SEMI-COMMUTATIVITY AND THE MCCOY CONDITION

PACE P. NIELSEN

ABSTRACT. We prove that all reversible rings are McCoy, generalizing the fact that both commutative and reduced rings are McCoy. We then give an example of a semi-commutative ring that is not right McCoy. At the same time, we also show that semi-commutative rings do have a property close to the McCoy condition.

1. INTRODUCTION

It is often taught in an elementary algebra course that if R is a commutative ring, and $f(x)$ is a zero-divisor in $R[x]$, then there is a nonzero element $r \in R$ with $f(x)r = 0$. This was first proved by McCoy [6, Theorem 2]. One can then make the following definition:

Definition. Let R be an associative ring with 1. We say that R is *right McCoy* when the equation $f(x)g(x) = 0$ over $R[x]$, where $f(x), g(x) \neq 0$, implies there exists a nonzero $r \in R$ with $f(x)r = 0$. We define *left McCoy* rings similarly. If a ring is both left and right McCoy we say that the ring is a McCoy ring.

As one would expect, reduced rings are McCoy. (In fact, reduced rings are Armendariz rings. A ring, R, is Armendariz if given $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ and $g(x) = \sum_{i=0}^{n} b_i x^i \in R[x]$ with $f(x)g(x) = 0$ this implies $a_i b_j = 0$ for all i, j.) A natural question is whether there is a class of rings that are McCoy, which also encompasses all reduced rings and all commutative rings. Recall that a ring R is called:

$$\begin{aligned} &\textit{symmetric} \text{ if } abc = 0 \Longrightarrow bac = 0, \text{ for all } a, b, c \in R,\\ &\textit{reversible} \text{ if } ab = 0 \Longrightarrow ba = 0, \text{ for all } a, b \in R,\\ &\textit{semi-commutative} \text{ if } ab = 0 \Longrightarrow aRb = 0, \text{ for all } a, b \in R.\end{aligned}$$

The following implications hold:

$$\left.\begin{array}{r}\text{reduced}\\ \text{commutative}\end{array}\right\} \Rightarrow \text{ symmetric } \Rightarrow \text{ reversible } \Rightarrow \text{ semi-commutative.}$$

In general, each of these implications is irreversible (see [5]). In [2, Corollary 2.3] it was claimed that all semi-commutative rings were McCoy. However, Hirano's claim assumed that if R is semi-commutative then $R[x]$ is semi-commutative, and this was later shown to be false in [3, Example 2]. However, the question of whether semi-commutativity implied the McCoy condition was left open. Herein we show

2000 *Mathematics Subject Classification.* Primary 16U80, Secondary 16S15.
Key words and phrases. Reduced Ring, Semi-commutative Ring, McCoy Condition.

that all reversible rings are McCoy, but give an example of a ring that is semi-commutative but not right McCoy, thus settling this issue. On the other hand, we do prove that semi-commutative rings satisfy a McCoy-like condition.

I wish to thank T. Y. Lam for introducing me to this problem. Also, while the proof of Theorem 2 is of my own construction, the result had been previously proven by T. Y. Lam, A. Leroy, and J. Matczuk in some unpublished joint work. I thank them for allowing me to include the result here.

2. A Large Class of McCoy Rings

We will shortly prove that all reversible rings are McCoy. To do so, we need to investigate what relations one can derive from $f(x)g(x) = 0$ when R is reversible, or more generally when R is semi-commutative.

Lemma 1. *Let R be a semi-commutative ring. Also let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be elements of $R[x]$. If $f(x)g(x) = 0$ then $a_i b_0^{i+1} = 0$ for all $i \in [0, m]$.*

Proof. The degree i part of the equation $f(x)g(x) = 0$ yields

$$(*)_i \qquad \sum_{j=0}^{i} a_j b_{i-j} = 0$$

for each $i \in [0, m]$. In particular, for $i = 0$ we have $a_0 b_0 = 0$.

Now, suppose by induction that $a_j b_0^{j+1} = 0$ for all $j < k$. In particular, $a_j b_0^k = 0$ and hence by semi-commutativity $a_j b_{k-j} b_0^k = 0$, for all $j < k$. Then taking $(*)_k$ and multiplying on the right by b_0^k yields

$$0 = \sum_{j=0}^{k} a_j b_{k-j} b_0^k = a_k b_0^{k+1}.$$

This finishes our inductive step, and the proof. □

Theorem 2. *If R is a reversible ring then R is a McCoy ring.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] - \{0\}$, and suppose $f(x)g(x) = 0$. Clearly it suffices to just prove that R is left McCoy. In fact, we will show something slightly stronger. For any polynomial $a(x) \in R[x]$ set C_a equal to the left ideal generated by the coefficients of $a(x)$. We will show, by induction, that there exists some $c \in C_f - \{0\}$ with $c \cdot g(x) = 0$, and this will imply R is left McCoy.

First, note that we may assume that $a_0, b_0 \neq 0$, after dividing $f(x)$ and $g(x)$ by powers of x if necessary. Also, we may assume that m and n are the actual degrees of $f(x)$ and $g(x)$ respectively. If $n = 0$ set $c = a_0$ and then $cg(x) = a_0 b_0 = 0$ and we are done. So we may assume $n \geqslant 1$. We also may suppose, by induction on the degree of $g(x)$, that for all $k < n$, if we have $a(x)b(x) = 0$ with $a(x), b(x) \in R[x] - \{0\}$ and $\deg(b(x)) = k$, then there is some $c \in C_a - \{0\}$ such that $c \cdot b(x) = 0$.

Choose $\ell \geqslant 0$ such that $f(x) b_0^{\ell+1} = 0 \neq f(x) b_0^{\ell}$; such an ℓ exists by Lemma 1 above. Now, using reversibility (in R), $a(x) := b_0^{\ell} f(x) \neq 0 = b_0^{\ell} f(x) b_0$. So we have both $a(x)g(x) = 0$ and $a(x)b_0 = 0$. If we set $b(x) := (g(x) - b_0)/x$ then the above equations imply $a(x)b(x) = 0$. Note that $b(x) \neq 0$ since $\deg(g(x)) = n > 0$, and also note $\deg(b(x)) = n - 1 < n$. So, by the inductive hypothesis, there is some $c \in C_a - \{0\}$ such that $c \cdot b(x) = 0$. Since $a(x)b_0 = 0$ this means $C_a b_0 = (0)$ and hence $cb_0 = 0$. Therefore, $c \cdot g(x) = 0$. On the other hand, by construction of $a(x)$ we have $C_a \subseteq C_f$. Thus $c \in C_f$, and this finishes our induction step. Therefore, we have proven that there is some nonzero $c \in C_f$ with $c \cdot g(x) = 0$ no matter what degree $g(x)$ has. □

Due to an example of Kim and Lee [4, Example 2.1], we know that if R is reversible then $R[x]$ may not even be semi-commutative. So the method of proof employed in [2] is not sufficient to prove Theorem 2. On the other hand, let $S = \mathbb{F}_2\langle a, b\rangle/\langle a^2, ab, b^2\rangle$. One can easily check that $S[x]$ is semi-commutative, but S is not reversible. Therefore, our result does not encompass Hirano's. This raises the new question:

Question: Is there a natural class of McCoy rings, which includes all reversible rings and all rings, R, where $R[x]$ is semi-commutative?

While we will show that semi-commutative rings are not McCoy in general, there is another nice condition that they do satisfy. The basic idea is to generalize Lemma 1 to construct a zero-divisor out of all the coefficients of $g(x)$.

Lemma 3. *Let R be semi-commutative. Also let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be elements of $R[x]$. If $f(x)g(x) = 0$, with $f(x) \neq 0$, then there exist non-negative integers $\ell_0, \ell_1, \dots, \ell_n \in \mathbb{N}$ that satisfy for each $k \in [0, n]$:*

$$f(x)b_k^{\ell_k} b_{k-1}^{\ell_{k-1}} \cdots b_0^{\ell_0} \neq 0 = f(x)b_k^{\ell_k+1} b_{k-1}^{\ell_{k-1}} \cdots b_0^{\ell_0}.$$

Proof. The existence of ℓ_0 follows from Lemma 1. Suppose, by induction, we have constructed $\ell_0, \dots, \ell_j$ satisfying the above conditions. Set $r = b_j^{\ell_j} b_{j-1}^{\ell_{j-1}} \cdots b_0^{\ell_0}$. Using $f(x)g(x) = 0$, we have the following $m + 1$ equations:

$$\begin{array}{llcccccccl}
(*)_{j+1} & a_0 b_{j+1} & + & a_1 b_j & + & \cdots & + & a_m b_{j-m+1} & = & 0, \\
(*)_{j+2} & a_0 b_{j+2} & + & a_1 b_{j+1} & + & \cdots & + & a_m b_{j-m+2} & = & 0, \\
 & \cdots & & & & & & & & \\
(*)_{j+m+1} & a_0 b_{j+m+1} & + & a_1 b_{j+m} & + & \cdots & + & a_m b_{j+1} & = & 0,
\end{array}$$

where $b_k = 0$ if $k < 0$ or $k > n$, and where the index on $(*)_\alpha$ just means we are looking at the degree α part of $f(x)g(x) = 0$. Notice that $a_i b_k r = 0$ for all i and all $k \leqslant j$, by semi-commutativity and from how $\ell_0, \dots, \ell_j$ have been chosen. We will use this fact, and the $m + 1$ equations above, to show $a_i b_{j+1}^{i+1} r = 0$ for all $i \leqslant m$. Once we establish this we just take ℓ_{j+1} to be the smallest non-negative integer such that $a_i b_{j+1}^{\ell_{j+1}+1} r = 0$ for all i.

Multiplying equation $(*)_{j+1}$ on the right by r yields

$$0 = a_0 b_{j+1} r + a_1 b_j r + \cdots + a_m b_{j-m+1} r = a_0 b_{j+1} r.$$

Now multiplying $(*)_{j+2}$ on the right by $b_{j+1} r$, and using semi-commutativity, we have

$$0 = a_0 b_{j+2} b_{j+1} r + a_1 b_{j+1}^2 r + \cdots + a_m b_{j-m+2} b_{j+1} r = a_1 b_{j+1}^2 r$$

since all but the second term must be zero. Continuing in this fashion, we obtain $a_i b_{j+1}^{i+1} r = 0$ for all $i \leqslant m$ as claimed. Thus ℓ_{j+1} can be defined. By induction, we have defined $\ell_0, \ell_1, \dots, \ell_n$. □

Theorem 4. *Let R be semi-commutative. Given $f(x)g(x) = 0$ with $f(x), g(x) \neq 0$ then (at least) one of $\operatorname{ann}_r^{R[x]}(f(x)) \cap R$ or $\operatorname{ann}_r^{R[x]}(g(x)) \cap R$ is nonzero. (Similarly, for the left annihilators.)*

Proof. Write $f(x)$ and $g(x)$ as before. Let $y = b_n^{\ell_n} \cdots b_0^{\ell_0}$ where $\ell_0, \dots, \ell_n$ are defined as in Lemma 3. If $g(x)y \neq 0$, let y' be a non-zero coefficient of this new polynomial. Then, by definition of $\ell_0, \dots, \ell_n$, and by semi-commutativity, $f(x)y' = 0$. □

3. Semi-commutative but not McCoy

Let $k = \mathbb{F}_2\langle a_0, a_1, a_2, a_3, b_0, b_1\rangle$ be the free associative algebra (with 1) over $\mathbb{F}_2$ generated by six indeterminates (as labelled above). Let I be the ideal generated by the following relations:

$$\begin{aligned}&\langle a_0b_0,\ \ a_0b_1 + a_1b_0,\ \ a_1b_1 + a_2b_0,\ \ a_2b_1 + a_3b_0,\ \ a_3b_1,\\ &\ a_0a_j\ \ (0 \leqslant j \leqslant 3),\ \ a_3a_j\ \ (0 \leqslant j \leqslant 3),\ \ a_1a_j + a_2a_j\ \ (0 \leqslant j \leqslant 3),\\ &\ b_ib_j\ \ (0 \leqslant i, j \leqslant 1),\ \ b_ia_j\ \ (0 \leqslant i \leqslant 1,\ \ 0 \leqslant j \leqslant 3)\rangle.\end{aligned}$$

We let $R = k/I$. Think of $\{a_0, a_1, a_2, a_3, b_0, b_1\}$ as elements (sometimes called letters) of R satisfying the relations in I, suppressing the bar notation.

Put $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $G(x) = b_0 + b_1x$. The first row of relations in I guarantees that $F(x) \cdot G(x) = 0$ in $R[x]$. We will show later that $F(x), G(x) \neq 0$ in $R[x]$. Further, we'll demonstrate that R is semi-commutative. Finally, we will prove that R is left McCoy but not right McCoy.

Notice I is a homogeneous ideal. Therefore, there is a notion of degree on the (non-zero) monomials of R. We will describe how each element of R can be written in a unique reduced form.

Claim 5. *Any element $\gamma \in R$ can be written uniquely in the form*

$$\begin{aligned}\gamma =& f_0 + f_1(a_2)a_1 + f_2(a_2)a_2 + g(a_2)a_0 + h(a_2)a_3\\ &+ (r_0 + r_1(a_2)a_1 + r_2(a_2)a_2 + r_3(a_2)a_3)b_0 + s_0b_1\\ &\qquad \textit{with}\ \ f_0, r_0, s_0 \in \mathbb{F}_2,\\ &\textit{and}\ \ f_1(x), f_2(x), g(x), h(x), r_1(x), r_2(x), r_3(x) \in \mathbb{F}_2[x].\end{aligned}$$

Proof. This is just a direct use of the diamond lemma, where one reduces any given monomial using the relations specified in the definition of I (see [1]). We describe how to make a reduction, and leave it to the reader to show that the hypotheses of the diamond lemma hold.

First, check whether the monomial we are reducing has any occurrence of a_0b_0, a_3b_1, b_ib_j, b_ia_j, a_0a_j, or a_3a_j. If so, then the monomial is zero. If not, repeatedly replace all occurrences of a_ib_1 with $a_{i+1}b_0$ and all occurrences of a_1a_j with a_2a_j. (Equivalently, we always try to reduce the index on b_j and increase the index on a_i.) The resulting monomial will be in reduced form. □

Claim 6. *The ring R is semi-commutative.*

Proof. Let $\gamma, \gamma' \in R$ with $\gamma\gamma' = 0$. Write γ in the unique form of Claim 5. For ease of notation, we will write f_1 for $f_1(a_2)$, and will do the same for all other polynomials in the variable a_2. We also write γ' in the unique form of Claim 5, so $\gamma' = f_0' + f_1'a_1 + \cdots + s_0'b_1$. Also, put $f = f_0 + f_1a_1 + f_2a_2$, and define f' similarly. Throughout we use the fact that I is a homogeneous ideal, so it follows that all the monomials of any given degree in $\gamma\gamma'$ must add to zero.

To prove that $\gamma r\gamma' = 0$ for all $r \in R$, we first show that this is true for the letters (i.e. monomials of degree 1). If γ or γ' is zero, then this is trivial. So, we may assume that $\gamma, \gamma' \neq 0$. Now $\gamma\gamma' = 0$ implies $f_0f_0' = 0$. Thus, $f_0 = 0$ or $f_0' = 0$.

First suppose that $f_0 = 0$. Let $\delta \neq 0$ be the sum of the (non-zero) terms of γ with lowest degree. Since I is homogeneous, $\delta f_0' = 0$. Therefore $f_0' = 0$. Similarly, if we assume $f_0' = 0$ we obtain $f_0 = 0$. So in all cases $f_0 = f_0' = 0$.

Notice that $b_i\gamma' = 0$ for $i = 0, 1$, since $f_0' = 0$. Therefore, $\gamma b_i \gamma' = 0$. So we only need to check whether $\gamma a_j \gamma' = 0$ for $0 \leqslant j \leqslant 3$. An easy computation shows that $\gamma a_j = (f_1 + f_2)a_2 a_j$, so if $f_1 = f_2$ then $\gamma a_j \gamma' = 0$. Therefore, we may also assume that $f_1 \neq f_2$. We will show below that this contradicts $\gamma' \neq 0$.

Calculating the reduced form for $\gamma\gamma'$ yields

$$\begin{aligned} 0 = \gamma\gamma' =& (f_1 + f_2)a_2(f_1' a_1 + f_2' a_2 + g' a_0 + h' a_3) \\ &+ (s_0' f_2 + r_0' h + (f_1 + f_2)a_2 r_3')a_3 b_0 \\ &+ (s_0' f_1 + r_0' f_2 + (f_1 + f_2)a_2 r_2')a_2 b_0 \\ &+ (s_0' g + r_0' f_1 + (f_1 + f_2)a_2 r_1')a_1 b_0. \end{aligned}$$

Since $f_1 + f_2 \neq 0$ we must have $f_1' = f_2' = g' = h' = 0$. Also, from the last three lines we obtain

$$s_0' f_2 + r_0' h + (f_1 + f_2)a_2 r_3' = 0, \tag{1}$$

$$s_0' f_1 + r_0' f_2 + (f_1 + f_2)a_2 r_2' = 0, \tag{2}$$

$$s_0' g + r_0' f_1 + (f_1 + f_2)a_2 r_1' = 0. \tag{3}$$

Suppose $s_0' = 1$. If $r_0' = 1$, then equation (2) implies $\deg((f_1 + f_2)a_2) \leqslant \deg(f_1 + f_2)$, which is impossible since $f_1 \neq f_2$. So $r_0' = 0$. But then adding equations (1) and (2) gives the same contradiction.

So we must have $s_0' = 0$. If $r_0' = 1$, then adding equations (2) and (3) we reach the same contradiction as before. Therefore $r_0' = 0$ also. But then since $f_1 \neq f_2$ we have $r_1' = r_2' = r_3' = 0$, and hence $\gamma' = 0$, contradicting our previous assumption that $\gamma' \neq 0$.

This shows that in all cases $\gamma r \gamma' = 0$ if r is a letter. Repeating the above argument, replacing γ by γr, the same is true if r is a monomial of any positive degree. But $\gamma\gamma' = 0$, so it also holds if $r = 1$. Since any element of R is just a sum of monomials, putting this all together yields $\gamma r \gamma' = 0$ for all $r \in R$. Therefore R is semi-commutative. □

Notice that Claim 5 implies each of the coefficients of $F(x)$ and $G(x)$ is non-zero. In particular, $F(x), G(x) \neq 0$ in $R[x]$. So we have the following:

Claim 7. *The ring R is not right McCoy.*

Proof. It suffices to show that if $F(x)r = 0$ for some $r \in R$, then $r = 0$. Thus, it suffices to show that if $a_2 r = 0$ then $r = 0$. This follows trivially from Claim 5, by calculating the reduced form for $a_2 r$ for any $r \in R$. □

Claim 8. *The ring R is left McCoy.*

Proof. Let $P(x), Q(x) \in R[x] - \{0\}$, with $P(x) \cdot Q(x) = 0$. Write $P(x) = \sum_{i=1}^m p_i x^i$ and $Q(x) = \sum_{i=1}^n q_i x^i$. If each q_i has zero constant term then $b_0 Q(x) = 0$, and we are done. So we may assume that some q_i has a non-zero constant term. Let k be the smallest index such that q_k has this property.

For each $p_i \neq 0$ let p_i' be the sum of the non-zero terms of p_i of smallest degree, and for each $p_i = 0$ put $p_i' = 0$. Let j be the smallest index such that, among the non-zero members of $\{p'_0, p'_1, \ldots, p'_m\}$, we have p'_j is of minimal degree. Notice that j exists since $P(x) \neq 0$.

Now, looking at the degree $j+k$ part of the equation $P(x) \cdot Q(x) = 0$ we have

$$\sum_{r,s:\ r+s=j+k} p_r q_s = 0. \tag{4}$$

Hence, I being an homogeneous ideal implies that the terms of any fixed degree in equation (4) must add to 0. But from our choice of j and k there is only one term in equation (4) of smallest degree, namely $p'_j \cdot 1 \neq 0$ coming from $p_j q_k$. Therefore this gives a contradiction, and we are done. □

4. Final Remarks

We note that there is a simpler ring that is left McCoy but not right McCoy. Set $k' = \mathbb{F}_2\langle c_0, c_1, d_0, d_1\rangle$, and put $R' = k'/J$ where J is the ideal generated by the relations:

$$\langle c_0d_0,\ \ c_0d_1 + c_1d_0,\ \ c_1d_1,\ \ d_id_j\ \ (0 \leqslant i, j \leqslant 1),\ \ d_ic_j\ \ (0 \leqslant i, j \leqslant 1)\rangle.$$

If $C(x) = c_0 + c_1x$ and $D(x) = d_0 + d_1x$, then $C(x) \cdot D(x) = 0$. One now proceeds as above: first, describing arbitrary elements in reduced form; second, showing that $C(x)r = 0$ for $r \in R'$ if and only if $r = 0$; and third, copying the ideas of Claim 8 to show R' is left McCoy.

References

[1] George M. Bergman. The diamond lemma for ring theory. *Adv. in Math.*, 29(2):178–218, 1978.

[2] Yasuyuki Hirano. On annihilator ideals of a polynomial ring over a noncommutative ring. *J. Pure Appl. Algebra*, 168(1):45–52, 2002.

[3] Chan Huh, Yang Lee, and Agata Smoktunowicz. Armendariz rings and semicommutative rings. *Comm. Algebra*, 30(2):751–761, 2002.

[4] Nam Kyun Kim and Yang Lee. Extensions of reversible rings. *J. Pure Appl. Algebra*, 185(1-3):207–223, 2003.

[5] Greg Marks. A taxonomy of 2-primal rings. *J. Algebra*, 266(2):494–520, 2003.

[6] N. H. McCoy. Remarks on divisors of zero. *Amer. Math. Monthly*, 49:286–295, 1942.

Department of Mathematics, University of California, Berkeley, CA 94720

E-mail address: pace@math.berkeley.edu