := p + s_0 \in I(\alpha)$. Then $x - x\,\alpha\,x = (p + s_0) - (p + s_0)\,\alpha\,(p + s_0) = p$.

Example 2.7. Let $\alpha \in \mathrm{sreg}\,(R)$; i.e. α is a strongly regular element. Then α has a (unique) group inverse s (so that $\alpha\,s = s\,\alpha$, $\alpha = \alpha s\alpha$, and $s = s\alpha s$). Write again $e := \alpha s = s\alpha$, which is called the "associated idempotent" of α, and $e' = 1 - e$ is the "spectral idempotent" of α. (The latter term is standard, while the former was introduced in [14].) In this case,

$$\mathrm{IAnn}\,(\alpha) = R\,e' + e'R = eR\,e' \oplus e'R\,e' \oplus e'R\,e$$

is the direct sum of all Peirce components of e except $eR\,e$. The analogue of this decomposition for *regular* elements in general will be given in §5.

Example 2.8. Let $R = \mathbb{M}_n(F)$ where F is a field. Any $\alpha \in R$ is unit regular, so we can write $\alpha = E\,U$ with $E = E^2 \in R$ and $U \in \mathrm{U}(R)$. Since $\mathrm{IAnn}\,(\alpha) = U^{-1}\,\mathrm{IAnn}\,(E)$ by Lemma 2.1, it suffices for us to consider the idempotent case $\alpha = E$. By the previous example, we see that $\mathrm{IAnn}\,(\alpha)$ is the sum of all Peirce components of E except ERE. Its (left and right) F-dimension is then $n^2 - r^2$ where $r = \text{rank}\,(E) = \text{rank}\,(\alpha)$. In particular, $\text{IAnn}\,(\alpha)$ behaves well with respect to scalar extensions: if $F' \supseteq F$ is any field extension then $\text{IAnn}\,(\alpha)$ computed in $\mathbb{M}_n(F')$ is equal to $F' \otimes_F \text{IAnn}\,(\alpha)$.

Example 2.9. Using the notation of the previous example, suppose $\alpha = J_n(\lambda)$ is an upper triangular Jordan matrix over F with diagonal $(\lambda, \ldots, \lambda)$. If $\lambda \neq 0$, then $\alpha \in \text{U}(R)$, so $\text{IAnn}\,(\alpha) = 0$. If $\lambda = 0$, then $\text{IAnn}\,(\alpha)$ consists of matrices $(\alpha_{i,j})$ where $\alpha_{i,j} = 0$ unless $i = 1$ or $j = n$. Thus, $\dim_F(\text{IAnn}\,(\alpha))$ is $n^2 - (n-1)^2 = 2n - 1$. Note that $\text{rank}\,(J_n(\lambda)) = n - 1$ in this case, so the formula from the previous example would have also given the same result.

3. Some Finiteness Results, and the Hartwig-Luh Maps on Inner Inverses

In this section, we begin our work on the set $I(\alpha)$ of inner inverses of a regular element $\alpha \in R$. Note that, since $I(\alpha) = s + \text{IAnn}\,(\alpha)$ for any inner inverse s of α, we have $|I(\alpha)| = |\text{IAnn}\,(\alpha)|$. (As usual, $|A|$ denotes the cardinality of a set A.) The main goal of this section is to present several results on elements with *finitely many* inner inverses or unit inner inverses in a ring R, and to introduce two self-maps φ and φ' on the set $I(\alpha)$ that were first implicitly constructed by Hartwig and Luh in [7]. These two mappings will play a substantial role in our later discussions in §6 on the "new Jacobson map" from $I(\alpha)$ to $I(\beta)$ for any Jacobson pair $\{\alpha, \beta\}$.

Our first result below is inspired by Kaplansky's observation that if an element $\alpha \in R$ is right-invertible but has only finitely many right inverses, then $\alpha \in \text{U}(R)$. (See [11, Exercise 1.14].)

Theorem 3.1. *Let $\alpha \in \text{reg}\,(R)$ with $n := |\,I(\alpha)\,| = |\,\text{IAnn}\,(\alpha)\,| < \infty$, then $\alpha \in \text{sreg}_\pi(R)$.*

Proof. Say $\alpha = \alpha s \alpha$. Letting $e = \alpha\, s$, $f = s\,\alpha$, $e' = 1 - e$, and $f' = 1 - f$ as in Example 2.6, we know that $\text{IAnn}\,(\alpha) = R\,e' + f'R$. Since $|\text{IAnn}\,(\alpha)| < \infty$, we have $|f'R| < \infty$. In particular, there exist integers $0 < j < i$ such that $f'\alpha^i = f'\alpha^j$; that is, $(1 - s\alpha)\,\alpha^i = (1 - s\alpha)\,a^j$. Therefore, $\alpha^j = s\alpha^{j+1} + (1 - s\alpha)\,\alpha^i \in R\,\alpha^{j+1}$. This shows that $aR \supseteq \alpha^2 R \supseteq \ldots$ stabilizes. A similar calculation using $|R\,e'| < \infty$ shows that $R\,\alpha \supseteq R\,\alpha^2 \supseteq \ldots$ stabilizes, so $\alpha \in \text{sreg}_\pi(R)$. □

In case the finite group $\text{IAnn}\,(\alpha)$ is of a very special nature, we have the following strengthening of Theorem 3.1.

Theorem 3.2. *If $\alpha \in \text{reg}\,(R)$ is such that $G := \text{IAnn}\,(\alpha)$ is a finite cyclic group or a group of order p^2 where p is a prime, then $\alpha \in \text{sreg}\,(R)$. (In particular, this conclusion holds if $|\,G\,| \leq 7$.)*

Proof. Let $\alpha = \alpha s\, \alpha$ and use the notations in the proof of Theorem 3.1. If $e' = 0$, then $1 = e = \alpha\, s$ implies that $\alpha \in \text{U}(R) \subseteq \text{sreg}\,(R)$, for otherwise α would have infinitely many right inverses, in contradiction to $|\,G\,| < \infty$. Thus, we may assume in the following that $e' \neq 0 \neq f'$. *We claim that either $\alpha\, e' = 0$ or $f'\alpha = 0$.* Once we know this, then we'll have $\alpha = \alpha^2 s$ or $\alpha = s\,\alpha^2$. Recalling Theorem 3.1, $\alpha \in \text{sreg}_\pi(R)$ has then Drazin index 1 (by [5]), which means that $\alpha \in \text{sreg}\,(R)$.

First assume that G is a finite cyclic group. Say e' has (additive) order n. Then e' generates the subgroup $\{g \in G : n\,g = 0\}$. Since $\alpha\, e' \in R\,e' \subseteq G$ and $n\,(\alpha\, e') = 0$, we have $\alpha\, e' = k\, e'$ for some integer k. Then $0 = (e'\alpha)\, e' = e'k\, e' = k\, e'$ implies that $\alpha\, e' = 0$, so we are done. Next, assume that $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ for some prime p. If $\alpha\, e'$ lies in the cyclic group generated by e', we are done as before. If otherwise, then G is generated by e' and $\alpha\, e'$, so $f' = k\, e' + \ell\, \alpha\, e'$ for some integers k, ℓ. Here we have $f'\alpha = (k + \ell\, \alpha)\,(e'\alpha) = 0$, so we are also done. □

Remark. In general, the assumptions placed on G in Theorem 3.2 cannot be removed. Indeed, if we take $R = \mathbb{M}_2(\mathbb{F}_p)$ and $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in R$, then $G = \text{IAnn}\,(\alpha)$ consists of all matrices (α_{ij}) with $\alpha_{21} = 0$. Therefore, $G \cong (\mathbb{Z}_p)^3$. Here, $\alpha^2 = 0$ implies that $\alpha \notin \text{sreg}\,(R)$.

Hartwig and Luh [7] have shown that if an element α in a ring R has a *unique* unit inner inverse, then $\alpha \in \operatorname{sreg}(R)$. We generalize this result into the following.

Theorem 3.3. *Suppose $\alpha \in R$ has exactly n unit inner inverses, where $1 \le n < \infty$. If R has no (additive) k-torsion for any positive integer $k \,|\, n$ then $\alpha \in \operatorname{sreg}(R)$.*

Proof. Say $\alpha = \alpha s \alpha$, where $s \in R$. Without assuming yet that s is a unit, let $f = s\,\alpha$, $f' = 1 - f$ (as before) and $r = f'\alpha$. Then $r^2 = f'(\alpha f')\,\alpha = 0$, so $u := 1 + r \in \mathrm{U}(R)$ (with inverse $1 - r$). Moreover $\alpha\, u = \alpha + \alpha f' \alpha = \alpha$, so for any $i \ge 1$ we have $\alpha\,(u^i s)\,\alpha = \alpha s \alpha = \alpha$; that is, $u^i s \in I(\alpha)$.

For the rest of the proof, assume that $s \in \mathrm{U}(R)$ and that R has no k-torsion for any $k \,|\, n$. Since $r^2 = 0$, we have $u^m = (1+r)^m = 1 + m\,r$ for any $m \ge 1$. As α has exactly n unit inner inverses, which are acted on freely by the cyclic group $\langle u \rangle$, the unit u must have multiplicative order $k \,|\, n$. Thus, $k\,r = 0$ and our assumption on torsion imply that $r = 0$. Since $r = (1 - s\,\alpha)\,\alpha$, it follows that $\alpha = s\,\alpha^2$. By symmetry, we also have $\alpha = \alpha^2 s$, so $\alpha \in \operatorname{sreg}(R)$. □

Remark. The assumption in Theorem 3.3 on the torsion structure of R cannot be dropped. To see this, we can take the same example $R = \mathbb{M}_2(\mathbb{F}_p)$ in the Remark after Theorem 3.2. Surely $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \operatorname{ureg}(R)$ has only finitely many (unit) inner inverses, but as we have pointed out before, $\alpha \notin \operatorname{sreg}(R)$. Here, additively, R is the elementary p-group $(\mathbb{Z}_p)^4$.

The discussion in the first paragraph in the proof of Theorem 3.3 suggests that, for any regular element $\alpha \in R$, we can define the *Hartwig-Luh map* $\varphi : I(\alpha) \to I(\alpha)$ by taking $\varphi(s) := us$ for any $s \in I(\alpha)$, where u is the unipotent unit $1 + f'\alpha$. Altogether, we have the following three different descriptions for the map φ:

$$\varphi(s) = (1 + f'\alpha)\,s = s + f'e = s + \alpha s - s\,\alpha^2 s. \tag{1}$$

Proposition 3.4. *The map $\varphi : I(\alpha) \to I(\alpha)$ takes units to units, and takes each $s \in I(\alpha)$ commuting with α to itself.*

Proof. If $s \in I(\alpha) \cap \mathrm{U}(R)$ (of course such units exist if and only if $\alpha \in \operatorname{ureg}(R)$), then by the equation (1) above, $\varphi(s) = (1 + f'\alpha)\,s \in \mathrm{U}(R)$ too since $1 + f'\alpha \in \mathrm{U}(R)$. And if $s \in I(\alpha)$ commutes with α, then $e = f$ and so $f'e = 0$, whence $\varphi(s) = s$ again by the equation (1). □

The mapping φ has a number of other interesting properties too. These will be proved later in a somewhat more general setting. We should point out that $\varphi : I(\alpha) \to I(\alpha)$ is in general neither injective nor surjective. For example, let $R = \mathbb{M}_2(k)$ where k is any commutative ring, and let α be the idempotent matrix $\left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}\right) \in R$. By a straightforward computation, the matrices $s = \left(\begin{smallmatrix} 0 & x \\ 0 & 1 \end{smallmatrix}\right)$ all lie in $I(\alpha)$, and are all mapped to α by φ. On the other hand, any matrix $\gamma = \left(\begin{smallmatrix} 0 & y \\ 0 & 1 \end{smallmatrix}\right)$ lies in $I(\alpha)$, but is easily seen to be *not* in the image of φ if $\gamma \ne \alpha$; that is, if $y \ne -1$.

Informally, we may refer to the Hartwig-Luh map φ on $I(\alpha)$ as a "left scaling" since, for every $s \in I(\alpha)$, $\varphi(s)$ is defined to be a left multiple of s by a unit (which depends on s). From this viewpoint, there is also a twin map $\varphi' : I(\alpha) \to I(\alpha)$ defined by the analogous formulas:

$$\varphi'(s) = s\,(1 + \alpha\, e') = s + f e' = s + s\,\alpha - s\,\alpha^2 s \quad (\text{for any } s \in I(\alpha)). \tag{2}$$

To keep track of these two maps, we can call φ the left Hartwig-Luh map, and φ' the right Hartwig-Luh map. However, we'll use this terminology only when it turns out to be necessary.

4. Inner Annihilators of Jacobson Pairs

In this section (and for much of the ensuing sections), we study two elements $\alpha = 1 - ab$ and $\beta = 1 - ba$ in a given ring R. As in the Introduction, we say that $\{\alpha, \beta\}$ is a *Jacobson pair* (with respect to a, b) in R. We'll show that the inner annihilator groups of α and β are very closely related, and they are always equipotent. By developing and refining a technique used by Huanyin Chen in [3], we'll prove further a stable isomorphism theorem for these two inner annihilator groups.

For simplicity we'll write $P = \mathrm{IAnn}\,(\alpha)$, $P_\ell = \mathrm{Ann}_\ell(\alpha)$, and $P_r = \mathrm{Ann}_r(\alpha)$, with $P_\ell + P_r \subseteq P$. Similarly, we define Q, Q_ℓ, and Q_r in terms of the three annihilators of β.

Proposition 4.1. *The following hold:*

1. $P_\ell \cong Q_\ell$ *as left ideals.*
2. $P_r \cong Q_r$ *as right ideals.*
3. $P_\ell \cap P_r \cong Q_\ell \cap Q_r$ *as abelian groups.*
4. $P/(P_\ell + P_r) \cong Q/(Q_\ell + Q_r)$ *as abelian groups.*

Proof. Right multiplication by a defines a left ideal map $P_\ell \to Q_\ell$, for if we are given $p \in P_\ell$ we have $p\,\alpha = 0$ and so $(p\,a)\,\beta = p\,\alpha\,a = 0$ shows that $p\,a \in Q_\ell$. Similarly, right multiplication by b defines a left ideal map $Q_\ell \to P_\ell$. We compute that $pab = p\,(1-\alpha) = p$, and similarly, given $q \in Q_\ell$ we have $qba = q$. Thus the two maps here are mutually inverse left ideal isomorphisms. In the same vein, left multiplication by b defines a right ideal isomorphism $P_r \to Q_r$, with inverse given by left multiplication by a. Thus, we have proved (1) and (2).

The additive group homomorphism $\lambda : P \to Q$, given by the rule $p \mapsto bpa$, defines a group homomorphism from $P_\ell \cap P_r$ to $Q_\ell \cap Q_r$. The map $\mu : Q \to P$, given by $q \mapsto aqb$, is its inverse map when we restrict to $Q_\ell \cap Q_r$. This proves (3).

To prove (4), note that while λ takes P_ℓ to Q_ℓ, and P_r to Q_r, it does not do so *isomorphically.* However, λ induces a homomorphism $P/(P_\ell + P_r) \to Q/(Q_\ell + Q_r)$. Similarly, μ induces a homomorphism the other way. For any $p \in P$, we have

$$\begin{aligned} \mu\,(\lambda\,(p)) &= abpab \;=\; (1-\alpha)\,p\,(1-\alpha) \\ &= p - \alpha\,p - p\,\alpha \;\equiv\; p \pmod{P_\ell + P_r}. \end{aligned}$$

By symmetry, we also have $\lambda\,(\mu\,(q)) \equiv q \pmod{Q_\ell + Q_r}$ for any $q \in Q$. Thus, the induced maps are mutually inverse group isomorphisms, proving (4). □

Using the above lemma, we arrive at the following computation of Q in terms of P, P_ℓ, and P_r.

Corollary 4.2. *We have* $Q = b\,P\,a + Q_\ell + Q_r = b\,P\,a + P_\ell a + b\,P_r$.

Corollary 4.3. *If* $\mathrm{IAnn}\,(\alpha)$ *is decomposable then so is* $\mathrm{IAnn}\,(\beta)$.

Proof. This follows from the group isomorphism in part (4) of Proposition 4.1. □

Example 4.4. It is not the case, generally, that if $P = P_r$ then $Q = Q_r$. For example, suppose $ab = 1 \neq ba$. We then have $\alpha = 1 - ab = 0$ and $\beta = 1 - ba \neq 0$. In this case, $P = P_\ell = P_r = R$. Now, $a\beta = a\,(1-ba) = (1-ab)\,a = 0$ so $a \in Q_\ell \subseteq Q$. However, $\beta\,a = (1-ba)\,a = a - b\,a^2$ need not be zero in general. Thus we may have $a \notin Q_r$. Similarly, $\beta\,b = 0$ but $b \notin Q_\ell$ in general. This proves that even if $P = P_\ell = P_r$, we can have $Q \neq Q_\ell$ and $Q \neq Q_r$.

This also gives an example showing that the maps λ and μ constructed in Proposition 4.1 need not be injective or surjective. Nevertheless, the truth of the following can be ascertained.

Proposition 4.5. *The groups* $\mathrm{IAnn}\,(\alpha)$ *and* $\mathrm{IAnn}\,(\beta)$ *are always equipotent.*

Proof. We know by Proposition 4.1, part (4), that $P/(P_\ell + P_r) \cong Q/(Q_\ell + Q_r)$. Thus it suffices to show that $|P_\ell + P_r| = |Q_\ell + Q_r|$. Parts (1) and (2) of the same Proposition tell us that $P_\ell \cong Q_\ell$ and $P_r \cong Q_r$. So it suffices to show that $|\,P_\ell \cap P_r\,| = |\,Q_\ell \cap Q_r\,|$, which follows from part (3). □

While we do not know in general whether $\mathrm{IAnn}\,(\alpha)$ is isomorphic to $\mathrm{IAnn}(\beta)$, we can show that they become isomorphic after "adding" the group $P_\ell \oplus P_r$. One may view the methods used below as an analogue of "Whitehead's Lemma" in algebraic K-theory: see [12, Chapter I, (5.1)]. For this purpose, we work in $\mathbb{M}_2(R)$ and let $A = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} \beta & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be the "suspensions" of α and β, respectively. Letting $U = \left(\begin{smallmatrix} a & \alpha \\ 1 & -b \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} a & 1 \\ \beta & -b \end{smallmatrix}\right)$ in $\mathbb{M}_2(R)$, we note that these are *invertible*, with inverses $U^{-1} = \left(\begin{smallmatrix} b & \beta \\ 1 & -a \end{smallmatrix}\right)$ and $V^{-1} = \left(\begin{smallmatrix} b & 1 \\ \alpha & -a \end{smallmatrix}\right)$. (As in Whitehead's Lemma, both U and V are products of a small number of elementary matrices. But we shall not make use of that fact here.)

We shall exploit the basic matrix identity

$$UB = AV, \tag{3}$$

which is easily seen by right-scaling the first column of U by β and left-scaling the first row of V by α, and then using the basic commutation rule $a\beta = \alpha a$. Thus, *A and B are always equivalent in* $\mathbb{M}_2(R)$, although α and β may not be equivalent in R. This very interesting observation has its origin in the familiar linear algebra fact (sometimes referred to as "Sylvester's Determinant Formula") that two $n \times n$ matrices of the form XY and YX have the same characteristic polynomials. Starting with one of the standard proofs of this fact using block elementary transformations on $2n \times 2n$ matrices (e.g. as in [9]), it is fairly routine to come up with the equation (3); cf. also the proof of Lemma 2.1 in [3]. For related discussions on "Vaserstein's Lemma", see [12, Chapter I, §9].

Now equation (3) and Lemma 2.1 imply that

$$\mathrm{IAnn}\,(B) = \mathrm{IAnn}\,(U^{-1}AV) = V^{-1}\,\mathrm{IAnn}\,(A)\,U = \begin{pmatrix} b & 1 \\ \alpha & -a \end{pmatrix} \mathrm{IAnn}\,(A) \begin{pmatrix} a & \alpha \\ 1 & -b \end{pmatrix}.$$

By an easy computation, we have $\mathrm{IAnn}\,(A) = \left(\begin{smallmatrix} P & P_r \\ P_\ell & 0 \end{smallmatrix}\right)$, where $P = \mathrm{IAnn}\,(\alpha)$, $P_\ell = \mathrm{Ann}_\ell(\alpha)$, and $P_r = \mathrm{Ann}_r(\alpha)$ are the sets we defined previously, and similarly for B. We then get an abelian group isomorphism $\mathrm{IAnn}\,(A) \to \mathrm{IAnn}\,(B)$ by left-multiplying by V^{-1} and right-multiplying by U, which gives the action

$$\begin{pmatrix} p & p_r \\ p_\ell & 0 \end{pmatrix} \mapsto \begin{pmatrix} bpa + p_\ell a + bp_r & bp\alpha - bp_r b \\ \alpha pa - ap_\ell a & 0 \end{pmatrix}.$$

This explicitly defined isomorphism proves, in particular, the following.

Theorem 4.6. *For any Jacobson pair of elements $\alpha, \beta \in R$, we have*

$$\mathrm{IAnn}\,(\alpha) \oplus P_\ell \oplus P_r \cong \mathrm{IAnn}\,(\beta) \oplus Q_\ell \oplus Q_r \quad (\textit{as abelian groups}).$$

If we can "cancel" the extra summands $P_\ell \oplus P_r$ and $Q_\ell \oplus Q_r$ (which are isomorphic to each other by Proposition 4.1), we would then have an isomorphism of the two inner annihilators. The following, for instance, would be a typical case in which such a "cancellation" can be accomplished.

Corollary 4.7. *If $\mathrm{IAnn}\,(\alpha)$ is a finitely generated abelian group, then $\mathrm{IAnn}\,(\alpha) \cong \mathrm{IAnn}\,(\beta)$.*

We conjecture that the two inner annihilator groups $\mathrm{IAnn}\,(\alpha)$ and $\mathrm{IAnn}\,(\beta)$ are *always* isomorphic, for every Jacobson pair $\{\alpha, \beta\}$. Of course, this would be true (by Lemma 2.1) if α and β happen to be *equivalent.* This is classically known to be the case if R is a semisimple ring. More generally, we'll show in §6 that this is also the case if R is a unit-regular ring. And, independently of the nature of the ground ring R, we'll show (in Theorem 5.1 below) that the conjecture made at the beginning of this paragraph is true when α (and hence β) is a *regular element* in R.

Another thing worth mentioning about a Jacobson pair (α, β) is the following. Without any suspension considerations we always have $R/\alpha R \cong R/\beta R$ and $\mathrm{Ann}_r(\alpha) \cong \mathrm{Ann}_r(\beta)$, where the explicit module isomorphisms are (well) defined by multiplications by a and by b. As a nice application of this fact, we note that α is right morphic (see [16]) iff so is β.

5. Regular Jacobson Pairs

We say that a Jacobson pair $\{\alpha, \beta\}$ (over a ring R) is regular if $\alpha \in \text{reg}\,(R)$ (and hence $\beta \in \text{reg}\,(R)$). In this case, we'll prove in this section an isomorphism theorem for the inner annihilators of α and β. We then go on to construct the "new Jacobson map" $\Phi : I(\alpha) \to I(\beta)$ mentioned in the Introduction, and offer some of its applications to the study of the sets of unit inner inverses $I_u(\alpha)$ and $I_u(\beta)$.

We shall use freely all the notations in the previous sections. For the rest of the paper, it will be convenient to also adopt the following additional notations for any regular element $\alpha \in \text{reg}\,(R)$:

$$\begin{array}{rcl} I(\alpha) & = & \text{the set of inner inverses for } \alpha \\ & = & s + \text{IAnn}\,(\alpha) \,=\, s + P \quad \text{(for any given inner inverse } s), \\ I_r(\alpha) & = & \{s \in I(\alpha) \,:\, s\alpha s = s\} \quad \text{(the "reflexive inverses")}, \\ I_u(\alpha) & = & \{s \in I(\alpha) \,:\, s \in \mathrm{U}(R)\} \quad \text{(the "unit inner inverses")}, \\ I_c(\alpha) & = & \{s \in I(\alpha) \,:\, \alpha s = s\alpha\} \quad \text{(the "commuting inner inverses")}. \end{array}$$

Note that $I_r(\alpha)$ is always nonempty (assuming $\alpha \in \text{reg}\,(R)$), but $I_u(\alpha) \neq \emptyset$ iff $\alpha \in \text{ureg}\,(R)$, and $I_c(\alpha) \neq \emptyset$ iff $\alpha \in \text{sreg}\,(R)$. Finally,

$$I_c(\alpha) \cap I_r(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \notin \text{sreg}\,(R), \\ \{\text{group inverse of } \alpha\} & \text{if } \alpha \in \text{sreg}\,(R). \end{cases}$$

We start by considering a given inner inverse s of α. As in Example 2.6 and in other previous sections, we write $e = \alpha s$, $f = s\alpha$, $e' = 1 - e$ and $f' = 1 - f$. Letting $X := P_\ell \cap P_r = f'R\,e'$, we have

$$P_\ell = R\,e' = fR\,e' \oplus X, \quad \text{and} \quad P_r = f'R = X \oplus f'R\,e. \tag{4}$$

Therefore

$$P = P_\ell + P_r = fR\,e' \oplus X \oplus f'R\,e, \tag{5}$$

and hence $P \oplus X \cong P_\ell \oplus P_r$ as *external* direct sums. Of course, this group isomorphism also follows from the split short exact sequence

$$0 \longrightarrow X \xrightarrow{\ \delta\ } P_\ell \oplus P_r \xrightarrow{\ \pi\ } P \longrightarrow 0$$

where $\delta(x) = (x, -x)$, and $\pi(y, z) = y + z$ is split by the map $p \in P \mapsto (pe', pe) \in P_\ell \oplus P_r$. (See Example 2.6.)

Theorem 5.1. *If* $\{\alpha, \beta\}$ *is a regular Jacobson pair over* R*, then* $\text{IAnn}\,(\alpha) \cong \text{IAnn}\,(\beta)$ *as abelian groups.*

Proof. Fix an inner inverse $t \in I(\beta)$, and write $g = \beta t$, $h = t\beta$, $g' = 1 - g$, and $h' = 1 - h$. Also set $Y = Q_\ell \cap Q_r$. Applying Proposition 4.1, we have $P_\ell \cong Q_\ell$, $P_r \cong Q_r$, and $X \cong Y$ (as groups). Thus, using external direct sums would give

$$P \oplus X \cong P_\ell \oplus P_r \cong Q_\ell \oplus Q_r \cong Q \oplus Y.$$

Since $X \cong Y$, this gives a "stable isomorphism" between P and Q. However, arguing a little more carefully with (5), we'll get

$$\begin{array}{rcl} P & = & P_\ell \oplus f'R\,e \cong Q_\ell \oplus f'R\,e \\ & = & hR\,g' \oplus Y \oplus f'R\,e \\ & \cong & hR\,g' \oplus X \oplus f'R\,e \\ & = & hR\,g' \oplus P_r \cong hR\,g' \oplus Q_r = Q, \end{array}$$

where the last equality follows from the analogue of (5) in terms of Q. $\square$

Remark. Of course, it would have been more preferable simply to find an isomorphism between each of the three direct summands of P in (5) and the corresponding direct summands of Q. However, this turns out to be *impossible* in general. For example, take a ring R with two elements a, b such that $ba = 1 \neq ab$, and let $p = ab$. (We can think of p as a "big idempotent" in R.) Here $\beta = 0$ but $\alpha = 1 - p \neq 0$. From the former, $Q = Q_\ell = Q_r = R$ and so $hR\,g' = 0$. Choosing $s = \alpha \in I(\alpha)$ (which is possible since α is an idempotent), we have $e = f = \alpha = 1 - p$. Thus $fR\,e' = (1-p)\,R\,p$, and this summand is *nonzero* as it contains $(1-ab)\,b\,(ab) = b - ab^2$ which is, generically, not zero.

Partly inspired by the above remark, we arrive at the following possibly surprising consequence of Theorem 5.1.

Corollary 5.2. *If* $b\,a = 1$*, then* $\mathrm{IAnn}\,(1 - ab) \cong R$ *as abelian groups.*

Note that in Theorem 5.1, we did not get a *canonical map* $\mathrm{IAnn}\,(\alpha) \to \mathrm{IAnn}\,(\beta)$ that is an additive group isomorphism. The isomorphism we constructed in the proof of that theorem depended on the choices of s and t, and also on which summands to replace first, so it is far from being "canonical." To remedy this situation, we shall attempt in the following to give a more "choice-independent" construction. *At this point, it is convenient to switch our attention from* $\mathrm{IAnn}\,(\alpha)$ *to* $I(\alpha)$.

In trying to construct a "new Jacobson map" $\Phi : I(\alpha) \to I(\beta)$, we start by considering an inner inverse $s \in I(\alpha)$. Let $A := \mathrm{diag}\,(\alpha, 1) \in \mathbb{M}_2(R)$, which has inner inverse $S := \mathrm{diag}\,(s, 1)$. Using our earlier notations in §4, we have $B := \mathrm{diag}\,(\beta, 1) = U^{-1}AV$, so a *natural* inner inverse for B arising from s is given by

$$W := V^{-1}S\,U = \begin{pmatrix} b & 1 \\ \alpha & -a \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & \alpha \\ 1 & -b \end{pmatrix} = \begin{pmatrix} 1 + bsa & -bf' \\ -e'a & 1 \end{pmatrix} \tag{6}$$

with $e = \alpha s$, $f = s\alpha$, and $e' = 1 - e$, $f' = 1 - f$ as usual. (Basically, we are using here the analogue of the first formula in Lemma 2.1, with the inner annihilator group there replaced by the set of inner inverses.) Notice that the "original" Jacobson map is given by assigning to s the $(1,1)$-entry of the matrix W above. It is thus no surprise that this map is far from optimal: we are dropping from our consideration three other important pieces of information about the matrix W!

To make better use of the equation (6), we take advantage of the fact that W has $(2,2)$-entry 1, which enables us to bring it to upper triangular form by an elementary column transformation:

$$W \begin{pmatrix} 1 & 0 \\ e'a & 1 \end{pmatrix} = \begin{pmatrix} 1 + bsa - bf'e'a & -bf' \\ 0 & 1 \end{pmatrix}. \tag{7}$$

This brings to light the expression $1 + bsa - bf'e'a \in R$, which emerges as a "(noncommutative) determinant" of the matrix W. This expression has also appeared earlier in the work of Castro-González, Mendes-Araújo and Patrício; see [2, Theorem 3.4]. Note that $\beta\,(bf'e'a) = b\,(\alpha f'e'a) = 0$, so $bf'e'a \in \mathrm{Ann}_r(\beta) \subseteq \mathrm{IAnn}\,(\beta)$. Thus, the fact that $1 + bsa \in I(\beta)$ implies that $1 + bsa - bf'e'a \in I(\beta)$. This suggests that we consider the mapping $\Phi : I(\alpha) \to I(\beta)$ defined by

$$\Phi\,(s) = 1 + bsa - bf'e'a \quad (\text{for all } s \in I(\alpha)). \tag{8}$$

We'll call Φ the *new Jacobson mapping*, since the right side of this equation has the extra term $-bf'e'a$ added to the "original" Jacobson map $s \mapsto 1 + bsa$. The following result gives the first couple of significant properties of Φ. (More properties will be given in Theorem 5.4 and in §7.)

Theorem 5.3. *The new Jacobson map* $\Phi : I(\alpha) \to I(\beta)$ *preserves and reflects units. More precisely, if* $s \in I_u(\alpha)$*, then* $\Phi\,(s) \in I_u(\beta)$*, with* $\Phi\,(s)^{-1} = \beta^2 + bs^{-1}a$*. Conversely, if* $s \in I(\alpha)$ *is such that* $t := \Phi\,(s) \in I_u(\beta)$*, then* $s \in I_u(\alpha)$*, with*

$$s^{-1} = \alpha^2 + (1 + \alpha\,e')\,a\,t^{-1}b\,(1 + f'\alpha), \tag{9}$$

where, as before, $e' = 1 - \alpha s$*, and* $f' = 1 - s\,\alpha$*.*

Proof. Let $s \in I(\alpha)$, $t := \Phi(s)$, and keep the previous notations. Recalling that $S := \operatorname{diag}(s, 1)$, we combine the two equations (6) and (7) into

$$V^{-1} S\, U \begin{pmatrix} 1 & 0 \\ e'a & 1 \end{pmatrix} = T := \begin{pmatrix} t & -bf' \\ 0 & 1 \end{pmatrix}. \tag{10}$$

First assume that $s \in \mathrm{U}(R)$. Then $S \in \mathrm{GL}_2(R)$, and so (10) implies that $T \in \mathrm{GL}_2(R)$. Thus, $t \in \mathrm{U}(R)$, and t^{-1} is given by the $(1,1)$-entry of the inverse of the left side of (10), which is easily seen to be $\beta^2 + bs^{-1}a$. Conversely, if $t \in \mathrm{U}(R)$, then T has inverse $\left(\begin{smallmatrix} t^{-1} & t^{-1}bf' \\ 0 & 1 \end{smallmatrix}\right)$, and (10) shows that $S \in \mathrm{GL}_2(R)$, which amounts to $s \in \mathrm{U}(R)$. Thus, s^{-1} can be gotten as the $(1,1)$-entry of the matrix

$$S^{-1} = U \begin{pmatrix} 1 & 0 \\ e'a & 1 \end{pmatrix} T^{-1} V^{-1},$$

which is easily seen to be given by (9). □

Remark. In the case where $s \in I_u(\alpha)$, the existence of $\Phi(s)^{-1}$ and its computation could have also been gotten from the work of Lam and Murray [13] on unit regular elements in corner rings. Indeed, for $s \in I_u(\alpha)$, the matrix B has a *unit* inner inverse W in (6). According to [13] (applied to the corner ring $e_{11}\mathbb{M}_2(R)\, e_{11} \cong R$), a *unit* inner inverse of β is then provided by the $(1,1)$-entry of $W - We_{2,2}W$, which is easily computed to be $1 + bsa - bf'e'a$. (This is yet another natural way in which this expression arises.) By [13] again, the inverse of this unit is given by the $(1,1)$-entry of W^{-1}, which is easily computed from (6) to be $\beta^2 + bs^{-1}a$.

Concerning the definition (8), of course we also have a similar map $\Psi : I(\beta) \to I(\alpha)$. The map Ψ is not necessarily the inverse of Φ as Φ may not be injective or surjective, even when we restrict to unit inner inverses. However, the remarkable simplicity of the equation $\Phi(s)^{-1} = \beta^2 + bs^{-1}a$ (for every $s \in I_u(\alpha)$) can be exploited to give the following.

Theorem 5.4. *Assume that $a, b \in R$ are not zero-divisors in R. Then the following hold.*

1. *The new Jacobson map $\Phi : I_u(\alpha) \to I_u(\beta)$ is injective.*
2. *The two sets $I_u(\alpha)$ and $I_u(\beta)$ are equipotent.*
3. *If $I_u(\alpha)$ is finite then $\Phi : I_u(\alpha) \to I_u(\beta)$ is a bijective map.*

Proof. (1) Suppose $\Phi(r) = \Phi(s)$ where $r, s \in I_u(\alpha)$. We then have

$$\beta^2 + b\, s^{-1} a = \Phi(s)^{-1} = \Phi(r)^{-1} = \beta^2 + b\, r^{-1} a.$$

Since a and b are not zero-divisors, we get $r^{-1} = s^{-1}$ and hence $r = s$.

(2) By symmetry, the corresponding map $\Psi : I_u(\beta) \to I_u(\alpha)$ is also injective. Therefore, (2) follows from the Schröder-Bernstein theorem.

(3) Any injection from a finite set to an equipotent finite set is a bijection. □

Of course, for a general analysis of the relationship between $I_u(\alpha)$ and $I_u(\beta)$, we should hope to avoid making the "not zero-divisors" assumption in Theorem 5.4. Some results of this nature will be given in the next section.

6. New Jacobson Pairs from Old, and Applications

This section reiterates a theme from our earlier work [14] that it is often possible to get new information on a Jacobson pair $\{\alpha, \beta\}$ by constructing from it some other Jacobson pairs. By exploiting this theme in the regular case, we'll get a number of new results on *equivalent pairs of elements* over special rings. In the second half of the section, we'll use the same technique on $\{\alpha, \beta\}$ to relate the new Jacobson map $\Phi : I(\alpha) \to I(\beta)$ to the left and right Hartwig-Luh maps φ and φ' constructed in §3. In this way, we'll arrive at suitable commutation rules for the "graph points" $(s, \Phi(s))$ of the mapping Φ (for any $s \in I(\alpha)$).

To begin with, we make the following general observation.

Lemma 6.1. *For a regular Jacobson pair $\alpha = 1 - ab$ and $\beta = 1 - ba$, let $s \in I(\alpha)$ and $t = \Phi(s)$. Let e, f, e', f' be as in Example 2.6, and define similarly $g = \beta t$, $h = t\beta$, $g' = 1 - g$, and $h' = 1 - h$. Then $g' = b e' a$, and $h' = b f' a$. Moreover, the following four pairs of idempotents are Jacobson pairs over R:*

$$\{e', f'\}, \quad \{g', h'\}, \quad \{e, g\}, \quad \textit{and} \quad \{f, h\}. \tag{11}$$

Schematically, these are the two rows of $A' = \begin{pmatrix} e' & f' \\ g' & h' \end{pmatrix}$, *and the two columns of* $A = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

Proof. Since $e' = 1 - \alpha s$ and $f' = 1 - s\alpha$, the pair $\{e', f'\}$ is obviously a Jacobson pair. Similarly, so is $\{g', h'\}$. Next, using the fact that $\alpha f' = 0$, we have

$$\begin{aligned} g &= \beta(1 + bsa - bf'e'a) = \beta + b\,\alpha sa - b\,\alpha f'e'a \\ &= 1 - b\,a + b\,e\,a = 1 - b\,e'a. \end{aligned}$$

Thus, $g' = b\,e'a$. Also, from the equations above, g forms a Jacobson pair with $1 - e'ab = 1 - e'(1-\alpha) = e$ (since $e'\alpha = 0$). The remaining proofs (for $h' = b\,f'a$ and for $\{f, h\}$ forming a Jacobson pair) can be given similarly. ☐

Quite generally, two *idempotents* in a ring form a Jacobson pair iff their complementary idempotents are isomorphic; see e.g. [10, (21.20)]. Therefore, the lemma above implies the following.

Corollary 6.2. *In the notations of Lemma 6.1, the two rows of A and the two columns of A' are isomorphic pairs of idempotents.*

In dealing with the rows and columns of the matrices A and A', some cancellation issues would arise naturally. In [8], a ring R is said to be **IC** (satisfying "internal cancellation") if, for any right ideal decompositions $R_R = J \oplus J' = K \oplus K'$, $J \cong K \Rightarrow J' \cong K'$. Phrased in terms of idempotents, this condition amounts to asking that *complements of isomorphic idempotents are isomorphic.* According to [8, (1.4)], this also amounts to the condition that *isomorphic idempotents are similar.* (In particular, "IC" is a left-right symmetric condition.) Using Lemma 6.1 and Corollary 6.2, we can now provide a new characterization for IC rings in terms of the equivalence of Jacobson pairs. Neither the "if" part nor the "only if" part of the theorem below seemed to have appeared before in the literature.

Theorem 6.3. *A ring R is IC iff every regular Jacobson pair of elements $\alpha = 1 - ab$ and $\beta = 1 - ba$ are equivalent over R. In this case, there exists a one-one correspondence between $I_u(\alpha)$ and $I_u(\beta)$.*

Proof. We start with the "if" part, since it turns out to be a bit easier (and does not really depend on the results in this paper). Suppose we are granted the (ostensibly weaker) condition that every Jacobson pair of *idempotents* over R are equivalent. According to a result of Song and Guo [20], this amounts to granting that every Jacobson pair of idempotents are *similar.* To check that R is IC, consider any pair of isomorphic idempotents, say $p = ab$ and $q = ba$. Then their complements $p' = 1 - ab$ and $q' = 1 - ba$ form a Jacobson pair of idempotents. By assumption, these are similar. It follows easily that p and q are also similar, so we have checked that R is IC.

Next, we come to the (harder) "only if" part, which depends crucially on (6.1) and (6.2). Consider *any* regular Jacobson pair over an IC ring R. Since $\alpha \in \operatorname{reg}(R)$, [8, (1.5)] implies that $\alpha \in \operatorname{ureg}(R)$. Fixing a *unit* inner inverse $s \in I_u(\alpha)$, let $t = \Phi(s) \in I_u(\beta)$, and use the notations in (6.1) and (6.2). By (6.2), $\{e', g'\}$ are isomorphic idempotents, so the IC assumption implies that $\{e', g'\}$ are similar. As in the last paragraph, it follows that e and g are similar, and in particular, equivalent. Now $s, t \in \mathrm{U}(R)$

implies that α is equivalent to $\alpha\, s = e$, and β is equivalent to $\beta\, t = g$. We can conclude, therefore, that α and β are equivalent!

To prove the last statement in the theorem, we continue to assume that R is IC, and use the (now proved) fact that any regular Jacobson pair α and β are equivalent; that is, $\beta = u\,\alpha\, v$ for some $u, v \in \mathrm{U}(R)$. By the analogue of Lemma 2.1 for $I(\alpha)$ and $I(\beta)$, the map $x \mapsto v^{-1}x\, u^{-1}$ is a bijection from $I(\alpha)$ to $I(\beta)$. Since this map obviously preserves and reflects units, it restricts to a bijection from $I_u(\alpha)$ to $I_u(\beta)$. ☐

On the strength of Theorem 6.3, we now obtain a number of nice applications. The first major consequence of that theorem is the following.

Proposition 6.4. *If R is an abelian ring, a quasi-continuous ring, a right* FIR *("free ideal ring"), a ring of stable range* 1 *(e.g. a semilocal ring or a strongly π-regular ring), or a matrix ring over a finite von Neumann algebra, then every pair of regular elements $1 - ab$, $1 - ba$ are equivalent.*

Proof. This follows from Theorem 6.3 as soon as we can justify the fact that all rings listed above are IC rings. For abelian rings, this is trivial. For quasi-continuous rings, right FIR's, and rings of stable range 1, see [8, (2.1)(4), (2.1)(6), (5.9)(6)]. For the parenthetical cases of rings of stable range 1, see [10, (20.9)] and [1]. Finally, as is observed in [8, Theorem 5.12], any matrix ring over a finite von Neumann algebra is IC, in view of a result of Lück [15, Corollary 3.2]. ☐

Another specialization of Theorem 6.3 gives a new characterization of unit-regular rings.

Proposition 6.5. *A regular ring is unit-regular iff every Jacobson pair of elements are equivalent.*

Proof. This follows from Theorem 6.3 since a regular ring is unit-regular iff it is IC, by [8, (1.5)]. ☐

We should point out that the Proposition above has a known analogue for strongly regular rings that is well known to experts. We take this opportunity to state and prove this analogue too, since it does not seem to be available from the standard text [6].

Proposition 6.6. *A regular ring R is strongly regular iff R is a reversible ring (see (2.5)), iff every pair of elements $1 - ab$ and $1 - ba$ are similar over R.*

Proof. First assume R is strongly regular. To prove the other two statements, it suffices to show that any two elements $a\,b$ and $b\,a$ are similar. By [6, (4.2)], R is unit-regular, so we can write $a = e\,u$ where e is an idempotent and $u \in \mathrm{U}(R)$. Here, e must be a *central* idempotent by [6, (3.5)]. Therefore,

$$u^{-1}\,(ab)\,u = e\,b\,u = b\,e\,u = b\,a.$$

For the rest, we need only show that, if R is reversible, then it is strongly regular. For any $\alpha \in R$, write $\alpha = \alpha\, s\, \alpha$ and let $e = \alpha\, s$. For $e' = 1 - e$, we have $e'\alpha = 0$, so reversibility gives $\alpha\, e' = 0$. This implies that $\alpha = \alpha^2 s \in \alpha^2 R$ (for every $\alpha \in R$), so R is strongly regular, as desired. ☐

For the rest of this section, we return to the theme of constructing new Jacobson pairs from old ones, and give another application of this method to the study of the commutation rules associated with the new Jacobson map $\Phi : I(\alpha) \to I(\beta)$. The key idea here is to find a nontrivial relationship between the map Φ and the two Hartwig-Luh self-maps φ and φ' on $I(\alpha)$ defined in §3. Indeed, for any $s \in I(\alpha)$, $\Phi\,(s)$ happens to be a "Jacobson twin" for both $\varphi\,(s)$ and $\varphi'(s)$, as the following result shows.

Theorem 6.7. *For any $s \in I(\alpha)$, $\{\varphi\,(s), \Phi\,(s)\}$ is a Jacobson pair. Indeed, for $x := (f'e' - s)\,a$, we have $\Phi\,(s) = 1 - bx$, and $\varphi\,(s) = 1 - xb$. Similarly, for $y := b\,(f'e' - s)$, we have $\Phi\,(s) = 1 - ya$ and $\varphi'(s) = 1 - ay$, so $\{\varphi'(s), \Phi\,(s)\}$ is also a Jacobson pair.*

Proof. For x as defined above, we have $\Phi(s) = 1 - b\,(f'e' - s)\,a = 1 - b\,x$, while

$$\begin{aligned} 1 - x\,b &= 1 - (f'e' - s)\,a\,b \\ &= 1 - f'e'\,(1-\alpha) + s\,(1-\alpha) \\ &= 1 - f'\,(1-e) + s - f \quad (\text{since } e'\alpha = 0) \\ &= 1 - f' - f + s + f'e = \varphi(s), \end{aligned}$$

where the last equality follows from equation (1) in §3. Checking that $\{\varphi'(s), \Phi(s)\}$ is a Jacobson pair (arising from a and $y := b\,(f'e' - s)$) is completely similar. □

A very natural and desirable consequence of Theorem 6.7 is now the following.

Corollary 6.8. (Generalized Commutation Rules) *Given a Jacobson pair $\alpha = 1 - ab$, $\beta = 1 - ba$ in R with $\alpha \in \operatorname{reg}(R)$, the following two commutation rules hold for any $s \in I(\alpha)$:*

$$\Phi(s)\,b = b\,\varphi(s), \quad \textit{and} \quad a\,\Phi(s) = \varphi'(s)\,a. \tag{12}$$

Proof. These are just two of the basic commutation rules for the two new Jacobson pairs obtained in Theorem 6.7. (The other two would be $x\,\Phi(s) = \varphi(s)\,x$ and $\Phi(s)\,y = y\,\varphi'(s)$, for any $s \in I(\alpha)$.) □

Another (this time entirely obvious) consequence of Theorem 6.7 is that, if $b \in R$ happens to be a unit, then *up to a conjugation by* b, the new Jacobson map $\Phi : I(\alpha) \to I(\beta)$ is exactly the map φ. (Needless to say, there is also a similar result relating Φ to φ' in the case where $a \in \mathrm{U}(R)$.)

Corollary 6.9. *If $b \in \mathrm{U}(R)$, then $\Phi(s) = b\,\varphi(s)\,b^{-1}$ for every $s \in I(\alpha)$. In particular, taking $b = 1$ shows that $\Phi = \varphi$ as self-maps on $I(\alpha)$, where $\alpha = 1 - a$.*

Making no assumptions on a, b, we have the following further consequence of Theorem 6.7.

Corollary 6.10. *If $s \in I_c(\alpha)$ and $t := \Phi(s)$, we have the "usual" commutation rules $at = sa$ and $tb = bs$. Conversely, if s, t are arbitrary elements in R satisfying $at = sa$ and $tb = bs$, then s commutes with α, and t commutes with β. In particular, we have $\Phi\big(I_c(\alpha)\big) \subseteq I_c(\beta)$.*

Proof. If $s \in I_c(\alpha)$, we have $\varphi(s) = s$ by Proposition 3.4, so the equations (12) boil down to $at = sa$ and $tb = bs$. Conversely, if s and t are any two elements in R such that $at = sa$ and $tb = bs$, then

$$\alpha s = s - abs = s - atb = s - sab = s\alpha,$$

and similarly, $\beta t = t\beta$. Applying the above two facts to $t = \Phi(s)$ where $s \in I_c(\alpha)$, we see that $t \in I_c(\beta)$, so $\Phi\big(I_c(\alpha)\big) \subseteq I_c(\beta)$. □

Using the Corollary above, we can now relate the restricted map $\Phi : I_c(\alpha) \to I_c(\beta)$ to the corresponding map $\Psi : I_c(\beta) \to I_c(\alpha)$.

Theorem 6.11. *For every element $s \in I_c(\alpha)$, we have $\Psi\,\Phi(s) = s$. Consequently, Φ and Ψ are mutually inverse maps on the level of the commuting inner inverses.*

Proof. Writing $t = \Phi(s)$ again, we compute

$$\begin{aligned} \Psi(t) &= 1 + a\,(t + t\,\beta - 1)\,b \\ &= 1 - a\,b + (a\,t)\,(1+\beta)\,b \\ &= \alpha + (s\,a)\,b\,(1+\alpha) \\ &= \alpha + s\,(1-\alpha^2) = s. \end{aligned}$$

By symmetry, we also have $\Phi\,\Psi(t) = t$ for every $t \in I_c(\beta)$, so the second conclusion in the theorem follows. □

As an afterthought, we can also show that Φ satisfies an axiomatic description when we restrict its domain to the commuting inner inverses.

Corollary 6.12. *For $s \in I_c(\alpha)$, $t := \Phi(s)$ is the unique element in $I_c(\beta)$ such that $a\,t = s\,a$.*

Proof. Let $t_0 \in I_c(\beta)$ be such that $at_0 = sa$. Since the calculation in the proof of Theorem 6.11 used only the properties $t \in I_c(\beta)$ and $a\,t = s\,a$, we also have $\Psi\,(t_0) = s$. Now $\Psi\,(t_0) = \Psi\,(t)$ and Ψ is injective on $I_c(\beta)$, so $t_0 = t$. □

Corollary 6.13. *If R is an abelian ring, then $\Phi : I(\alpha) \to I(\beta)$ is a bijection with inverse Ψ.*

Proof. In an abelian ring R, a theorem of Raphael [19] implies that every regular element $\alpha \in R$ commutes with each of its inner inverses. In other words, $I(\alpha) = I_c(\alpha)$. Therefore, the desired result follows from Theorem 6.11. (*Note.* After the von Neumann product operation is introduced on $I(\alpha)$ and $I(\beta)$ in the next section, it will follow from Theorem 7.1 below that, in the abelian case, Φ is a *semigroup isomorphism* from $I(\alpha)$ to $I(\beta)$.) □

To conclude this section, we'll give another application of the main part of Theorem 6.7, which states that $\{\varphi\,(s), \Phi\,(s)\}$ is a Jacobson pair for any $s \in I(\alpha)$. From the definition of φ, we recall that $\varphi\,(s)$ is always an *associate* of s (in the sense that it is an element of R that is equivalent to s). Thus, $\Phi\,(s)$ *is a Jacobson twin of an associate of* s. In particular, if $\mathcal{P}$ is any ring element property that is preserved by taking associates and taking Jacobson twins, then for any regular Jacobson pair $\{\alpha, \beta\}$ and any $s \in I(\alpha)$, s *satisfies the property* $\mathcal{P}$ *if and only if* $\Phi\,(s)$ *does.* From this general observation, we can then come up with a few other nice facts about the mapping Φ. For instance, the ring element property of being *regular* (or being *unit-regular*) is preserved by taking associates, and we also know that it is shared by Jacobson twins. Therefore, by the reasonings given in the remarks above, we have the following result in parallel to the fact (in Theorem 5.3) that Φ preserves and reflects units.

Theorem 6.14. *For any regular Jacobson pair $\{\alpha, \beta\}$, the map $\Phi : I(\alpha) \to I(\beta)$ preserves and reflect regular elements as well as unit-regular elements. In other words, using pre-image notations:*

$$\Phi^{-1}\big(I(\beta) \cap \operatorname{reg}(R)\big) = I(\alpha) \cap \operatorname{reg}(R), \quad \textit{and} \quad \Phi^{-1}\big(I(\beta) \cap \operatorname{ureg}(R)\big) = I(\alpha) \cap \operatorname{ureg}(R).$$

7. Inner Inverses under the von Neumann Product

In this final section, we'll prove the Homomorphism Theorem 7.1 for the new Jacobson mapping Φ with respect to the von Neumann product operation on the sets of inner inverses for any regular Jacobson pair $\{\alpha, \beta\}$. We then conclude our discussions by applying this result to show quickly that Φ preserves and reflects the reflexive inverses. (The same statement then holds also for the left and right Hartwig-Luh mappings, which are special cases of Φ according to Corollary 6.9.) From this overarching viewpoint, the passage of group invertibility and Drazin invertibility from α to β becomes virtually an inescapable conclusion.

For any $\alpha \in \operatorname{reg}(R)$, we define the *von Neumann product* as the following binary operation on $I(\alpha)$:

$$x * y = x\alpha y \qquad (\text{for all } x, y \in I(\alpha)).$$

The fact that $x\alpha y$ belongs to $I(\alpha)$ was essentially noticed (and effectively used) by J. von Neumann ($\alpha(x\alpha y)\alpha = \alpha y\alpha = \alpha$), hence the nomenclature. This binary operation $*$ makes $I(\alpha)$ into a *semigroup*; the associativity of $*$ is just the associativity of the "α-homotope" of the ring R. (In general, however, $(I(\alpha), *)$ is not a monoid.) The following three things are worth noting about the semigroup $(I(\alpha), *)$.

A. We check easily that all products in $(I(\alpha), *)$ land in $I_r(\alpha)$. In particular, $I_r(\alpha)$ is a subsemigroup of $I(\alpha)$.

B. $I_r(\alpha)$ consists of precisely the "idempotents" in the semigroup $(I(\alpha), *)$.

C. $I_c(\alpha)$ is also a subsemigroup of $(I(\alpha), *)$.

Another very nice feature of the Jacobson map Φ is the following remarkable property.

Theorem 7.1. *For every Jacobson pair $\{\alpha, \beta\}$ in R with $\alpha \in \operatorname{reg}(R)$, the new Jacobson map $\Phi : I(\alpha) \to I(\beta)$ is a semigroup homomorphism (with respect to the von Neumann products). In particular, the left and right Hartwig-Luh self-maps φ and φ' on $I(\alpha)$ are semigroup endomorphisms.*

Proof. The second statement follows from the first since φ and φ' are special cases of Φ. For the first statement, we want to check that

$$\Phi\,(s_1 * s) = \Phi\,(s_1) * \Phi\,(s) \quad (\text{for all } s, s_1 \in I(\alpha)). \tag{13}$$

We define e, f, e', f' as usual, and also set $e_1 = \alpha s_1$, $f_1 = s_1\alpha$, $e_1' = 1 - e_1$, and $f_1' = 1 - f_1$. After noting the identity

$$f_1' f = f - s_1\alpha s\alpha = f - f_1 = f_1' - f', \tag{14}$$

we proceed to compute the left-hand side of (13):

$$\begin{aligned} \Phi\,(s_1 * s) &= 1 + b\,(s_1\alpha s)\,a - b\,(1 - s_1\alpha s \cdot \alpha)\,(1 - \alpha \cdot s_1\alpha s)\,a \\ &= 1 + b\,(s_1\alpha s)\,a - b\,(1 - s_1\alpha)\,(1 - \alpha s)\,a \\ &= 1 + bf_1 sa - bf_1' e' a. \end{aligned}$$

Finally, letting $t = \Phi\,(s)$ and $t_1 = \Phi\,(s_1)$, we compute the right-hand side of (13) to be

$$\begin{aligned} \Phi\,(s_1) * \Phi\,(s) &= t_1\beta\, t = (1 + bs_1 a - bf_1' e_1' a)\,\beta\, t \\ &= (1 - ba + bs_1\alpha a - bf_1' e_1' \alpha a)\, t \\ &= t - b\,(1 - s_1\alpha)\, a\, t \quad (\text{since } e_1'\alpha = 0) \\ &= t - bf_1' s\,(1 + \alpha\, e')\, a \quad (\text{by the equations (12) and (2)}) \\ &= t - bf_1' sa - bf_1' f e' a \\ &= 1 + bsa - bf'e'a - bf_1' sa - b\,(f_1' - f')\, e' a \quad (\text{by (14)}) \\ &= 1 + bf_1 sa - bf_1' e' a = \Phi\,(s_1 * s). \end{aligned}$$

□

Using Theorem 7.1, we can establish yet another desirable property of the mapping Φ. The "only if" part of the following result is Theorem 3.4 in the paper of Castro-González, Mendes-Araújo and Patrício [2], but the "if" part seems to be new.

Theorem 7.2. *For any Jacobson pair $\{\alpha, \beta\}$, the map $\Phi : I(\alpha) \to I(\beta)$ preserves and reflects reflexive inverses. More precisely, for any $s \in I(\alpha)$, we have $s \in I_r(\alpha)$ if and only if $\Phi\,(s) \in I_r(\beta)$. In particular, the Hartwig-Luh self-maps on $I_r(\alpha)$ preserves and reflects reflexive inverses.*

Proof. Since semigroup homomorphisms preserve idempotents, and the idempotents in $(I(\alpha), *)$ form the subset $I_r(\alpha)$ (and similarly for $(I(\beta), *)$), Theorem 7.1 implies that $\Phi\,\big(I_r(\alpha)\big) \subseteq \big(I_r(\beta)\big)$.

To prove that Φ *reflects* reflexive inverses, consider any $s \in I(\alpha)$, and let $t = \Phi\,(s)$. Keeping the notations in the proof of Theorem 7.1 and specializing the last display in that proof by setting $s_1 = s$, we have an explicit formula for the element $t\,\beta\, t = \Phi\,(s) * \Phi\,(s)$. Using this formula, we have

$$\begin{aligned} t - t\,\beta\, t &= 1 + bsa - bf'e'a - (1 + bfsa - bf'e'a) \\ &= b\,(s - fs)\, a \;=\; b\,(s - s\,\alpha s)\, a. \end{aligned}$$

This shows once more that $s \in I_r(\alpha) \Rightarrow t \in I_r(\beta)$ (though of course this is not a new proof). However, if we assume instead that $t \in I_r(\beta)$, then the above display gives $b\,(s - s\,\alpha\, s)\, a = 0$. Now $s - s\,\alpha\, s \in \operatorname{Ann}_\ell(\alpha) \cap \operatorname{Ann}_r(\alpha)$, and we know from the proof of Proposition 4.1 that the map sending

$x \in \mathrm{Ann}_\ell(\alpha) \cap \mathrm{Ann}_r(\alpha)$ to $bxa \in \mathrm{Ann}_\ell(\beta) \cap \mathrm{Ann}_r(\beta)$ is a group isomorphism. Thus, we must have $s = s\,\alpha\,s$, which implies that $s \in I_r(\alpha)$, as desired. ☐

Theorem 7.2 above coupled with Corollary 6.10 leads to a quick conceptual proof of the known fact that $\alpha \in \mathrm{sreg}\,(R) \Rightarrow \beta \in \mathrm{sreg}\,(R)$, and in the meantime recaptures the formula relating the group inverses of α and β in [2, Theorem 3.5] and [14, Corollary 2.7].

Corollary 7.3. *If α has a group inverse $\alpha^{\#}$, then β has group inverse $\beta^{\#} = 1 + b\,(\alpha^{\#} - p)\,a$ where p is the spectral idempotent $1 - \alpha^{\#}\alpha$ of α.*

Proof. Since $\alpha^{\#} \in I_c(\alpha) \cap I_r(\alpha)$, applying Corollary 6.10 and Theorem 7.2 shows that β has group inverse $\Phi\,(\alpha^{\#})$, which is by definition the element given in this Corollary (noting that the spectral idempotent p of α is *exactly* what we have denoted by $e' = f'$). ☐

Assuming an elementary result from [2], we can also quickly deduce the following known result, where the element α is now assumed to be Drazin invertible instead of "regular".

Theorem 7.4. *If $\{\alpha, \beta\}$ are a Jacobson pair and α is Drazin invertible, then so is β.*

Proof. By [5], α^n is group invertible for some $n \geq 1$. As $\{\alpha^n, \beta^n\}$ is also a Jacobson pair according to [2, Lemma 2.3] (cf. [14, Theorem 3.7]), Corollary 7.3 above implies that β^n is group invertible. By [5] again, β is Drazin invertible. (Of course, to compute explicitly the Drazin inverse of β from that of α would have required additional work such as done, e.g. in [2, Theorem 3.6], or [14, Theorem 2.1]. ☐

While the map Φ behaves pretty well on the sets $I_u(\alpha)$, $I_c(\alpha)$ and $I_r(\alpha)$ (thus preserving group inverses if they exist), it is in general neither injective nor surjective. This is clearly shown by the following example, the explicit nature of which also makes it useful for checking the truth or falsehood of any other claims about the relationship between α and β.

Example 7.5. Let k be a commutative ring, $R = \mathbb{M}_2(k)$, $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $b = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$\alpha = 1 - ab = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = 1 - ba = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are both idempotents. By straightforward computations, we can easily determine the following sets:

$$\begin{array}{ll}
\mathrm{IAnn}\,(\alpha) = \left\{ \begin{pmatrix} x & y \\ z & z \end{pmatrix} \,:\, x, y, z \in k \right\}, & \mathrm{IAnn}\,(\beta) = \left\{ \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \,:\, x, y, z \in k \right\}, \\
I(\alpha) = \left\{ \begin{pmatrix} x & y \\ z & 1+z \end{pmatrix} \,:\, x, y, z \in k \right\}, & I(\beta) = \left\{ \begin{pmatrix} x & y \\ z & 1 \end{pmatrix} \,:\, x, y, z \in k \right\}, \\
I_c(\alpha) = \left\{ \begin{pmatrix} x & x-1 \\ 0 & 1 \end{pmatrix} \,:\, x \in k \right\}, & I_c(\beta) = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \,:\, x \in k \right\}, \\
I_r(\alpha) = \{ s \in I(\alpha) \,:\, \det(s) = 0 \}, & I_r(\beta) = \{ s \in I(\beta) \,:\, \det(s) = 0 \}, \\
I_u(\alpha) = \{ s \in I(\alpha) \,:\, \det(s) \in \mathrm{U}(k) \}, & I_u(\beta) = \{ s \in I(\beta) \,:\, \det(s) \in \mathrm{U}(k) \}.
\end{array}$$

By a quick computation, the "original" Jacobson map from $I(\alpha)$ to $I(\beta)$ is given by

$$s = \begin{pmatrix} x & y \\ z & 1+z \end{pmatrix} \in I(\alpha) \;\mapsto\; 1 + b\,s\,a = \begin{pmatrix} \mathrm{tr}\,(s) & 0 \\ 0 & 1 \end{pmatrix} \in I(\beta).$$

Clearly, this map fails to preserve (or reflect) units, reflexive inverses, regular inverses, and (unit) regular inner inverses. On the other hand, the *new* Jacobson map turns out to be

$$\Phi : s = \begin{pmatrix} x & y \\ z & 1+z \end{pmatrix} \in I(\alpha) \;\mapsto\; \Phi\,(s) = \begin{pmatrix} \det\,(s) & 0 \\ 0 & 1 \end{pmatrix} \in I(\beta).$$

From this, we can see directly that Φ has all the properties proved in the preceding sections. However, Φ is in general neither injective not surjective, even if we restrict it to (say) $I_u(\alpha) \to I_u(\beta)$. Indeed, we have here

$$\Phi\big(I(\alpha)\big) = \Phi\big(I_c(\alpha)\big) = I_c(\beta), \qquad \Phi\big(I_r(\alpha)\big) = \{\beta\},$$

and $\Phi(I_u(\alpha))$ is the set of all diagonal matrices $\operatorname{diag}(\mathrm{U}(k), 1)$. In particular, we see (from the first equation above) that Φ does not reflect commuting inner inverses (unless $k = \{0\}$).

Is there a "better" Jacobson map here? Yes, for example the map

$$s = \begin{pmatrix} x & y \\ z & 1+z \end{pmatrix} \in I(\alpha) \;\mapsto\; t = \begin{pmatrix} x+z & 1+y-x \\ z & 1 \end{pmatrix} \in I(\beta)$$

is a *bijection,* and $\det(t) = \det(s)$ so it preserves (and reflects) unit (and other kinds of) inner inverses too. This map is obtained by simply using the conjugation relation $\beta = u\alpha u^{-1}$, where $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. This bijection is, however, not "canonical"; e.g. we could have used many other conjugations too.

Question. *Let R be a ring and let $\{\alpha, \beta\}$ be a regular Jacobson pair in R. Is there a canonical bijection $I(\alpha) \to I(\beta)$ which preserves units, reflexive inverses, commuting inner inverses, etc., and satisfies a reasonable pair of commutation laws?*

Even if there is no such canonical bijection, we propose the following:

Conjecture. For any Jacobson pair $\{\alpha,\, \beta\}$, there is a natural bijection between their sets of unit inner inverses $I_u(\alpha)$ and $I_u(\beta)$.

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Department of Mathematics, University of California, Berkeley, CA 94720, United States
E-mail address: lam@math.berkeley.edu

Department of Mathematics, Brigham Young University, Provo, UT 84602, United States
E-mail address: pace@math.byu.edu