SEMI-COMMUTATIVITY AND THE MCCOY CONDITION

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Abstract. We prove that all reversible rings are McCoy, generalizing the fact that both commutative and reduced rings are McCoy. We then give an example of a semi-commutative ring that is not right McCoy. At the same time, we also show that semi-commutative rings do have a property close to the McCoy condition.

1. Introduction

It is often taught in an elementary algebra course that if \( R \) is a commutative ring, and \( f(x) \) is a zero-divisor in \( R[x] \), then there is a nonzero element \( r \in R \) with \( f(x)r = 0 \). This was first proved by McCoy [6, Theorem 2]. One can then make the following definition:

Definition. Let \( R \) be an associative ring with 1. We say that \( R \) is right McCoy when the equation \( f(x)g(x) = 0 \) over \( R[x] \), where \( f(x), g(x) \neq 0 \), implies there exists a nonzero \( r \in R \) with \( f(x)r = 0 \). We define left McCoy rings similarly. If a ring is both left and right McCoy we say that the ring is a McCoy ring.

As one would expect, reduced rings are McCoy. (In fact, reduced rings are Armendariz rings. A ring, \( R \), is Armendariz if given \( f(x) = \sum_{i=0}^{m} a_i x^i \in R[x] \) and \( g(x) = \sum_{i=0}^{n} b_i x^i \in R[x] \) with \( f(x)g(x) = 0 \) this implies \( a_i b_j = 0 \) for all \( i, j \).) A natural question is whether there is a class of rings that are McCoy, which also encompasses all reduced rings and all commutative rings. Recall that a ring \( R \) is called:

- symmetric if \( abc = 0 \implies bac = 0 \), for all \( a, b, c \in R \),
- reversible if \( ab = 0 \implies ba = 0 \), for all \( a, b \in R \),
- semi-commutative if \( ab = 0 \implies aRb = 0 \), for all \( a, b \in R \).

The following implications hold:

\[
\text{reduced} \quad \Rightarrow \quad \text{symmetric} \quad \Rightarrow \quad \text{reversible} \quad \Rightarrow \quad \text{semi-commutative}.
\]

In general, each of these implications is irreversible (see [5]). In [2, Corollary 2.3] it was claimed that all semi-commutative rings were McCoy. However, Hirano’s claim assumed that if \( R \) is semi-commutative then \( R[x] \) is semi-commutative, and this was later shown to be false in [3, Example 2]. However, the question of whether semi-commutativity implied the McCoy condition was left open. Herein we show
that all reversible rings are McCoy, but give an example of a ring that is semi-commutative but not right McCoy, thus settling this issue. On the other hand, we do prove that semi-commutative rings satisfy a McCoy-like condition.

I wish to thank T. Y. Lam for introducing me to this problem. Also, while the proof of Theorem 2 is of my own construction, the result had been previously proven by T. Y. Lam, A. Leroy, and J. Matczuk in some unpublished joint work. I thank them for allowing me to include the result here.

2. A LARGE CLASS OF MCCOY RINGS

We will shortly prove that all reversible rings are McCoy. To do so, we need to investigate what relations one can derive from \( f(x)g(x) = 0 \) when \( R \) is reversible, or more generally when \( R \) is semi-commutative.

**Lemma 1.** Let \( R \) be a semi-commutative ring. Also let \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \) be elements of \( R[x] \). If \( f(x)g(x) = 0 \) then \( a_i b_{i+1}^0 = 0 \) for all \( i \in [0, m] \).

**Proof.** The degree \( i \) part of the equation \( f(x)g(x) = 0 \) yields

\[
(*)_i \quad \sum_{j=0}^{i} a_j b_{i-j} = 0
\]

for each \( i \in [0, m] \). In particular, for \( i = 0 \) we have \( a_0 b_0 = 0 \).

Now, suppose by induction that \( a_j b_{j+1}^0 = 0 \) for all \( j < k \). In particular, \( a_j b_k^0 = 0 \) and hence by semi-commutativity \( a_j b_{k-j} b_0^j = 0 \), for all \( j < k \). Then taking \((*)_k\) and multiplying on the right by \( b_0^k \) yields

\[
0 = \sum_{j=0}^{k} a_j b_{k-j} b_0^j = a_k b_0^{k+1}.
\]

This finishes our inductive step, and the proof. \( \square \)

**Theorem 2.** If \( R \) is a reversible ring then \( R \) is a McCoy ring.

**Proof.** Let \( f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] - \{0\} \), and suppose \( f(x)g(x) = 0 \). Clearly it suffices to just prove that \( R \) is left McCoy. In fact, we will show something slightly stronger. For any polynomial \( a(x) \in R[x] \) set \( C_a \) equal to the left ideal generated by the coefficients of \( a(x) \). We will show, by induction, that there exists some \( c \in C_f - \{0\} \) with \( c \cdot g(x) = 0 \), and this will imply \( R \) is left McCoy.

First, note that we may assume that \( a_0, b_0 \neq 0 \), after dividing \( f(x) \) and \( g(x) \) by powers of \( x \) if necessary. Also, we may assume that \( m \) and \( n \) are the actual degrees of \( f(x) \) and \( g(x) \) respectively. If \( n = 0 \) set \( c = a_0 \) and then \( c g(x) = a_0 b_0 = 0 \) and we are done. So we may assume \( n \geq 1 \). We also may suppose, by induction on the degree of \( g(x) \), that for all \( k < n \), if we have \( a(x) b(x) = 0 \) with \( a(x), b(x) \in R[x] - \{0\} \) and \( \deg(b(x)) = k \), then there is some \( c \in C_a - \{0\} \) such that \( c \cdot b(x) = 0 \).

Choose \( \ell \geq 0 \) such that \( f(x)b_0^{\ell+1} = 0 \neq f(x)b_0^\ell \); such an \( \ell \) exists by Lemma 1 above. Now, using reversibility (in \( R \)), \( a(x) := b_0^\ell f(x) \neq 0 = b_0^\ell f(x) b_0 \). So we have both \( a(x)g(x) = 0 \) and \( a(x)b_0 = 0 \). If we set \( b(x) := (g(x) - b_0)/x \) then the above equations imply \( a(x)b(x) = 0 \). Note that \( b(x) \neq 0 \) since \( \deg(g(x)) = n > 0 \), and also note \( \deg(b(x)) = n - 1 < n \). So, by the inductive hypothesis, there is some \( c \in C_a - \{0\} \) such that \( c \cdot b(x) = 0 \). Since \( a(x)b_0 = 0 \) this means \( C_a b_0 = (0) \) and
have defined for the left annihilators.) Then (at least) one of
Theorem 4. Let \( a \) since all but the second term must be zero. Continuing in this fashion, we obtain

\[
\text{Now multiplying (} b_j 0 = 0 = 0 = 0 \text{) yields}
\]

Once we establish this we just take \( f(x)g(x) = 0 \) with \( f(x)g(x) = 0 \) no matter what degree

Due to an example of Kim and Lee [4, Example 2.1], we know that if \( R \) is reversible then \( R[x] \) may not even be semi-commutative. So the method of proof employed in [2] is not sufficient to prove Theorem 2. On the other hand, let \( S = \mathbb{F}_2(a, b)/\langle a^2, ab, b^2 \rangle \). One can easily check that \( S[x] \) is semi-commutative, but \( S \) is not reversible. Therefore, our result does not encompass Hirano’s. This raises the new question:

**Question:** Is there a natural class of McCoy rings, which includes all reversible rings and all rings, \( R \), where \( R[x] \) is semi-commutative?

While we will show that semi-commutative rings are not McCoy in general, there is another nice condition that they do satisfy. The basic idea is to generalize Lemma 1 to construct a zero-divisor out of all the coefficients of \( g(x) \).

**Lemma 3.** Let \( R \) be semi-commutative. Also let \( f(x) = \sum_{i=0}^m a_ix^i \) and \( g(x) = \sum_{j=0}^n b_jx^j \) be elements of \( R[x] \). If \( f(x)g(x) = 0 \), with \( f(x) \neq 0 \), then there exist non-negative integers \( \ell_0, \ell_1, \ldots, \ell_n \in \mathbb{N} \) that satisfy for each \( k \in [0, n] \):

\[
f(x)b_k^k b_{k-1} \cdots b_0 = 0 = f(x)b_k^k b_{k-1} \cdots b_0.
\]

**Proof.** The existence of \( \ell_0 \) follows from Lemma 1. Suppose, by induction, we have constructed \( \ell_0, \ldots, \ell_j \) satisfying the above conditions. Set \( r = b_j^j b_{j-1}^j \cdots b_0^j \). Using \( f(x)g(x) = 0 \), we have the following \( m + 1 \) equations:

\[
\begin{align*}
(*)&_{j+1} & a_0b_{j+1} + a_1b_j + \cdots + a_mb_{j-m+1} = 0, \\
(*)&_{j+2} & a_0b_{j+2} + a_1b_{j+1} + \cdots + a_mb_{j-m+2} = 0, \\
& \cdots \\
(*)&_{j+m+1} & a_0b_{j+m+1} + a_1b_{j+m} + \cdots + a_mb_{j+1} = 0,
\end{align*}
\]

where \( b_k \) is the smallest non-negative integer such that \( a_i b_{j+i+1}^j r = 0 \) for all \( i \).

Multiplying equation \( (*)&_{j+1} \) on the right by \( r \) yields

\[
0 = a_0b_{j+1}r + a_1b_jr + \cdots + a_mb_{j-m+1}r = a_0b_{j+1}r.
\]

Now multiplying \( (*)&_{j+2} \) on the right by \( b_{j+1}r \), and using semi-commutativity, we have

\[
0 = a_0b_{j+2}b_{j+1}r + a_1b^2_{j+1}r + \cdots + a_mb_{j-m+2}b_{j+1}r = a_1b^2_{j+1}r
\]

since all but the second term must be zero. Continuing in this fashion, we obtain \( a_i b_{j+i+1}^j r = 0 \) for all \( i \leq m \) as claimed. Thus \( \ell_{j+1} \) can be defined. By induction, we have defined \( \ell_0, \ell_1, \ldots, \ell_n \).

**Theorem 4.** Let \( R \) be semi-commutative. Given \( f(x)g(x) = 0 \) with \( f(x), g(x) \neq 0 \) then (at least) one of \( \text{ann}^R(f(x)) \cap R \) or \( \text{ann}^R(g(x)) \cap R \) is nonzero. (Similarly, for the left annihilators.)
Proof. Write $f(x)$ and $g(x)$ as before. Let $y = b'_n \cdots b'_0$ where $\ell_0, \ldots, \ell_n$ are defined as in Lemma 3. If $g(x)y \neq 0$, let $y'$ be a non-zero coefficient of this new polynomial. Then, by definition of $\ell_0, \ldots, \ell_n$, and by semi-commutativity, $f(x)y' = 0$. \hfill \Box

3. Semi-commutative but not McCoy

Let $k = \mathbb{F}_2(a_0, a_1, a_2, a_3, b_0, b_1)$ be the free associative algebra (with 1) over $\mathbb{F}_2$ generated by six indeterminates (as labelled above). Let $I$ be the ideal generated by the following relations:

\[
\begin{align*}
&\langle a_0b_0, \ a_0b_1 + a_1b_0, \ a_1b_1 + a_2b_0, \ a_2b_1 + a_3b_0, \ a_3b_1, \\
&a_0a_j \ (0 \leq j \leq 3), \ a_3a_j \ (0 \leq j \leq 3), \ a_1a_j + a_2a_j \ (0 \leq j \leq 3), \\
&b_ib_j \ (0 \leq i, j \leq 1), \ b_ia_j \ (0 \leq i \leq 1, \ 0 \leq j \leq 3).
\end{align*}
\]

We let $R = k/I$. Think of $\{a_0, a_1, a_2, a_3, b_0, b_1\}$ as elements (sometimes called letters) of $R$ satisfying the relations in $I$, suppressing the bar notation.

Put $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $G(x) = b_0 + b_1x$. The first row of relations in $I$ guarantees that $F(x) \cdot G(x) = 0$ in $R[x]$. We will show later that $F(x), G(x) \neq 0$ in $R[x]$. Further, we’ll demonstrate that $R$ is semi-commutative. Finally, we will prove that $R$ is left McCoy but not right McCoy.

Notice $I$ is a homogeneous ideal. Therefore, there is a notion of degree on the (non-zero) monomials of $R$. We will describe how each element of $R$ can be written in a unique reduced form.

Claim 5. Any element $\gamma \in R$ can be written uniquely in the form

\[
\gamma = f_0 + f_1(a_2)a_1 + f_2(a_2)a_2 + g(a_2)a_0 + h(a_2)a_3
\]

\[
+ (r_0 + r_1(a_2)a_1 + r_2(a_2)a_2 + r_3(a_2)a_3)b_0 + s_0b_1
\]

with $f_0, r_0, s_0 \in \mathbb{F}_2$.

and $f_1(x), f_2(x), g(x), h(x), r_1(x), r_2(x), r_3(x) \in \mathbb{F}_2[x]$.\newline

Proof. This is just a direct use of the diamond lemma, where one reduces any given monomial using the relations specified in the definition of $I$ (see [1]). We describe how to make a reduction, and leave it to the reader to show that the hypotheses of the diamond lemma hold.

First, check whether the monomial we are reducing has any occurrence of $a_0b_0$, $a_3b_1$, $b_ib_j$, $b_ia_j$, $a_0a_j$, or $a_3a_j$. If so, then the monomial is zero. If not, repeatedly replace all occurrences of $a_i b_1$ with $a_{i+1} b_0$ and all occurrences of $a_1 a_j$ with $a_2 a_j$. (Equivalently, we always try to reduce the index on $b_j$ and increase the index on $a_i$.) The resulting monomial will be in reduced form. \hfill \Box

Claim 6. The ring $R$ is semi-commutative.

Proof. Let $\gamma, \gamma' \in R$ with $\gamma \gamma' = 0$. Write $\gamma$ in the unique form of Claim 5. For ease of notation, we will write $f_1$ for $f_1(a_2)$, and will do the same for all other polynomials in the variable $a_2$. We also write $\gamma'$ in the unique form of Claim 5, so $\gamma' = f'_0 + f'_1 a_1 + \cdots + s'_0 b_1$. Also, put $f = f_0 + f_1 a_1 + f_2 a_2$, and define $f'$ similarly. Throughout we use the fact that $I$ is a homogeneous ideal, so it follows that all the monomials of any given degree in $\gamma \gamma'$ must add to zero.

To prove that $\gamma \gamma' = 0$ for all $r \in R$, we first show that this is true for the letters (i.e. monomials of degree 1). If $\gamma$ or $\gamma'$ is zero, then this is trivial. So, we may assume that $\gamma, \gamma' \neq 0$. Now $\gamma \gamma' = 0$ implies $f_0 f'_0 = 0$. Thus, $f_0 = 0$ or $f'_0 = 0$. If $f_0 = 0$, then $f_1 a_1 + f_2 a_2$ is semi-commutative, and so is $f'$. We will now show that $f' a_1 + f_2 a_2$ is semi-commutative. To prove this, we will use the fact that $I$ is a homogeneous ideal. Therefore, there is a notion of degree on the (non-zero) monomials of $R$. We will describe how each element of $R$ can be written in a unique reduced form.\hfill \Box
First suppose that \( f_0 = 0 \). Let \( \delta \neq 0 \) be the sum of the (non-zero) terms of \( \gamma \) with lowest degree. Since \( I \) is homogeneous, \( \delta f_0 = 0 \). Therefore \( f_0 = 0 \). Similarly, if we assume \( f_k = 0 \) we obtain \( f_0 = 0 \). So in all cases \( f_0 = f_k = 0 \).

Notice that \( b_i \gamma' = 0 \) for \( i = 0, 1 \), since \( f_0 = 0 \). Therefore, \( \gamma b_i \gamma' = 0 \). So we only need to check whether \( \gamma a_j \gamma' = 0 \) for \( 0 \leq j \leq 3 \). An easy computation shows that \( \gamma a_j = (f_1 + f_2) a_2 a_j \), so if \( f_1 = f_2 \) then \( \gamma a_j \gamma' = 0 \). Therefore, we may also assume that \( f_1 \neq f_2 \). We will show below that this contradicts \( \gamma' \neq 0 \).

Calculating the reduced form for \( \gamma \gamma' \) yields

\[
0 = \gamma \gamma' = (f_1 + f_2) a_2 (f_1' a_1 + f_2' a_2 + g' a_0 + h' a_3) + (s_0' f_2 + r_0' h + (f_1 + f_2) a_2 r_3') a_3 b_0 + (s_0' f_1 + r_0' f_2 + (f_1 + f_2) a_2 r_2') a_2 b_0 + (s_0' g + r_0' f_1 + (f_1 + f_2) a_2 r_1') a_1 b_0.
\]

Since \( f_1 + f_2 \neq 0 \) we must have \( f_1' = f_2' = g' = h' = 0 \). Also, from the last three lines we obtain

\[
\begin{align*}
(1) & \quad s_0' f_2 + r_0' h + (f_1 + f_2) a_2 r_3' = 0, \\
(2) & \quad s_0' f_1 + r_0' f_2 + (f_1 + f_2) a_2 r_2' = 0, \\
(3) & \quad s_0' g + r_0' f_1 + (f_1 + f_2) a_2 r_1' = 0.
\end{align*}
\]

Suppose \( s_0' = 1 \). If \( r_0' = 1 \), then equation (2) implies \( \deg((f_1 + f_2) a_2) \leq \deg(f_1 + f_2) \), which is impossible since \( f_1 \neq f_2 \). So \( r_0' = 0 \). But then adding equations (1) and (2) gives the same contradiction.

So we must have \( s_0' = 0 \). If \( r_0' = 1 \), then adding equations (2) and (3) we reach the same contradiction as before. Therefore \( r_0' = 0 \) also. But then since \( f_1 \neq f_2 \) we have \( r_1' = r_2' = r_3' = 0 \), and hence \( \gamma' = 0 \), contradicting our previous assumption that \( \gamma' \neq 0 \).

This shows that in all cases \( \gamma r \gamma' = 0 \) if \( r \) is a letter. Repeating the above argument, replacing \( \gamma \) by \( \gamma r \), the same is true if \( r \) is a monomial of any positive degree. But \( \gamma r \gamma' = 0 \), so it also holds if \( r = 1 \). Since any element of \( R \) is just a sum of monomials, putting this all together yields \( \gamma r \gamma' = 0 \) for all \( r \in R \). Therefore \( R \) is semi-commutative.

Notice that Claim 5 implies each of the coefficients of \( F(x) \) and \( G(x) \) is non-zero.

In particular, \( F(x), G(x) \neq 0 \) in \( R[x] \). So we have the following:

**Claim 7.** The ring \( R \) is not right McCoy.

**Proof.** It suffices to show that if \( F(x) r = 0 \) for some \( r \in R \), then \( r = 0 \). Thus, it suffices to show that if \( a_2 r = 0 \) then \( r = 0 \). This follows trivially from Claim 5, by calculating the reduced form for \( a_2 r \) for any \( r \in R \).

**Claim 8.** The ring \( R \) is left McCoy.

**Proof.** Let \( P(x), Q(x) \in R[x] \setminus \{0\} \), with \( P(x) \cdot Q(x) = 0 \). Write \( P(x) = \sum_{i=1}^m p_i x^i \) and \( Q(x) = \sum_{i=1}^n q_i x^i \). If each \( q_i \) has zero constant term then \( b_0 Q(x) = 0 \), and we are done. So we may assume that some \( q_i \) has a non-zero constant term. Let \( k \) be the smallest index such that \( q_k \) has this property.

For each \( p_i = 0 \) let \( p'_i \) be the sum of the non-zero terms of \( p_i \) of smallest degree, and for each \( p_i = 0 \) put \( p'_i = 0 \). Let \( j \) be the smallest index such that, among the
non-zero members of \( \{p'_0, p'_1, \ldots, p'_m\} \), we have \( p'_j \) is of minimal degree. Notice that \( j \) exists since \( P(x) \neq 0 \).

Now, looking at the degree \( j + k \) part of the equation \( P(x) \cdot Q(x) = 0 \) we have

\[
\sum_{r,s: \ r+s=j+k} p_rq_s = 0.
\]

Hence, \( I \) being an homogeneous ideal implies that the terms of any fixed degree in equation (4) must add to 0. But from our choice of \( j \) and \( k \) there is only one term in equation (4) of smallest degree, namely \( p'_j \cdot 1 \neq 0 \) coming from \( p_jq_k \). Therefore this gives a contradiction, and we are done. \( \square \)

4. Final Remarks

We note that there is a simpler ring that is left McCoy but not right McCoy. Set \( k' = \mathbb{F}_2 \langle c_0, c_1, d_0, d_1 \rangle \), and put \( R' = k'/J \) where \( J \) is the ideal generated by the relations:

\[
\langle c_0d_0, \ c_0d_1 + c_1d_0, \ c_1d_1, \ d_id_j \ (0 \leq i, j \leq 1), \ d_ic_j \ (0 \leq i, j \leq 1) \rangle.
\]

If \( C(x) = c_0 + c_1x \) and \( D(x) = d_0 + d_1x \), then \( C(x) \cdot D(x) = 0 \). One now proceeds as above: first, describing arbitrary elements in reduced form; second, showing that \( C(x)r = 0 \) for \( r \in R' \) if and only if \( r = 0 \); and third, copying the ideas of Claim 8 to show \( R' \) is left McCoy.

References


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