MCCOY RINGS AND ZERO-DIVISORS

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ABSTRACT. We investigate relations between the McCoy property and other standard ring theoretic properties. For example, we prove that the McCoy property does not pass to power series rings. We also classify how the McCoy property behaves under direct products and direct sums. We prove that McCoy rings with 1 are Dedekind finite, but not necessarily Abelian. In the other direction, we prove that duo rings, and many semi-commutative rings, are McCoy. Degree variations are defined, studied, and classified. The McCoy property is shown to behave poorly with respect to Morita equivalence and (infinite) matrix constructions.

1. Introduction and Definitions

N. H. McCoy proved in 1942 [12] the now folklore result that if two polynomials annihilate each other over a commutative ring then each polynomial has a non-zero annihilator in the base ring. Following a suggestion by T. Y. Lam, the second author made the following definition.

Definition 1.1. A ring R is said to be *right McCoy* (respectively *left McCoy*) if for each pair of non-zero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$ then there exists a non-zero element $r \in R$ with $f(x)r = 0$ (respectively $rg(x) = 0$). A ring is *McCoy* if it is both left and right McCoy.

Thus N. H. McCoy's result states (in modern terminology) that commutative rings are McCoy. There are many ways to generalize his theorem. The following zero-divisor conditions are all standard, and we direct the reader to the excellent papers [1], [10], and [11] for a nice introduction to these topics.

Definition 1.2. A ring R is *Armendariz* if given polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$ then $ab = 0$ for each coefficient a of $f(x)$ and b of $g(x)$. Let S_n be the group of permutations on n-elements. A ring R is

$$\begin{aligned}
&\textit{reduced} && \text{if } a^2 = 0 \Longrightarrow a = 0, \text{ for all } a \in R,\\
&\textit{symmetric} && \text{if } a_1a_2\cdots a_n = 0 \Longrightarrow a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)} = 0,\\
& && \text{for all } n \in \mathbb{N},\, a_i \in R, \sigma \in S_n,\\
&\textit{reversible} && \text{if } ab = 0 \Longrightarrow ba = 0, \text{ for all } a, b \in R,\\
&\textit{semi-commutative} && \text{if } ab = 0 \Longrightarrow aRb = 0, \text{ for all } a, b \in R.
\end{aligned}$$

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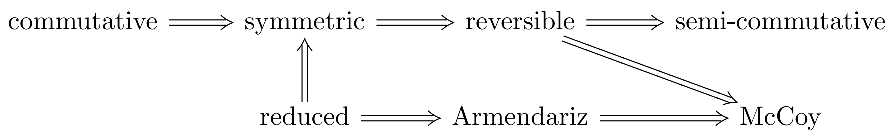
The following diagram shows all implications among these properties (with no other implications holding, except by transitivity):



Note that some authors define symmetric rings by $abc = 0 \implies bac = 0$. This definition is equivalent to ours in rings with 1, but in general is strictly a weaker property, and not even left-right symmetric, so we eschew that definition. In particular, under our definition, the implication "symmetric $\implies$ reversible" holds even for rings without 1.

All of the implications given above are straightforward to prove, except "reversible $\implies$ McCoy" which is [14, Theorem 2]. The proof was stated in the context of rings with 1, but all results from that paper hold for general rings. The non-existence of additional implications in our diagram takes a little work. Most are found in [11], while an example of a semi-commutative, non-McCoy ring is given in [14]. We will give an example of an Armendariz ring which is not semi-commutative later in the paper.

Throughout, by the word "ring" we mean an associative ring, possibly without 1, and by $\mathbb{N}$ we mean the non-negative integers. In Section 2 we discuss the Diamond Lemma, which is machinery to simplify computations in quotients of free algebra. Section 3 uses this machinery to prove that power series rings over McCoy rings need not be McCoy. In Section 4 we classify exactly when direct products and sums are McCoy. Sections 5 and 6 delve into the situation when degree conditions accompany the McCoy condition. We ultimately prove that there are no relations amongst these properties, except trivial ones. We introduce a few more common ring-theoretic conditions in Sections 7 through 9 and investigate their relationships to the McCoy property. We prove in Section 10 that matrix constructions, and more generally Morita invariance, behaves poorly with regards to the McCoy condition. In the last section we end with a summary of the results, in the form of a diagram of implications, and raise a few open questions.

2. The Diamond Lemma and Notations

The Diamond Lemma in ring theory provides sufficient conditions for a list of reductions on monomials to result in a unique normal form for elements of a ring. Since this lemma is essential to most of the results in this paper, we supply the version we will use. There are, of course, more general statements in the literature.

Let k be a commutative ring with 1, let X be a set, and let $R = k\langle X\rangle$ be the free associative algebra over k in the letters from X. By a *semigroup partial ordering* $\leqslant$ on a semigroup A, we mean that if $a, b, b', c \in A$ and $b \leqslant b'$ then $abc \leqslant ab'c$. A partial ordering has the *descending chain condition* if all descending chains stabilize. We will in practice take $A = \langle X\rangle$, which is the free semigroup of words on the letters in X (including the empty word, 1). There is a natural grading on elements of $\langle X\rangle$ given by counting (with multiplicities) the number of variables which appear in a given monomial. (This grading is sometimes called the "degree" of a monomial, but we reserve that word throughout for the degree of polynomials.) In this case we usually take $\leqslant$ to be ordering by grade on monomials, and for monomials of the same grading we use some lexicographical ordering on the elements of X.

By a *reduction system* for R, we mean a set of pairs $S = \{(m_i, n_i), \mid i \in I\}$ indexed by some set I, where $m_i \in \langle X \rangle$ is a monomial, and $n_i \in R$, for each $i \in I$. If effect, the reduction system tells us that we can replace each occurrence of the monomial m_i with the element n_i. We say S is *compatible* with a partial ordering if, for each $i \in I$, each monomial m in the support of n_i satisfies $m < m_i$. Given two monomials $a, b \in \langle X \rangle$, and an element $(m_i, n_i) \in S$, we can define a map $r_{am_ib} : R \to R$, which acts on the k-basis $\langle X \rangle$ by fixing all basis elements, except the map sends am_ib to an_ib. Such maps are called *reductions*, and we say an element of R is *reduced* under the reduction system S if every reduction fixes that element. An element $s \in R$ is *reduction finite* if for every sequence of reductions $r_1, r_2, r_3, \ldots$ the sequence $r_1(s), r_2r_1(s), r_3r_2r_1(s), \ldots$ stabilizes (to a reduced element).

Lemma 2.1 (Bergman's Diamond Lemma, cf. [2])**.** *Let k be a commutative ring with 1, let X be a set, and let $R = k\langle X \rangle$ be the free associative algebra over k. Let $\leqslant$ be a semigroup partial ordering on $\langle X \rangle$, having the descending chain condition. Let $S = \{(m_i, n_i)\}_{i \in I}$ be a reduction system compatible with $\leqslant$. Consider the following two conditions:*

1. *(Overlaps are resolvable.) If $a, b, c \in \langle X \rangle$, and there exist $i, j \in I$ with $ab = m_i$ and $bc = m_j$, then there exists a sequence of reductions sending n_ic and an_j to a common element.*
2. *(Inclusions are resolvable.) If $a, b, c \in \langle X \rangle$, and there exist $i, j \in I$ with $abc = m_i$ and $b = m_j$, then there exists a sequence of reductions sending n_i and an_jc to a common element.*

If the two conditions occur then every element of R is reduction finite with respect to S. Moreover, let J be the ideal generated by the relations $\{m_i = n_i\}_{i \in I}$. Each element in R/J can be uniquely written in the form $r + J$, where r is the unique element in the coset $r + J$ which is reduced with respect to S. The representation can be found by reducing any element in the coset $r + J$.

Whenever we have a quotient R/J, where $R = k\langle X \rangle$ is as above and $J \cap k = (0)$, we abuse notation and identify the elements of X with their images in the quotient. We will either write $J = (\{m_i = n_i\}_{i \in I})$ where $S = \{(m_i, n_i) \mid i \in I\}$ is a reduction system satisfying the conditions of the Diamond Lemma, or will explicitly tell the reader what reductions to make. The semigroup partial ordering is usually clear from context.

Example 2.2. We will never write $\mathbb{Z}\langle a, b, c \rangle/(ab = 1, bc = 0)$ because the system $S = \{(ab, 1), (bc, 0)\}$ doesn't satisfy condition (1) of the lemma. (Try reducing abc in two different ways. The overlap is not resolvable using only the system S.) Instead, we will write $\mathbb{Z}\langle a, b, c \rangle/(ab = 1, c = 0)$, because $T = \{(ab, 1), (c, 0)\}$ is a reduction system satisfying the two conditions of the Diamond Lemma. This means that any word can be put into normal form by (repeatedly) replacing any occurrence of ab by 1 and c by 0.

3. Power Series

By a nice specialization argument, one can show that the polynomial ring over a McCoy ring is always McCoy. While preparing this paper we were introduced to three independent manuscripts proving this fact (see [4], [7], or [15]), so we do not include the proof here. Similarly, one might ask if the power series ring is McCoy, over a McCoy ring. In this section we construct an example giving a negative answer. We begin by setting

$$R = K\langle a_i,\, b_i,\, c_i,\, d_i \,|\, i \in \mathbb{N}\rangle$$

with K an arbitrary field. Since we want the power series ring over R not to be McCoy we first need two non-zero polynomials in $R[\![t]\!]$ which multiply to zero. Setting

$$f(x) = \sum_{i=0}^{\infty} a_i t^i + \sum_{i=0}^{\infty} b_i t^i x,\;\; g(x) = \sum_{i=0}^{\infty} c_i t^i + \sum_{i=0}^{\infty} d_i t^i x \in R[\![t]\!][x]$$

we wish to force $f(x)g(x) = 0$, and so take I_0 to be the ideal generated by the relations

$$\sum_{i=0}^{n} a_i c_{n-i} = 0,\;\; \sum_{i=0}^{n} (a_i d_{n-i} + b_i c_{n-i}) = 0,\;\; \sum_{i=0}^{n} b_i d_{n-i} = 0$$

for each $n \in \mathbb{N}$. (A reduction system will be given below.) Fix $R_0 = R/I_0$, and equate the variables with their images in this ring. We might naively suppose that R_0 is the ring we are looking for. However, one can check that the polynomial $a_0 + b_0 x$ is annihilated on the right by $c_0 + d_0 x$, and is not annihilated by any element of R_0. So, at the very least, we need to make sure a_0 and b_0 have a common right annihilator.

Let F_0 be the set of all finite subsets of variables in R_0. For every set $S \in F_0$, adjoin two new variables x_S and y_S to R_0 and let I_1 be the ideal generated by the relations:

$$x_S a_i = x_S b_i = c_i y_S = d_i y_S = 0,\; \forall\, i \in \mathbb{N}$$
$$x_S s = s y_S = 0,\; \forall\, s \in S.$$

Now we construct the ring

$$R_1 = \mathbb{F}_2\langle a_i, b_i, c_i, d_i, x_S, y_S \,|\, i \in \mathbb{N},\; S \in F_0\rangle / I_0 \cup I_1.$$

To see that the ring R_0 sits inside R_1, one can (momentarily) specialize all the new variables x_S and y_S to zero.

Repeat the construction inductively, taking finite subsets of all the variables (including the new ones) and adjoining two new variables for each such subset, using exactly the same relational equations as above. Finally take $R_\infty = \cup_i R_i$. Notice that each of the ideals I_i is homogeneous, and hence we may grade the monomials in R_∞. Further, since the construction above is left-right symmetric, it suffices to deal with the case of the right McCoy property.

Lemma 3.1. *Monomials in the ring R_∞, above, can be put into normal form by replacing all occurrences of $a_0 c_n$ by $-\sum_{i=1}^{n} a_i c_{n-i}$, all occurrences of $a_0 d_n$ by $-\sum_{i=1}^{n} a_i d_{n-i} - \sum_{i=0}^{n} b_i c_{n-i}$, all occurrences of $b_0 d_n$ by $-\sum_{i=1}^{n} b_i d_{n-i}$, and finally by removing any monomial containing a product of two letters which have been chosen to annihilate each other.*

Proof. Apply the Diamond Lemma. Note that all reductions involve monomials of grade 2, so condition (2) of the lemma is automatically satisfied vacuously. Also, it is clear that if we limit ourselves to the relations in I_0, condition (1) is satisfied vacuously. If we limit ourselves to the relations in $\cup_{i>0} I_i$, all such reductions immediately result in zero, so this case presents no problems. Thus, we only need consider the interaction between the relations in I_0 and the relations in $\cup_{i>0} I_i$. But we chose x_S to annihilate each a_i and b_i on the left. So, for example, monomials such as $x_S a_0 c_n$ still uniquely reduce to 0 (even if we first replace $a_0 c_n$ by $-\sum_{i=1}^{n} a_i c_{n-i}$). Similarly, the variables y_S were chosen to annihilate each c_i and d_i on the right. Thus, it is an easy exercise to verify that condition (1) in the Diamond Lemma also holds. □

Proposition 3.2. *The ring R_∞, above, is right McCoy.*

Proof. Fix $p(x), q(x) \in R_\infty \setminus \{0\}$ with $p(x)q(x) = 0$. Write $p(x) = \sum_{i=0}^{m} p_i x^i$ and $q(x) = \sum_{j=0}^{n} q_j x^j$ where we may assume $p_0, q_0 \neq 0$, dividing by x if necessary, and also we may assume that the coefficients are written in normal form. If each of the coefficients of $p(x)$ consists of sums of monomials of grading larger than 0 (i.e. none of the coefficients have 1 in their supports) then since there are only finitely many coefficients and only finitely many monomials in each coefficient, our inductive construction implies the existence of a variable y_S which annihilates all these coefficients on the right. Further, since all the ideals we quotient by to obtain the ring R_∞ consist of sums of monomials of grade 2, $y_S \neq 0$. Since $p(x)y_S = 0$ we are done in this case.

So we may assume 1 occurs in the support of p_k (with $k \leqslant m$ minimal). Further, we can assume $\ell \leqslant n$ is minimally chosen so that q_ℓ has a non-zero monomial in its support of smallest possible grade. Set q'_ℓ equal to the sum of the monomials of smallest grade in q_ℓ. If we now compute the $(k+\ell)$-degree coefficient of $p(x)q(x)$, we have

$$\sum_{(u,v)\,:\,u+v=k+\ell} p_u q_v = 0.$$

Due to the fact that S is the quotient of a free algebra by a homogeneous ideal, we know that the monomials of any fixed grade in the previous equation must sum to zero. But this implies $1 \cdot q'_\ell = 0$, a contradiction. Hence, S is right McCoy in any case. □

Proposition 3.3. *The ring $T = R_\infty[\![t]\!]$, with R_∞ as above, is not right McCoy.*

Proof. Take $\alpha = \sum_{i=0}^{\infty} a_i t^i$, $\beta = \sum_{i=0}^{\infty} b_i t^i$, $\gamma = \sum_{i=0}^{\infty} c_i t^i$, and $\delta = \sum_{i=0}^{\infty} d_i t^i$. The relations in I_0 were chosen so that $f(x)g(x) = 0$ with $f(x) = \alpha + \beta x$ and $g(x) = \gamma + \delta x$. Further note that $f(x), g(x) \neq 0$, since no reduction involves monomials of grade less than 2.

Assume by contradiction that T is right McCoy, so there exists some non-zero power series $\zeta = \sum_{i=0}^{\infty} z_i t^i \in T$ with $f(x)\zeta = 0$. Dividing by t if necessary, we may assume $z_0 \neq 0$. We also write all coefficients in normal form. Since $z_0 \neq 0$, it can only annihilate finitely many of the a_i and b_i from the right. Thus there exists some index n with a_n and b_n not annihilated on the right by any of the monomials occurring in z_0. But from $\alpha\zeta = 0$, we need

$$\sum_{i=0}^{n} a_i z_{n-i} = 0.$$

Writing everything in normal form implies that $a_n z_0$ (but not $b_n z_0$) must occur in the support of $a_0 z_n$ (after it is reduced), and hence z_n has in its support a monomial, z_n', beginning with c_n. But from the equation $\beta\zeta = 0$, we have

$$\sum_{i=0}^{n} b_i z_{n-i} = 0$$

which is impossible, because no reduction will cancel out the monomial $b_0 z_n'$. □

The argument given in the last paragraph of Proposition 3.2 will be used throughout this paper, and thus bears repeating and generalizing.

Definition 3.4. An $\mathbb{N}$-*graded ring* R is a ring where as an abelian additive group $R = \bigoplus_{n\in\mathbb{N}} I_n$, and multiplicatively $I_m I_n \subseteq I_{m+n}$ for each $m, n \in \mathbb{N}$. The set I_n is called the grade n component of R.

Let R be an $\mathbb{N}$-graded ring and fix $n \in \mathbb{N}$. Given an element $r \in R$ we write r_n for the grade n component of r. We further can grade $R[x]$ by letting x have grade 0.

Lemma 3.5. *Let R be an $\mathbb{N}$-graded ring, and let $f(x)$, $g(x) \in R[x]$ be non-zero polynomials with $f(x)g(x) = 0$. If we choose k and ℓ to be minimal so that $f(x)_k$, $g(x)_\ell \neq 0$, then $f(x)_k g(x)_\ell = 0$. In particular, the first non-zero coefficient of $f(x)_k$ annihilates the first non-zero coefficient of $g(x)_\ell$, and hence is a left zero-divisor.*

Proof. In the product $f(x)g(x) = 0$, each graded component is zero, But the grade $(k+\ell)$ component is exactly $f(x)_k g(x)_\ell$ by minimality, finishing the theorem. □

In practice, the ring R will be a quotient of a free K-algebra, over a field K, by an ideal consisting of homogeneous relations, and the $\mathbb{N}$-grading on R will be given by grading on monomials. In this situation, the grade 0 component of R has no zero-divisors, so $f(x)_0 = 0 = g(x)_0$. Also note that we could have worked with maximal indices.

4. Direct Products and Sums

One can describe exactly when a direct product or direct sum is right McCoy. First we need a new definition. Call a ring *right finite annihilated* (RFA) if every finite subset has a non-zero right annihilator. For example, rings with 1 (including the zero ring) are never RFA.

Lemma 4.1 (cf. [15]). *A direct product of rings $R = \prod_{i\in I} R_i$ is right McCoy if and only if either one of the rings is RFA, or all the rings are right McCoy.*

Proof. Let

$$f(x) = (f_i(x))_{i\in I},\ g(x) = (g_i(x))_{i\in I} \in \prod_{i\in I} R_i[x] = R[x]$$

be non-zero polynomials with $f(x)g(x) = 0$. Suppose each ring R_i is right McCoy. Since $g(x) \neq 0$ there exists some index $i_0 \in I$ with $g_{i_0}(x) \neq 0$. In particular, there exists some non-zero $r_{i_0} \in R_{i_0}$ with $f_{i_0}(x) r_{i_0} = 0$ by the McCoy property (unless $f_{i_0}(x) = 0$, in which case take r_{i_0} to be any non-zero coefficient of $g(x)$). On the other hand, suppose for some $i_0 \in I$ that R_{i_0} is a RFA ring. In this case there is again some non-zero element $r_{i_0} \in R_{i_0}$ which annihilates $f_{i_0}(x)$ on the right. In any case, let r be the sequence with r_{i_0} in the i_0th coordinate, and zeros elsewhere. Clearly $f(x)r = 0$, and $r \neq 0$, so R is right McCoy.

Conversely, suppose R is right McCoy, and assume none of the rings R_i is RFA. For each $i \in I$, fix a polynomial $f_i(x) \in R_i[x]$ whose coefficients do not have a simultaneous non-zero right annihilator in R_i. Fix $i_0 \in I$, and suppose $p(x)q(x) = 0$ holds for non-zero polynomials $p(x), q(x) \in R_{i_0}$. Let $P(x) \in R[x]$ be the sequence with $p(x)$ in the i_0th coordinate, and f_i in the ith coordinate for each $i \neq i_0$. Let $Q(x) \in R[x]$ be the sequence with $q(x)$ in the i_0th coordinate, and zeros elsewhere. Clearly $P(x)Q(x) = 0$ and $P(x), Q(x) \neq 0$. Since R is right McCoy, there exists a non-zero element $r = (r_i)_{i\in I} \in R[x]$ with $P(x)r = 0$. In particular, $f_i(x)r_i = 0$ for $i \neq i_0$, and because of how we chose $f_i(x)$ we have $r_i = 0$ for $i \neq i_0$. But $r \neq 0$ and hence $r_{i_0} \neq 0$, with $p(x)r_{i_0} = 0$. This proves that R_{i_0} is right McCoy. Since $i_0 \in I$ was arbitrary, we are done. □

Corollary 4.2. *If a direct product of rings is right McCoy, then one of the factors is right McCoy.*

Proof. This follows from the fact that RFA rings are tautologically right McCoy. □

The case for direct sums turns out to be quite different. First, we may assume that none of the rings in our sum is the zero ring, since they make no contributions. Second, we may assume our sum is infinite, otherwise we reduce to the direct product. In this case, we have the following (perhaps surprising) result:

Proposition 4.3. *If I is an infinite set, and R_i is a non-zero ring for each $i \in I$, then the ring $R = \bigoplus_{i\in I} R_i$ is right McCoy.*

Proof. Given any polynomial $f(x) \in R[x]$, there exists some coordinate $i_0 \in I$ such that $f(x)$ is zero in the i_0-coordinate. Fix $r_{i_0} \in R_{i_0} \setminus \{0\}$, and take r to be the sequence with r_{i_0} in the i_0th coordinate and zero elsewhere. Clearly $f(x)r = 0$ and $r \neq 0$. In fact, this proof can easily be generalized to show R is an RFA ring. □

This proposition implies that there is no nontrivial ring theoretic property $\mathcal{P}$ forced upon a McCoy ring, if $\mathcal{P}$ is inherited by summands, and if the zero ring is not the only ring without $\mathcal{P}$. For example, we could take $\mathcal{P}$ to be semi-commutativity. By taking an infinite direct sum of rings which are not semi-commutative, we arrive at a ring (without 1) which is McCoy but not semi-commutative. In fact, any nontrivial property expressible in terms of equations which can be checked componentwise, and not necessitating the existence of 1, will necessarily not hold for some McCoy ring.

One might interpret this fact as saying that the McCoy property is ill-behaved for general rings, and we should restrict our attention to the case of rings with 1 when looking for necessary conditions. In fact, if we do restrict ourselves to rings with 1, we immediately have the following nice facts.

Theorem 4.4. *Let I be an indexing set, and for each $i \in I$ let R_i be a ring with 1. The direct product ring $\prod_{i\in I} R_i$ is (right) McCoy if and only if each R_i is (right) McCoy.*

Proof. Follows from what was done above, and the fact that rings with 1 are never RFA. □

Example 4.5. Let $R = K\langle a, b, c, d\rangle/(ac = 0, ad = c, bc = d, bd = 0)$, where K is any ring with 1. If we define polynomials $f(x) = a - x + bx^2$ and $g(x) = c + dx$, then $f(x)g(x) = 0$. A straightforward application of the Diamond lemma allows us to see that $f, g \neq 0$. Further, since only 0 can annihilate 1, it must be the case that $\mathrm{ann}_r^{R[x]}(f(x)) \cap R = (0)$, and so R is not right McCoy. Further, R cannot embed *unitally* into a right McCoy ring, for the same reason.

5. Linearly McCoy Rings

If one wants two non-zero polynomials to annihilate each other, and also for one of the polynomials to have a unit as a coefficient, the previous example involves polynomials of minimal degree. However, if one studies the previous example carefully, and merely desires that all unital embeddings are non-McCoy, the construction can be generalized. We might ask whether the degree on $f(x)$ is minimal in this case. Perhaps surprisingly, the answer is no. We could have used the slightly easier example

$$R = K\langle u, v\rangle/(uv = 1)$$

and taken the polynomials to be $f(x) = u + (1 - vu)x$, $g(x) = (1 - vu) - v(1 - vu)x$.

Similarly, we might ask if the McCoy condition can be checked in some minimal degree. Admittedly, this question is loosely stated, and needs clarification. We begin by making the following definition:

Definition 5.1. A ring R is said to be *right linearly McCoy* if given non-zero linear polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, then there exists a non-zero element $r \in R$ with $f(x)r = 0$.

A ring is commonly called *Dedekind finite* (or directly finite) if $uv = 1$ implies $vu = 1$. A ring is said to be *Abelian* if all idempotents are central. Semicommutative rings are Abelian, and Abelian rings are Dedekind finite, but neither implication is reversible in general. The first example of this section leads us to the following nice necessary condition for a ring with 1 to be right linearly McCoy.

Theorem 5.2. *If R is a right linearly McCoy ring with 1, then R is Dedekind finite.*

Proof. If R is not Dedekind finite, then there exist $u, v \in R$ with $uv = 1$ but $vu \neq 1$. Taking $f(x) = u + (1 - vu)x$, $g(x) = (1 - vu) - v(1 - vu)x$ we have $f(x)g(x) = 0$ and $f(x), g(x) \neq 0$. But $f(x)$ has no non-zero right annihilator in R. □

Now, fix two positive integers $m, n \geqslant 1$, and form the universal ring

$$R_{m,n} = K\langle a_i,\, b_j \mid i \leqslant m,\, j \leqslant n\rangle/I$$

where K is a field, and I is the (minimal) ideal of relations forcing $f(x)g(x) = 0$, with $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$. More concretely, we can write a reduction system by taking $a_m b_j = -\sum_{i<m} a_i b_{j+m-i}$ for each $j \leqslant n$, and $a_\ell b_0 = -\sum_{i<\ell} a_i b_{\ell-i}$ for each $\ell \leqslant m$. (Note: We set $b_j = 0$ if j is negative or greater than n.) There are no overlaps or inclusions, so the Diamond Lemma holds vacuously. This ring is not right McCoy, since $\mathrm{ann}_r(a_0) = b_0 R$, $\mathrm{ann}_r(a_m) = b_n R$, and $b_0 R \cap b_n R = (0)$.

One might ask whether the condition of being right linearly McCoy is actually weaker than being right McCoy. The natural example to first consider is the ring $R_{m,n}$, but there $a(x) = a_0 + a_0 b_n a_m x$ and $b(x) = b_0 - b_n a_m b_0 x$ annihilate each other. A straightforward use of the Diamond Lemma demonstrates $\mathrm{ann}_r(a_0 b_n a_m) = b_n R$. Thus, $R_{m,n}$ is not right linearly McCoy (for any m, n).

One could attempt to modify the ring above, forcing all linear polynomials with non-zero right annihilators to contain right annihilators over R, and try to prove that the resulting ring is still not right McCoy. In fact, we will do something of the sort in a later section. But for now there is a short-cut.

Proposition 5.3. *All semi-commutative rings are right linearly McCoy.*

Proof. Let R be semi-commutative, and take two non-zero linear polynomials

$$f(x) = a_0 + a_1 x,\ g(x) = b_0 + b_1 x \in R[x]$$

with $f(x)g(x) = 0$. If $a_0 = 0$ then each coefficient of $g(x)$ will annihilate $f(x)$ on the right. So we may assume $a_0 \neq 0$. If $b_0 = 0$, we can divide $g(x)$ by x if necessary, and assume $b_0 \neq 0$. Now, if $f(x)b_0 = 0$ then we are done, so we may assume $a_1 b_0 \neq 0$.

The equation $f(x)g(x) = 0$ implies $a_0 b_0 = 0$, so by semi-commutativity $a_0 a_1 b_0 = 0$, and similarly $a_1 a_0 b_1 = 0$. The linear term in $f(x)g(x) = 0$ gives the relation $a_1 b_0 + a_0 b_1 = 0$, so multiplying on the left by a_1 yields

$$0 = a_1(a_1 b_0 + a_0 b_1) = a_1^2 b_0 + a_1 a_0 b_1 = a_1^2 b_0.$$

In particular $f(x)(a_1 b_0) = 0$. □

By the main result of [14], there exists a semi-commutative ring, hence right linearly McCoy ring, which is not right McCoy. In that example, the construction relies on the existence of polynomials of degree three and one (respectively) which annihilate each other. One wonders if a simpler example exists, and we now develop the machinery to show the answer is no.

Lemma 5.4. *Let R be a semi-commutative ring, let $m, n \in \mathbb{N}$, and let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$. If there exists $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ with $f(x)g(x) = 0$ then $a_0^{n+1} g(x) = 0$ and $a_m^{n+1} g(x) = 0$.*

Proof. This is a restatement of [14, Lemma 1]. For completeness we include the proof here.

First note that by setting $f^*(x) = x^m f(x^{-1})$ and $g^*(x) = x^n g(x^{-1})$, we have just "reversed" the coefficients on the polynomials and obtain the equation $f^*(x)g^*(x) = 0$. To finish the lemma it suffices to prove $a_0^{n+1} g(x) = 0$, because we then obtain $a_m^{n+1} g^*(x) = 0$ by symmetry, and this last equations is equivalent to $a_m^{n+1} g(x) = 0$.

Clearly $a_0 b_0 = 0$. Assume by induction that $a_0^{\ell+1} b_\ell = 0$ for all $\ell < j$. Looking at the degree j coefficient of the equation $f(x)g(x) = 0$ yields

$$\sum_{i=0}^{j} a_i b_{j-i} = 0.$$

Multiplying on the left by a_0^j, we have

$$0 = \sum_{i=0}^{j} a_0^j a_i b_{j-i} = a_0^{j+1} b_j + \sum_{i=1}^{j} a_0^j a_i b_{j-i} = a_0^{j+1} b_j$$

with the last equality following from the semi-commutativity property, and the induction hypothesis. Therefore $a_0^{j+1} b_j = 0$, and by induction we are done. □

Theorem 5.5. *Let R be a semi-commutative ring, and let $f(x)$, $g(x) \in R[x]$ be non-zero polynomials satisfying $f(x)g(x) = 0$. If $\operatorname{ann}_r^{R[x]}(f(x)) \cap R = (0)$, then $\deg(f(x)) > 2$.*

Proof. We will prove the contrapositive statement. Let $R' = \mathbb{Z} \oplus R$ be the ring obtained by adjoining an identity to R (if needed) in the usual way. Set $f(x) = a_0 + a_1x + a_2x^2$. After dividing $g(x)$ by a power of $x - 1$ (over R') if necessary, we may assume $g(1) \neq 0$. Note that $g(1)$ belongs to the left ideal generated by the b_i (even if we first had to divide by a power of $x - 1$ in R'), and in particular $g(1) \in R$. By Lemma 5.4, and using semi-commutativity of R, there exists some maximal integer $n_1 \geqslant 0$ so that $a_0^{n_1}g(1) \neq 0$. By the same lemma, and using semi-commutativity, there exists some maximal integer $n_2 \geqslant 0$ so that $a_2^{n_2}a_0^{n_1}g(1) \neq 0$. Since $f(1)g(1) = 0$, which is an equation in R, by semi-commutativity

$$(a_0 + a_1 + a_2)a_2^{n_2}a_0^{n_1}g(1) = 0.$$

Thus $a_ia_2^{n_2}a_0^{n_1}g(1) = 0$ for each $i \leqslant 2$, and hence $f(x)a_2^{n_2}a_0^{n_1}g(1) = 0$. □

6. Degree Considerations

We saw in the last section that semi-commutative rings are linearly, and even quadratically, McCoy, but not in general fully McCoy. This leads us to the following definition.

Definition 6.1. Let R be a ring and fix positive integers $m, n \geqslant 1$. We say that R is (m,n)-*right McCoy* if for each pair of non-zero polynomials $f(x), g(x) \in R[x]$ the conditions $f(x)g(x) = 0$, $\deg(f) = m$, and $\deg(g) = n$ imply there exists some non-zero $r \in R$ with $f(x)r = 0$.

If R is (m,n)-right McCoy, then R is (m',n')-right McCoy as long as $m' \leqslant m$ and $n' \leqslant n$, which can be seen as follows. If $f_1(x)g_1(x) = 0$ with $\deg(f_1) = m' \leqslant m$ and $\deg(g_1) = n' \leqslant n$ then taking $f(x) = f_1(x) + x^{m-m'}f_1(x)$ and $g(x) = g_1(x) + g_1(x)x^{n-n'}$ we see $f(x)g(x) = 0$. Any right annihilator in R for $f(x)$ will also annihilate $f_1(x)$.

One may wonder if there are any other relations among these properties. The rings $R_{m,n}$ we constructed in the previous section are of nearly no use to us, since they are not even linearly McCoy. However, if we modify the definition of $R_{m,n}$ slightly, we can prove there are no more relations among these relative McCoy properties. Before we do that, we do collect an important fact about these rings.

Fix $m, n \geqslant 1$, and let $F_{m,n} = K\langle a_i, b_j \,|\, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n\rangle$ where K is a field. Recall that if we set $f(x) = \sum_{i=0}^m a_ix^i$ and $g(x) = \sum_{j=0}^n b_jx^j$, then $R_{m,n} = F_{m,n}/I$ where I is the ideal generated by the relations $\sum_{i+j=k} a_ib_j = 0$ for all $0 \leqslant k \leqslant m + n$. These are exactly the degree k coefficients of $f(x)g(x)$. (A reduction system was given previously.) Let $A = \sum_{i=0}^m Ka_i$ and $B = \sum_{j=0}^n Kb_j$ be the grade 1 components of $F_{m,n}$ generated by the a's and b's, respectively.

Lemma 6.2. *In the notations above, if $a \in A$, $b \in B$, and $ab \in I$ then there exists some $c \in K \cup \{\infty\}$ so that $a \in Kf(c)$, $b \in Kg(c)$.*

Remark 6.3. By $f(\infty)$ we mean a_m, or in other words the value of the reversed polynomial $x^mf(1/x)$ at $x = 0$.

Proof. Note that $ab \in I$ means exactly that when we write all the monomials appearing in the product ab (without using any reductions from I) we have a K-linear combination of the grade 2 relations in I. Each monomial $a_i b_j$ appears in one and only one such relation, namely the relation coming from the degree $i+j$ coefficient of $f(x)g(x)$.

Let i_0 and j_0 be maximal with $a_{i_0} \in \operatorname{supp}(a)$ and $b_{j_0} \in \operatorname{supp}(b)$. Note that $a_{i_0} b_{j_0} \in \operatorname{supp}(ab)$, and hence so are all terms in the relation in I in which $a_{i_0} b_{j_0}$ appears. But, the only relations where $a_{i_0} b_{j_0}$ appears with both i_0 and j_0 maximal are $a_0 b_0 = 0$ and $a_m b_n = 0$. The same remarks apply to minimal indices.

We may as well suppose $(a, b) \notin \{(ka_0, k'b_0), (ka_m, k'b_n)\}$. The above maximality and minimality argument implies that both a_0 and a_m appear in the support of a, while b_0 and b_n appear in the support of b. Scaling a and b if necessary, we can assume the coefficients of a_0 and b_0 are 1. If $a_i \in \operatorname{supp}(a)$, with $i < m$, then $a_i b_n \in \operatorname{supp}(ab)$, and hence $a_{i+1} b_{n-1} \in \operatorname{supp}(ab)$, whence $a_{i+1} \in \operatorname{supp}(a)$. Repeating this argument, $a_i \in \operatorname{supp}(a)$ for all i, and by symmetry $b_j \in \operatorname{supp}(b)$ for all j.

Let $c \in K$ be the coefficient of a_1 in a. Since $ca_1 b_0$ appears in ab, then so does $ca_0 b_1$, and hence c is the coefficient of b_1 in b. But then $c^2 a_1 b_1$ appears in ab, and therefore so do $c^2 a_2 b_0$ and $c^2 a_0 b_2$, whence c^2 is the coefficient of a_2 in a, and of b_2 in b. Continuing in this fashion, we arrive at

$$a = \sum_{i=0}^{m} a_i c^i = f(c), \ \ b = \sum_{j=0}^{n} b_j c^j = g(c).$$

□

Lemma 6.4. *In the notations above, if $p(x) \in A[x]$, $q(x) \in B[x]$, and $p(x)q(x) \in I[x]$ then either:*

1. $\deg(p) \geqslant m$ *and* $\deg(q) \geqslant n$,
2. $p(x) \in K[x]a$ *and* $q(x) \in K[x]b$ *for some* $a \in A$ *and* $b \in B$ *with* $ab \in I$, *or*
3. $p(x) = 0$ *or* $q(x) = 0$.

Proof. We may assume $p(0) \neq 0$, $q(0) \neq 0$, and that $p(x)$ and $q(x)$ are not multiplies of zero-divisors modulo I, and we will show condition 1 occurs. Given c, c_1, $c_2 \in K^*$, there is an automorphism of $R_{m,n}[x]$ sending $c_1 f(c)$ to a_0 and $c_2 g(c)$ to b_0 (just take the inverse of the map sending the coefficients of $f(x)$ to those of $c_1 f(x+c)$, and sending the coefficients of $g(x)$ to those of $c_2 g(x+c)$). The map reversing the coefficients of $f(x)$ and those of $g(x)$ is also an automorphism. From $p(0)q(0) \in I$ and using the previous lemma, after applying such an automorphism if necessary we may assume $p(0) = a_0$ and $q(0) = b_0$.

Let $K(z)$ be a transcendental extension of K. The constructions of $F_{m,n}$ and $R_{m,n}$ are functorial in the base field, meaning that tensoring up to a new field just extends the field of coefficients. In particular, we have $p(z)q(z) \in I \otimes K(z) = I \cdot K(z)$, which is a zero product in the ring $R_{m,n} \otimes_K K(z)$. From the previous lemma, we can write

$$p(z) = h_1(z) f(r(z)/s(z)),\ q(z) = h_2(z) g(r(z)/s(z))$$

for polynomials $h_1(z)$, $h_2(z)$, $r(z)$, $s(z) \in K[z]$, with $\gcd(r(z), s(z)) = 1$.

The constant coefficient of $p(x)$ is a_0, and hence

$$a_0 + zp'(z) = p(z) = h_1(z)\left(\sum_{i=0}^{m} a_i \left(\frac{r(z)}{s(z)}\right)^i\right),$$

for some polynomial $p'(z)$. Therefore $\gcd(h_1(z), z) = 1$ and $s(z)^m | h_1(z)$. Because $p(x)$ is not a multiple of a_0, $r(z) \neq 0$. But this implies $z|r(z)$ since a_1 is not in the constant term of $p(x)$. The coefficient of a_m, which is $(r(z)/s(z))^m h_1(z)$, is then divisible by z^m, and thus $\deg(p(x)) \geqslant m$. By similar reasoning $\deg(q(x)) \geqslant n$. □

Now consider the ring $F_{m,n} = K\langle a_i, b_j \,|\, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n\rangle$ and the two polynomials $f(x)$ and $g(x)$ as before. We let $S_{m,n}$ be the ring $F_{m,n}$ modulo the (minimal) ideal of relations forcing $f(x)g(x) = 0$ and $b_j a_i = b_j b_{j'} = 0$.

Proposition 6.5. *Given positive integers $m, n \geqslant 1$, the ring $S_{m,n}$ constructed above is not (m, n)-right McCoy, is (m', n')-right McCoy if $m' < m$ or $n' < n$, and is left McCoy.*

Proof. All elements are identified with their images in $S_{m,n}$. First we describe how to find the normal form for words in $S_{m,n}$. Any monomial containing a product $b_j a_i$ or $b_j b_{j'}$ is zero. Finally, use the same relations as in $R_{m,n}$ by replacing $a_m b_j$ by $-\sum_{i=0}^{m-1} a_i b_{j+m-i}$ (for each $j \leqslant n$) and replacing $a_\ell b_0$ by $-\sum_{i=0}^{\ell-1} a_i b_{\ell-i}$ (for each $\ell \leqslant m$). Thus, given $\gamma \in S_{m,n}$ we can write it in normal form as

$$\gamma = c + \sum_{i=0}^{m} w_i a_i + \sum_{j=0}^{n} c_j b_j + \sum_{i=0}^{m-1}\sum_{j=1}^{n} w_{i,j} a_i b_j$$

where $c, c_j \in K$ and $w_i, w_{i,j} \in K\langle a_k \,|\, 0 \leqslant k \leqslant m\rangle$.

It is clear $f(x)g(x) = 0$. Also, $\text{ann}_r(a_0) = b_0 K$ by a simple calculation using normal forms. Therefore, $\text{ann}_r^{S_{m,n}[x]}(f(x)) \cap S_{m,n} = (0)$ since $f(x)b_0 \neq 0$. Hence $S_{m,n}$ is not (m, n)-right McCoy. Suppose we have two non-zero polynomials $p(x), q(x) \in S_{m,n}[x]$, with $p(x)q(x) = 0$. If one of the coefficients of either $p(x)$ or $q(x)$ has 1 in its support then by Lemma 3.5 we reach a contradiction. Thus each coefficient of $q(x)$ consists of sums of monomials of strictly positive grading, whence $b_0 q(x) = 0$. This proves that $S_{m,n}$ is left McCoy.

We wish to show now that $S_{m,n}$ is (m', n')-right McCoy, when either $m' < m$ or $n' < n$. Again let $p(x)$ and $q(x)$ be non-zero polynomials which annihilate each other. The same remarks in the previous paragraph apply. Without loss of generality we can assume that b_j doesn't appear in any monomial in any coefficient of $p(x)$. In other words, $p(x) \in K\langle a_i\rangle$. Lemma 3.5, with minimality replaced by maximality, shows that no monomial in any coefficient of $q(x)$ can have a_i appearing. This forces $q(x) \in \sum_{j=0}^{n} K b_j[x]$.

Fix a non-zero monomial wa_k appearing in one of the coefficients of $p(x)$, where $w \in K\langle a_i\rangle$ (and we may as well assume w has maximal grade r). Let $p_0(x)$ be the polynomial obtained from $p(x)$, by retaining exactly those monomials in the coefficients of grade $r+1$ beginning with w (i.e. which are of the form $wa_{i'}$). Let $p_1(x)$ be the polynomial obtained from $p_0(x)$ by removing w from the left of each monomial in each coefficient of $p_0(x)$. The equality $p_1(x)q(x) = 0$ follows from $p_0(x)q(x) = 0$, which in turn comes from looking at all coefficients of $p(x)q(x)$ of grade $r+2$ beginning with w. By construction, $\deg(p_1) \leqslant \deg(p)$, $p_1(x) \neq 0$, and $p_1(x) \in \sum_{i=0}^{m} K a_i[x]$.

Applying the previous lemma, either $q(x)$ is a multiple of a zero divisor (which means $p(x)$ is annihilated on the right by an element of $S_{m,n}$) or $\deg(p(x)) \geqslant \deg(p_1(x)) \geqslant m$ and $\deg(q(x)) \geqslant n$. In either case we are done. □

Lemma 6.6. *Let I be an index set, and for each $i \in I$ let R_i be a ring with 1. The direct product $\prod_{i\in I} R_i$ is (m,n)-right McCoy if and only if each ring R_i is (m,n)-right McCoy.*

Proof. The proof of Lemma 4.1 suffices, noting that the trivial polynomial 1 has no non-zero right annihilator and has degree 0. □

Theorem 6.7. *Given any two sets, $\mathcal{L}$ and $\mathcal{R}$, consisting of pairs of positive integers, there is a ring $R_{\mathcal{L},\mathcal{R}}$ which is not (m,n)-left McCoy (respectively (m,n)-right McCoy) if and only if there exists an element $(m',n') \in \mathcal{L}$ (respectively $(m',n') \in \mathcal{R}$) with $m \geqslant m'$ and $n \geqslant n'$.*

Proof. Just take $R_{\mathcal{L},\mathcal{R}} = \prod_{(m,n)\in\mathcal{L}} S_{m,n}^{\mathrm{op}} \times \prod_{(m,n)\in\mathcal{R}} S_{m,n}$, and use the previous proposition and lemma. □

This theorem proves that the only relations among the relative McCoy properties (on either side) are the trivial ones. Therefore, in general, it is hopeless to try and prove the McCoy property by checking it for only certain degrees.

7. Abelian and Linearly Armendariz Rings

Following the literature, a ring is *linearly Armendariz* if the Armendariz condition holds for linear polynomials: $(a_0 + a_1x)(b_0 + b_1x) = 0$ implies $a_ib_j = 0$ for all $i, j \in \{0, 1\}$. Clearly, Armendariz implies linearly Armendariz, which implies linearly McCoy. In [9] an example is constructed showing that the first implication is irreversible. We will construct an example showing the second implication is also irreversible. We note that in the literature, linearly Armendariz rings are sometimes called *weakly Armendariz*, but that name is also given to another class of rings, and so we avoid that terminology.

Recall that a ring is Abelian if its idempotents are central. Linearly Armendariz rings with 1 are always Abelian [9, Lemma 3.4] (cf. [5, Corollary 8]). We saw in the previous section that right linearly McCoy rings with 1 are Dedekind-finite, but left it unanswered whether they must be Abelian. We answer that question now (in the negative), even for McCoy rings (with 1). In particular, this shows that McCoy rings do not need to be linearly Armendariz.

Theorem 7.1. *There exists a McCoy ring with 1 which is not Abelian.*

Proof. Let K be a field, and let

$$R = K\langle e, x, y, z\rangle/(e^2 = e,\, ex = x,\, xe = 0,\, ey = ye = 0,\, ez = ze = z,\\ x^2 = y^2 = z^2 = xy = xz = yx = yz = zx = zy = 0).$$

We will show R is right McCoy, and by symmetry (sending $e \mapsto 1-e$), R is also left McCoy. As a vector space $\{1, e, x, y, z\}$ forms a basis (for elements in their reduced form). The element e is an idempotent, and e doesn't commute with x. We think of e as having grade 0, and let $M = Kx + Ky + Kz$ be the ideal of R with positive grading, and note $M^2 = (0)$ and $MR = RM = M$.

Suppose $f(w), g(w) \in R[w]$ are non-zero with $f(w)g(w) = 0$. If $f(w)y = 0$ we are done, so we can assume at least one of the coefficients of $f(w)$ is of the form

$\alpha + \alpha' e + m$ for some $m \in M$, $\alpha, \alpha' \in K$, $\alpha \neq 0$. Write $f(w) = \sum_{i=0}^{m} a_i w^i$ and $g(w) = \sum_{j=0}^{n} b_j w^j$ as usual, with $a_0, b_0 \neq 0$, and let $k \leqslant m$ be the smallest index with 1 in the support of a_k. If there exists a coefficient of $g(w)$ with 1 in its support then by an argument similar to that given in the last paragraph of Proposition 3.2 we obtain a contradiction. (Let ℓ be minimal with 1 in the support of b_ℓ and compute the $k + \ell$ degree coefficient of $f(w)g(w)$.) Similarly, y cannot be in the support of any of the coefficients of $g(w)$.

Case 1: Suppose one of the coefficients of $f(w)$ is of the form $\alpha + \alpha' e + m$ with $m \in M$, $\alpha, \alpha' \in K$ and $\alpha \neq -\alpha'$. In this case, repeating the argument at the end of the previous paragraph, replacing y with e, demonstrates that e doesn't appear in the support of any coefficient of $g(w)$. Thus $g(w) \in (Kx + Kz)[w]$, and repeating the same argument (two more times) we find x and z also do not appear in the supports, so $g(w) = 0$, a contradiction.

Case 2: All of the coefficients of $f(w)$ are of the form $\alpha(1-e)+m$, with $m \in M$, $\alpha \in K$. In this case $f(w)z = 0$, and we are done. □

8. Duo and Semi-commutative Rings

There is another important ring theoretic condition common in the literature related to the zero-divisor and annihilator conditions we have been studying.

Definition 8.1. A ring is said to be *right duo* if all right ideals are two-sided ideals. Left duo rings are defined similarly, and a ring is called *duo* if it is both left and right duo.

The following implications hold, and are irreversible:

$$\text{commutative} \Longrightarrow \text{duo} \Longrightarrow \text{one-sided duo} \Longrightarrow \text{semi-commutative}.$$

In Section 5, we saw that semi-commutative rings are quadratically McCoy; hence the same holds for duo rings. More is true.

Theorem 8.2. *Right duo rings are necessarily right McCoy.*

Proof. Let R be a right duo ring. For any polynomial $p(x) \in R[x]$ we let $I_{p(x)}$ denote the right ideal generated by the coefficients of $p(x)$. Suppose we are given polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$ and $g(x) \neq 0$. We will prove, by induction on the degree of $f(x)$, that there is some non-zero element in $I_{g(x)}$ which annihilates $f(x)$ on the right. Write $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ as usual, where we may assume $b_0 \neq 0$.

First, if $\deg(f(x)) = 0$ then the claim is trivial since $f(x)b_0 = 0$. So the base case of our induction is established. Now, suppose $\deg(f(x)) > 0$.

Case 1: Suppose $a_0 g(x) = 0$. This implies $a_0 I_{g(x)} = 0$. In this case we set $f_1(x) = (f(x) - a_0)/x$ and find $f_1(x)g(x) = 0$. But $\deg(f_1(x)) < \deg(f(x))$, and hence by induction there exists a non-zero element $b \in I_{g(x)}$ satisfying $f_1(x)b = 0$, whence $f(x)b = 0$.

Case 2: Suppose $a_0 g(x) \neq 0$. Let j be minimal so that $a_0 b_j \neq 0$. By Lemma 5.4, there exists an integer $n > 0$ satisfying $a_0^n b_j \neq 0 = a_0^{n+1} b_j$. Since R is right duo, there exists $r \in R$ with $a_0^n b_j = b_j r$. If we let $g_1(x) = g(x)r$, then clearly $f(x)g_1(x) = 0$, and $(0) \neq I_{g_1(x)} \subseteq I_{g(x)}$. This means we can replace $g(x)$ by $g_1(x)$ without any loss of generality. By construction, a_0 annihilates the first j coefficients of $g_1(x)$, so after repeating this process a finite number of times we reduce to the previous case. □

Hirano [3] proved that if $R[x]$ is semi-commutative then R is McCoy, using an idea similar to one used by McCoy in [13]. It is known that if R is semi-commutative then $R[x]$ is not necessarily semi-commutative [5], and the same is true even if we assume R is reversible [6]. We thus have three conditions which imply that R is semi-commutative (namely, R is reversible, R is duo, and $R[x]$ is semi-commutative) and each of them also implies R is McCoy. So it seems surprising that the semi-commutative property doesn't imply the McCoy property (especially in light of the work in Section 5).

The example of a semi-commutative ring which is not $(3,1)$-right McCoy, as constructed in [14], turns out to be an $\mathbb{F}_2$-algebra. Furthermore, if we choose a base field where 2 is invertible then the example fails, and R is always *forced* to be $(3,1)$-right McCoy in this case. These sorts of computations can be generalized, but first we need to introduce some standard notation.

Definition 8.3. A matrix of the form

$$V_n = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}$$

over a commutative ring is called a *Vandermonde* matrix. Its determinant is $\prod_{i>j}(c_i - c_j)$. In particular, the determinant is non-zero when the c_i are distinct, and the ring has no zero-divisors.

Theorem 8.4. *If R is a semi-commutative ring and embeds in a $\mathbb{Q}$-algebra, then $R[x]$ is semi-commutative.*

Proof. Given a polynomial $p(x) \in R[x]$ write $p(x)[i]$ for the ith coefficient of $p(x)$. Suppose we have two non-zero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$. Fixing $r \in R$, we wish to prove $f(x)rg(x) = 0$. We do know we can specialize x to anything in the center, and in particular $f(c)g(c) = 0$ for each $c \in \mathbb{Z}$. This equation lies over R, so by semi-commutativity $f(c)rg(c) = 0$. In particular, letting $m = \deg(f)$ and $n = \deg(g)$, we have $\sum_{k=0}^{m+n} c^k (f(x)rg(x))[k] = 0$ for each $c \in \mathbb{Z}$. Fixing $m+n+1$ distinct integers c_i, we find

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^{m+n} \\ 1 & c_1 & c_1^2 & \cdots & c_1^{m+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_{m+n} & c_{m+n}^2 & \cdots & c_{m+n}^{m+n} \end{pmatrix} \begin{pmatrix} f(x)rg(x)[0] \\ f(x)rg(x)[1] \\ \vdots \\ f(x)rg(x)[m+n] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

But R embeds in a $\mathbb{Q}$-algebra, so the Vandermonde matrix is invertible (in the embedding). In particular, each of the coefficients of $f(x)rg(x)$ is zero, which finishes the proof. □

Remark 8.5. The construction above works for a polynomial ring in any number of variables. In particular, this leads to a lot of examples of semi-commutative rings.

This theorem says, in effect, it is a vestige of possible zero-divisors in $\mathbb{Z}$ that $R[x]$ can be non-semi-commutative when R is semi-commutative. We finish this section by showing how deeply this pathology runs, giving an example of a ring R that is a symmetric ring with 1 but $R[x]$ is not semi-commutative.

Example 8.6. We follow and simplify the construction in [6, Example 2.1]. Let $R = \mathbb{F}_2\langle a_0, a_1, a_2, b_0, b_1, c\rangle$. Set $f(x) = a_0 + a_1x + a_2x^2$ and $g(x) = b_0 + b_1x$.

Let I be the ideal generated by the relations

$$\begin{aligned}&a_0b_0 = 0,\ a_0b_1 = a_1b_0,\ a_1b_1 = a_2b_0,\ a_2b_1 = 0,\ a_0cb_0 = a_2cb_1 = 0,\\ &b_0a_0 = 0,\ b_1a_0 = b_0a_1,\ b_1a_1 = b_0a_2,\ b_1a_2 = 0,\ b_0ca_0 = b_1ca_2 = 0\\ &a_0cb_1 = a_1cb_0 + a_1cb_1 + a_2cb_0,\ b_1ca_0 = b_0ca_1 + b_1ca_1 + b_0ca_2,\\ &a_iRa_{i'} = b_jRb_{j'} = cRc = 0.\end{aligned}$$

The first line of relations guarantees $f(x)g(x) = a_0cb_0 = a_2cb_1 = 0$, and the second line reverses all of these relations. The third line captures the reduction relations corresponding to $f(1)cg(1) = g(1)cf(1) = 0$. The last line simplifies the calculations and makes the ring finite.

We speak of three *types* of letters in R, namely the letters $\{a_0, a_1, a_2\}$, the letters $\{b_0, b_1\}$, and the letter $\{c\}$. Notice that in every reduction relation in I, the monomials all have the same ordering (and grade) on the types of letters. We will refer to this fact by saying I *preserves type orders*. The Diamond Lemma conditions are easily checked.

Let $S = R/I$. The product $f(x)cg(x)$ is non-zero in $S[x]$, while $f(x)g(x) = 0$, with $f(x), g(x) \neq 0$. Thus $S[x]$ is not semi-commutative. The ideal I is homogeneous, and we let H_i denote the $\mathbb{F}_2$-vector space of homogeneous words of grade i, in their normal form. Notice that $H_4 = 0$. Also, as a bit of notation, we let $A = \sum_i \mathbb{F}_2a_i$ and $B = \sum_j \mathbb{F}_2b_j$.

We will first show that H_1 is symmetric. Fix elements $\gamma_1, \ldots, \gamma_n \in H_1$ with $\gamma_1 \cdots \gamma_n = 0$, and let S_n denote the group of permutations on n elements. If $n = 1$ or $n > 3$ then trivially $\gamma_{\sigma(1)} \cdots \gamma_{\sigma(n)} = 0$ for each $\sigma \in S_n$. We may also assume that $\gamma_i \neq 0$ for each $i \leqslant n$. If $n = 2$, the only possibilities are

$$\begin{aligned}(\gamma_1, \gamma_2) \text{ or } (\gamma_2, \gamma_1) \in \{&(a_0 + \epsilon, b_0 + \epsilon'),\ (a_2 + \epsilon, b_1 + \epsilon'),\ (f(1) + \epsilon, g(1) + \epsilon'),\\ &(a, a'),\ (b, b'),\ (c, c)\}\end{aligned}$$

where

$$(\epsilon, \epsilon') \in \{(0, a),\ (b, 0),\ (b_0, a_0),\ (b_1, a_2),\ (g(1), f(1))\}$$

with $a, a' \in A$, $b, b' \in B$. In each case $\gamma_1R\gamma_2 = \gamma_2R\gamma_1 = 0$.

We now suppose $n = 3$. Write $\gamma_i = \delta_ic + \gamma_i'$ where δ_i is one or zero, depending on whether c is in the support of γ_i or not. From $\gamma_1\gamma_2\gamma_3 = 0$ and since I preserves type orders, we obtain $\delta_1c\gamma_2'\gamma_3' = 0$, $\delta_2\gamma_1'c\gamma_3' = 0$, and $\delta_3\gamma_1'\gamma_2'c = 0$. Further, we then obtain $\delta_3\gamma_1'\gamma_2' = 0$ and $\delta_1\gamma_2'\gamma_3' = 0$ (again, from our reduction relations). We saw when $n = 2$ that we can insert c into a zero product, and rearrange the terms however we like. Further, the only non-zero monomials of degree 3 contain all three types of variables. Thus, it suffices to assume $\delta_2 = 1$ and show that we can remove c from the equation $\gamma_1'c\gamma_3' = 0$, and then we will have $\gamma_{\sigma(1)}\gamma_{\sigma(2)}\gamma_{\sigma(3)} = 0$ for each $\sigma \in S_3$. Write $\gamma_1' = \alpha_1 + \beta_1$ and $\gamma_3' = \alpha_3 + \beta_3$ with $\alpha_1, \alpha_3 \in A$ and $\beta_1, \beta_3 \in B$. From $\gamma_1'c\gamma_3' = 0$ and since I preserves type orders, we have $\alpha_1c\beta_3 = 0$ and $\beta_1c\alpha_3 = 0$. From visually looking at the reduction relations, we see that these equations force $\alpha_1\beta_3 = 0$ and $\beta_1\alpha_3 = 0$, and in particular $\gamma_1'\gamma_3' = 0$. This finishes the proof that H_1 is symmetric.

Now fix a positive integer $n > 0$, and elements $r_k \in R$ for $0 < k \leqslant n$ with $Q = r_1r_2 \cdots r_n = 0$. We wish to show that any permutation of the factors in Q still results in a zero product. We will do so by a series of simplifications. We can write $r_k = \sum_{i=0}^{3} h_{i,k}$, where $h_{i,k} \in H_i$ for each i. The first simplification we make is by looking at the constant terms $h_{0,k}$ of the factors r_k. If $h_{0,k} = 1$ for all k, then $r_1 r_2 \cdots r_n \neq 0$, a contradiction. If only one of the constant terms is zero, say for $h_{0,k_1} = 0$, then $r_{k_1} = 0$ (just look at the smallest non-zero monomial among the products in Q) hence any permutation of the factors in Q will still be zero. If four or more of the constant terms are zero then (under any permutation) all monomials in Q have grade 4 or more, hence are zero.

We thus have just two cases to consider. If three of the constant terms are zero, say for indices $k_1 < k_2 < k_3$, then the only non-zero monomials (of grading less than 4) in Q come from the product $h_{1,k_1} h_{1,k_2} h_{1,k_3}$. But H_1 is symmetric, so this case leads to no problems.

We have thus simplified to the case where there are exactly two zero constant terms, say for indices $k_1 < k_2$. The only grade 2 monomials in Q with non-zero coefficients come from the product $h_{1,k_1} h_{1,k_2}$, and hence $h_{1,k_1} h_{1,k_2} = 0$. Notice that this zero product is reversible since H_1 is symmetric. We calculate that the grade 3 monomials in Q arise from

$$h_{1,k_1} h_{2,k_2} + h_{2,k_1} h_{1,k_2} + \sum_{k<k_1} h_{1,k} h_{1,k_1} h_{1,k_2} + \sum_{k_1<k<k_2} h_{1,k_1} h_{1,k} h_{1,k_2} + \sum_{k>k_2} h_{1,k_1} h_{1,k_2} h_{1,k}.$$

All of the terms in the last three sums are zero (under any permutation) so $P = h_{1,k_1} h_{2,k_2} + h_{2,k_1} h_{1,k_2} = 0$, and it suffices to show that we can reverse the products in the two summands of P and still obtain 0.

First, if $h_{1,k_2} = 0$ our claim reduces to showing $h_{1,k_1} h_{2,k_2} = 0$ can be reversed. Looking at any given type order in the product, this computation reduces to the fact that H_1 is symmetric. So, by symmetry, we may assume $h_{1,k_i} \neq 0$ for each i. Second, suppose c is in the support for h_{1,k_1}. Since $h_{1,k_1} h_{1,k_2} = 0$ this implies $h_{1,k_1} = c = h_{1,k_2}$. Again looking at possible type orders, the products in P can be rearranged. By symmetry, we may then suppose c is not in the support of h_{1,k_i} for either i.

Hence, one may assume $h_{1,k_i} = \alpha_i + \beta_i \neq 0$ with $\alpha_i \in A$ and $\beta_i \in B$. Furthermore, since monomials of grade three are zero (under any permutation) unless all three types of variables are present, we may write $h_{2,k_i} = c\alpha_i' + c\beta_i' + \alpha_i'' c + \beta_i'' c$, for $i \in \{1, 2\}$ (with the α's in A and β's in B).

The equation $P = 0$ is now equivalent to the equations

$$\alpha_1 \beta_2'' c = 0,\ \beta_1 \alpha_2'' c = 0,\ c\beta_1' \alpha_2 = 0,\ c\alpha_1' \beta_2 = 0,$$
$$\alpha_1 c \beta_2' + \alpha_1'' c \beta_2 = 0,\ \beta_1 c \alpha_2' + \beta_1'' c \alpha_2 = 0.$$

In the first four products, we can permute the products in any way and they are still zero (in fact, we can remove the c and they are still zero) since H_1 is symmetric. It therefore suffices to show that each of the summands in the last two equations is zero under any permutation. By symmetry it suffices to work with the equation

$$\alpha_1 c \beta_2' + \alpha_1'' c \beta_2 = 0$$

and show that both summands are zero. If either α_1 or β_2 are zero, then the claim is trivial. From the fact that $\alpha_1 \beta_2 = 0$, we then have three other possibilities

$$(\alpha_1, \beta_2) \in \{(a_0, b_0),\ (f(1), g(1)),\ (a_2, b_1)\}.$$

One checks, in each case, each of the summands must be zero, finishing the proof.

9. 2-primal Rings

In this section, we assume all rings are unital. Fix a ring R and recall the following standard notions:

Definition 9.1. An ideal $\mathfrak{p}$ is prime if for every pair of elements $a, b \in R$ with $aRb \in \mathfrak{p}$ then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. A prime ideal $\mathfrak{p}$ is called *completely prime* if $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

The *lower nilradical* (or Baer-McCoy radical) $\mathrm{Nil}_*(R)$ is the intersection of the (minimal) prime ideals. It is always a nil ideal (i.e. every element in it is nilpotent). In the literature, a ring is called 2*-primal* if every nilpotent element is contained in the lower nilradical. A ring is *semi-prime* if the lower nilradical is zero.

G. Shin proved that a ring is 2-primal if and only if all minimal prime ideals are completely prime, see either [16] or [8, Theorem 12.6′]. All semi-commutative rings are 2-primal, and the implication is irreversible [11]. In [14, Theorem 4] it was shown that all semi-commutative rings have a property close to that of McCoy rings, and we now show that this theorem holds for the larger class of 2-primal rings.

Theorem 9.2. *Let R be a 2-primal ring. If $f(x)$, $g(x) \in R[x]$ are two non-zero polynomials with $f(x)g(x) = 0$ then $f(x)$ or $g(x)$ has a non-zero right annihilator in R (and similarly for left annihilators).*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be as above, and let $\mathfrak{p}$ be a minimal prime ideal in R. We claim that each of the coefficients of f, or of g, are in $\mathfrak{p}$. If not, let k and ℓ be the minimal indices with $a_k, b_\ell \notin \mathfrak{p}$. Calculating the degree $k + \ell$ coefficient of $f(x)g(x)$, we have

$$\sum_{(i,j)\,:\,i+j=k+\ell} a_i b_j = a_k b_\ell + \sum_{(i,j)\neq(k,\ell)\,:\,i+j=k+\ell} a_i b_j = 0 \in \mathfrak{p}$$

but every term in the righthand sum belongs to $\mathfrak{p}$ by minimality on k and ℓ. Thus $a_k b_\ell \in \mathfrak{p}$, contradicting the fact that $\mathfrak{p}$ is completely prime.

We have thus shown $a_i b_j \in \mathfrak{p}$ for each i and j and all minimal primes $\mathfrak{p}$. In particular $a_i b_j \in \mathrm{Nil}_*(R)$. Let S be the set of products $a_i b_j$. It is well known that any finite subset of the lower nilradical is locally nilpotent. So there exists some number $t \geqslant 1$ with $S^t = (0)$ and t is minimal.

If $t = 1$ then all coefficients of f annihilate all coefficients of g, and we are done. So we may assume $t > 1$. In this case, fix $r \in S^{t-1}$ with $r \neq 0$. If $g(x)r \neq 0$, fix a coefficient b_j of $g(x)$ with $b_j r \neq 0$. We calculate $f(x)b_j r = 0$ by definition of t. □

This theorem leaves open the possibility that all of the annihilator conditions could be focused on $g(x)$, with $f(x)$ having neither left nor right annihilators over R. To find an example where this happens, one can use the methods employed in [14] and form the universal $\mathbb{F}_2$-algebra which is not $(3, 1)$-right McCoy but is semi-commutative (hence 2-primal). One checks that the polynomial of degree 3 has no left or right annihilators in the base ring.

In other words, consider the polynomials $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and $g(x) = b_0 + b_1$, and take R to be the free $\mathbb{F}_2$-algebra over their coefficients, modulo the ideal of relations forcing $f(x)g(x) = 0$, $b_j r b_{j'} = 0$, $a_0 a_i r b_j = a_3 a_i r b_j = 0$, and

$a_1a_irb_j = a_2a_irb_j$, for every monomial r. We claim without proof (leaving it to the interested reader–who may find the methods employed in [14] useful) that the ring R is semi-commutative, but $f(x)$ is not annihilated on either the left or the right by a non-zero element in R.

We noted earlier that Armendariz rings do not have to be semi-commutative, and promised an example. Note that a ring is semi-prime 2-primal if and only if it is reduced. Thus, the following example more than suffices.

Example 9.3. Let $R = \mathbb{C}\langle a, b\rangle/(a^2 = 0)$. By the Diamond Lemma, a $\mathbb{C}$-basis for R is given by monomials in which two copies of a never appear next to each other. When we speak of the support of an element, we mean with regards to this fixed basis. Notice that our ideal of relations is homogeneous as usual.

We begin by classifying zero-divisor pairs. Let $\alpha, \beta \in R$ be non-zero with $\alpha\beta = 0$. If $1 \in \mathrm{supp}(\alpha)$ then fix β' to be a monomial in the support of β with smallest grade. But this implies $1\cdot\beta'$ is in the support of $\alpha\beta = 0$, a contradiction. Thus $1 \notin \mathrm{supp}(\alpha)$, and similarly $1 \notin \mathrm{supp}(\beta)$.

We claim $\alpha \in Ra$ and $\beta \in aR$. We prove this by contradiction, so assume $\alpha \notin Ra$. Suppose first that $\beta \in aR$, and let $\alpha_1 \in \mathrm{supp}(\alpha) \cap Rb$ with the grade on α_1 maximal, and take $\beta_1 \in \mathrm{supp}(\beta)$ with maximal grading. We then find $\alpha_1\beta_1$ must appear in the support of $\alpha\beta$ (no other monomials in the product $\alpha\beta$ can cancel out $\alpha_1\beta_1$, due to maximality on the grading), a contradiction. Thus, we can also assume $\beta \notin aR$.

Let α_0 be a monomial in the support of α, of the form $b^{m_1}ab^{m_2}a\cdots ab^{m_r}$, which satisfies the following four conditions: (1) $m_r > 0$, (2) $b^{m_1}a\cdots ab^{m_i} \notin \mathrm{supp}(\alpha)$ for each $i < r$, (3) r is maximal with respect to these two properties, and (4) we choose m_r maximal with respect to the previous three properties. Similarly, let β_0 be a monomial in the support of β of the form $b^{n_s}ab^{n_{s-1}}a\cdots ab^{n_1}$ where: $n_s > 0$, $b^{n_i}a\cdots ab^{n_1} \notin \mathrm{supp}(\beta)$ for each $i < s$, s is maximal with respect to these properties, and then n_s is maximal also. The monomial $\alpha_0\beta_0$ must be canceled out in the product $\alpha\beta = 0$, so there must exist monomials $\alpha_1 \in \mathrm{supp}(\alpha)$, $\beta_1 \in \mathrm{supp}(\beta)$, satisfying $\alpha_1\beta_1 = \alpha_0\beta_0$ and (by symmetry) we may assume $\deg(\alpha_1) > \deg(\alpha_0)$.

If $\alpha_1 = \alpha_0 b^k$ this would contradict the maximality of m_r, and if

$$\alpha_1 = \alpha_0 b^{n_s}a\cdots ab^{n_{i+1}}ab^k,$$

for some $k > 0$, this would contradict the maximality on r (taking i to be minimal with such a product in the support of α). Thus α_1 is of the form $\alpha_0 b^{n_s}a\cdots ab^{n_i}a$ (for some $i \leqslant s$) and hence $\beta_1 = b^{n_{i-1}}a\cdots ab^{n_0}$ contradicting condition (2) for β_0. Thus, all cases lead to a contradiction. This means that $\alpha_0\beta_0$ cannot be canceled out of the product $\alpha\beta$. Hence, our assumption $\alpha \notin Ra$ leads to a contradiction.

We have thus proven that if $\alpha\beta = 0$ (with $\alpha, \beta \neq 0$) then $\alpha \in Ra$ and $\beta \in aR$. Suppose now we have two non-zero polynomials $f(x)$ and $g(x)$ satisfying $f(x)g(x) = 0$. Since we are working over an infinite field, there is some constant $c \in \mathbb{C}$ so that each monomial of each coefficient of $f(x)$ and $g(x)$ appears with non-zero support in $f(c)$ and $g(c)$ (respectively). In particular, $f(c)g(c) = 0$ implies that $f(x) \in R[x]a$ and $g(x) \in aR[x]$. Thus, R is Armendariz. It is straightforward to see that R is semi-prime but not reduced. We also note that, instead of $\mathbb{C}$, we could have used any field, since all fields embed into infinite fields and our construction of R is functorial.

10. Morita Invariance

We again assume throughout this section that rings are unital. Two rings are said to be *Morita equivalent* if their module categories are equivalent. A ring theoretic property is said to be a *Morita invariant* if it is preserved within any Morita equivalence class. Examples of Morita invariant properties include a ring being semisimple, Noetherian, Artinian, or simple. The property of being Dedekind-finite is not Morita invariant.

The Morita invariance of a property of R can be checked by testing if it passes to matrix rings $\mathbb{M}_n(R)$ and corner rings eRe, with $e^2 = e$ a full idempotent ($ReR = R$). It turns out that the McCoy property is badly behaved with regards to Morita invariance. In fact:

Theorem 10.1. *Let R be a ring and suppose there exists a non-trivial, full idempotent $e \in R$. The ring R is not McCoy.*

Proof. Write $1 = \sum_{i=1}^{m} r_i e s_i$. Let $f(x) = e + \sum_{i=1}^{m} e s_i (1-e) x^i$ and $g(x) = (1-e) - \sum_{i=1}^{m} e s_i (1-e) x^i$. The fact that e is non-trivial means $f(x), g(x) \neq 0$. One computes $f(x)g(x) = 0$. If $f(x)r = 0$ then $er = 0$ and $e s_i (1-e) r = 0$ for each i. In particular $r = \sum_{i=1}^{m} r_i e s_i r = 0$. □

Proposition 10.2 (cf. [4], [15])**.** *Matrix rings and upper triangular matrix rings (of any non-trivial size, indexed over a well-ordered set) over a non-zero ring are never linearly McCoy.*

Proof. Let e_{ij} be the usual matrix units, where the indices occur in a well-ordered set I, and let 1 be the first element in I. The polynomials $f_1(x) = e_{1,1} + e_{1,2}x$, $g_1(x) = (1 - e_{1,1}) - e_{1,2}x$ suffice to show such rings are never left linearly McCoy, and for right linearly McCoy the polynomials $f_2(x) = (1 - e_{2,2}) + e_{1,2}x$, $g_2(x) = e_{2,2} - e_{1,2}x$ suffice. □

Notice that the above argument will also work for rings which embed in matrix rings, containing the coefficients of the polynomials used in the proof. In particular, the ring of infinite matrices (over a non-zero ring) with each row and column having only finitely many non-zero entries is never (linearly) McCoy.

Corollary 10.3. *Simple rings with non-trivial idempotents are never McCoy.*

We have thus shown that the McCoy property does not hold in any ring which remotely behaves like a matrix ring. In the other direction, we might ask whether the McCoy property passes to corner rings. We know this is true for central idempotents, in unital rings (but not in general rings), by Theorem 4.4. However, for arbitrary idempotents this doesn't hold.

Example 10.4. Let $R_0 = K\langle e, a_0, a_1, b_0, b_1, y, z\rangle$ where K is a field. We think of e as a variable of grading 0, and all the other variables as having grade 1. Let I be the ideal generated by the relations

$$e^2 = e,\ a_0 b_0 = 0,\ a_0 b_1 = -a_1 b_0,\ a_1 b_1 = 0,\ e a_i = a_i e = a_i,\ e b_i = b_i e = b_i,$$
$$ey = 0,\ ye = y,\ ze = 0,\ ez = z,\ y^2 = yz = zy = z^2 = 0,$$
$$a_i y = y a_i = b_i y = y b_i = 0,\ a_i z = z a_i = b_i z = z b_i = 0.$$

One can check, via the Diamond Lemma, these relations form a reduction system. Notice also that I is homogeneous if we think of e as having grade 0.

As usual, identify the letters with their images in the ring $R = R_0/I$. The corner ring eRe is isomorphic to $R_{1,1} = k\langle a_0, a_1, b_0, b_1\rangle/(a_0b_0 = 0, a_0b_1 = -a_1b_0, a_1b_1)$ which is not (left or right linearly) McCoy.

Let $p(x), q(x) \in R[x]$ be non-zero polynomials with $p(x)q(x) = 0$. We may assume $q(0) \neq 0$. If $p(x)y = 0$ we are done, so we may assume one of the coefficients of $p(x)$ (in normal form) has 1 in its support. Just as in the proof of Theorem 7.1, we find that 1 and y are not in the support of any coefficients of $q(x)$. If $p(x)z = 0$ we are done, so we also may assume $p(x)$ has a coefficient of the form $\alpha+\alpha' e+\beta$ where $\alpha, \alpha' \in k$, $\alpha \neq -\alpha$, and β is composed of monomials of grading strictly greater than 0. Writing $p(x) = \sum_i p_i x^i$, we let i_0 be the smallest index with $p_{i_0} = \alpha + \alpha' e + \beta$ as above.

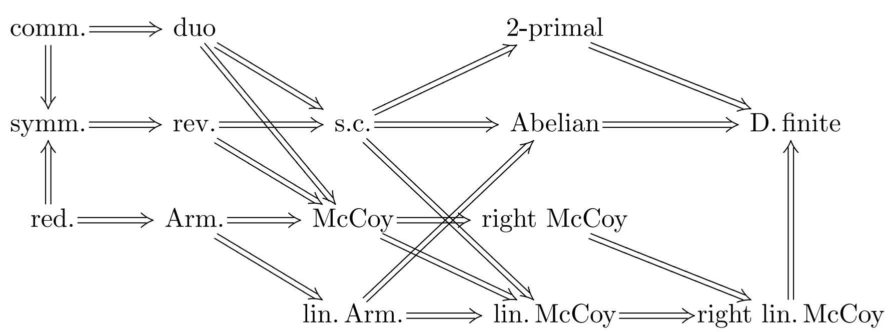
Again, as in the proof of Theorem 7.1, we see that e and z are not in the support of any coefficient of $q(x)$. Note that no non-zero monomial, in normal form, of grading greater than 1, involves e, y, or z. Thus, we have reduced to $q(x) \in R_{1,1}$. Hence

$$0 = p(x)q(x) = p(x)(eq(x)) = (p(x)e)q(x).$$

Note that the grade 0 component of $p(x)e$ is non-zero (in the i_0 degree coefficient). From $\mathrm{ann}_r((\alpha+\alpha')e) \cap R_{1,1} = (0)$ and Lemma 3.5, we reach a contradiction. Thus R must be right McCoy, and by left-right symmetry R is McCoy.

11. Extended Diagram and Open Questions

The implication chart we gave in the first section can now be expanded. First note that the ring $S_{1,1}$, constructed previously, is not right linearly McCoy but turns out to be 2-primal and abelian. The ring $S_{2,2}$ is linearly Armendariz but not right McCoy. Our extended chart is as follows:



No other implications hold (except by transitivity). Note that if we work with non-unital rings we must remove a few conditions from our chart.

We leave the reader with a few open questions (whose answers would extend the diagram further):

Question 11.1. Are left duo rings right McCoy?

Question 11.2. If R is a (symmetric) duo ring, is $R[x]$ semi-commutative?

Question 11.3. Are free algebras over McCoy rings still McCoy?

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