THE UPPER NILRADICAL AND JACOBSON RADICAL OF SEMIGROUP GRADED RINGS

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Abstract. Given a semigroup S, we prove that if the upper nilradical $\mathrm{Nil}^*(R)$ is homogeneous whenever R is an S-graded ring, then the semigroup S must be cancelative and torsion-free. In case S is commutative the converse is true. Analogs of these results are established for other radicals and ideals. We also describe a large class of semigroups S with the property that whenever R is a Jacobson radical ring graded by S, then every homogeneous subring of R is also a Jacobson radical ring. These results partially answer two questions of Smoktunowicz. Examples are given delimiting the proof techniques.

Introduction

Starting with Bergman's proof that the Jacobson radical is always homogeneous in a $\mathbb{Z}$-graded ring [2], numerous authors have studied the question of which (semi)groups can replace $\mathbb{Z}$, and whether the same result holds true for other radicals. Recently, in [16] Smoktunowicz proved that the upper nilradical is homogeneous in $\mathbb{Z}$-graded rings. Smoktunowicz in [16] and independently Lee and Puczyłowski in [10] also proved that if R is a Jacobson radical ring graded by the additive semigroup of positive integers, then every homogeneous subring is Jacobson radical.

Our paper is correspondingly split into two main sections in which we explore two questions posed by Smoktunowicz in [16]. In Section 1 we investigate the following question, slightly expanding [16, Question 0.1].

Question A. *For which semigroups S is the upper nilradical of any S-graded ring homogeneous?*

We first prove that S must be cancelative and torsion-free. Conversely, if S is commutative, cancelative, and torsion-free, then the upper nilradical is always homogeneous in any S-graded ring. Our work extends to many other radicals.

The central question explored in Section 2 is [16, Question 0.2], which we reproduce here.

Question B. *For which semigroups S are homogeneous subrings of S-graded Jacobson radical rings also Jacobson radical rings?*

We prove that semigroups for which all finitely generated subsemigroups $T \subseteq S$ satisfy the requirement $\bigcap_{n\geq 1} T^n = \emptyset$ have the property described in Question B. On the other hand, we provide some examples showing this is not a necessary condition. We finish by describing the implications between this and other well known semigroup conditions, such as "ACCPL" and "positively orderable."

Throughout the paper, all rings are associative, but do not necessarily contain 1. By $\mathbb{N}^+$ we will mean the semigroup of positive integers under addition. If R is a ring, then $\mathrm{Nil}^*(R)$ is the upper nilradical, i.e. the largest nil ideal in R. We write $I \trianglelefteq R$ to mean that I is a two-sided ideal of R. If S is a semigroup, we define

$$S^1 = \begin{cases} S & \text{if } S \text{ is a monoid,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

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Given a subset C of a semigroup S, we will write $\langle C\rangle$ for the subsemigroup of S generated by C.

1. When is the upper nilradical always homogeneous for an S-grading?

In this section we study those semigroups S for which the upper nilradical of any S-graded ring is homogeneous, providing a partial answer to Question A. While we will focus on the upper nilradical, our arguments apply to very general radical-like ideals. This section starts with an introduction to important examples which will be used throughout the paper. We then establish a few necessary conditions on a semigroup S for $\mathrm{Nil}^*(R)$ to be homogeneous whenever R is an S-graded ring, and show that they are sufficient when S is commutative. The section ends with applications of these methods to the bounded nilradical and the Wedderburn radical.

1.1. **Ideal functions.** By an *ideal function* $\mathfrak{F}$, we mean a function which assigns to each ring R a two-sided ideal $\mathfrak{F}(R)\trianglelefteq R$. Typical examples of ideal functions are Kurosh-Amitsur radicals, but this notion is much more general.

Example 1. The *Wedderburn radical* of a ring R, denoted $\mathfrak{W}(R)$, is the sum of all nilpotent ideals in R. This is not a Kurosh-Amitsur radical, as $\mathfrak{W}(R/\mathfrak{W}(R))$ may be nonzero [1]. The Wedderburn radical is always contained in the prime radical, and indeed the prime radical is the lower radical determined by the class of nilpotent rings. Clearly, $\mathfrak{W}$ is an ideal function.

Example 2. Let R be a ring and let

$$\mathfrak{Z}_\ell(R)=\{x\in R\,:\,xR=0\}$$

which we call the *left annihilator* of R. Define the *right annihilator*, $\mathfrak{Z}_r(R)$, similarly. In a ring with 1, these are both just the zero ideal. Finally, set $\mathfrak{Z}(R)=\mathfrak{Z}_\ell(R)+\mathfrak{Z}_r(R)\trianglelefteq R$, and note that $\mathfrak{Z}_\ell$, $\mathfrak{Z}_r$, and $\mathfrak{Z}$ are ideal functions.

In this paper, we often work with ideal functions prescribed by explicit lower and upper bounds.

Definition 3 (Sandwiched ideal function)**.** Let $\mathcal{C}$ be an isomorphically closed class of rings, and let $\mathfrak{F}$ and $\mathfrak{G}$ be ideal functions. We say the ideal function $\mathfrak{F}$ is *$\mathfrak{G}$-$\mathcal{C}$-sandwiched* if for any ring R we have $\mathfrak{G}(R)\subseteq\mathfrak{F}(R)$ and $\mathfrak{F}(R)\subseteq I$ for any ideal $I\trianglelefteq R$ for which $R/I\in\mathcal{C}$.

Example 4. Recall that the *Thierren radical* $\mathcal{F}$ is the upper radical of all division rings [6, Example 3.2.18]. Equivalently, if $\mathcal{D}=\{D\,:\,D\text{ is a division ring}\}$ then $\mathcal{F}(R)=\bigcap_{I\trianglelefteq R,\,R/I\in\mathcal{D}}I$. An ideal function $\mathfrak{F}$ is $\mathfrak{W}$-$\mathcal{D}$-sandwiched if and only if for every ring R we have $\mathfrak{W}(R)\subseteq\mathfrak{F}(R)\subseteq\mathcal{F}(R)$.

The $\mathfrak{W}$-$\mathcal{D}$-sandwiched ideal functions include many of the usual Kurosh-Amitsur radicals, such as the prime radical, Levitzki radical, upper nilradical, Jacobson radical, Behrens radical, the left and right strongly prime radicals, Jenkins radical, Brown-McCoy radical, the uniformly strongly prime radical, the upper radical $\mathcal{T}$ determined by all matrix rings over division rings, the generalized nil-radical, and Thierren radical (see the chart [6, p. 293]). The $\mathfrak{W}$-$\mathcal{D}$-sandwiched ideal functions also capture some of the other radical-like ideals that are not Kurosh-Amitsur radicals, such as the Wedderburn radical (and higher Wedderburn radicals) and the bounded nilradical. Finally, the function which assigns to each ring R the sum of all nil left ideals of R is a $\mathfrak{W}$-$\mathcal{D}$-sandwiched ideal function. Moreover, if Köthe's conjecture is false, then this is not a Kurosh-Amitsur radical [14], and the other ideals in the Andrunakievich chain [4] are also $\mathfrak{W}$-$\mathcal{D}$-sandwiched.

Remark 5. We will make use of the following observation a couple of times in this section. Fix an ideal function $\mathfrak{G}$ and an isomorphically closed class of rings $\mathcal{C}$. Suppose that $\mathfrak{F}$ is a $\mathfrak{G}$-$\mathcal{C}$-sandwiched ideal function and R is a subring of a semigroup ring $R_0[S]$ (given the usual S-grading), where $R_0\in\mathcal{C}$. Let $\omega: R_0[S]\to R_0$ be the augmentation map. Assuming $\omega(R)=R_0$, we then have $\mathfrak{F}(R)\subseteq\ker(\omega|_R)$. Hence either $\mathfrak{F}(R)=0$ or $\mathfrak{F}(R)$ is not homogeneous in $R_0[S]$, since $\ker(\omega|_R)$ contains no nonzero homogeneous elements.

1.2. **Cancelativity.** Our first result yields a necessary condition on those semigroups S that give a positive answer to Question A. Some of the arguments originate from the work of Clase and Kelarev [5], who focused on the Jacobson radical. Recall that a ring is *reduced* when it contains no nonzero nilpotent elements.

Theorem 6. *Let S be a semigroup and let $\mathfrak{F}$ be a $\mathfrak{Z}_\ell$-$\mathcal{C}$-sandwiched ideal function, where $\mathcal{C}$ is an isomorphically closed class of rings containing a non-reduced ring R_0. If for every S-graded ring R the ideal $\mathfrak{F}(R)$ is homogeneous, then the semigroup S is right cancelative.*

Proof. Assume the semigroup S is not right cancelative. Then there exist $u, v, w \in S$ with $u \neq v$ but $uw = vw$. It suffices to find a ring R which is S-graded, but such that $\mathfrak{F}(R)$ is not homogeneous. Let $A = R_0[S]$ be the semigroup ring over S with coefficients in R_0. As R_0 is not reduced, we can fix $x \in R_0$ with $x^2 = 0$ but $x \neq 0$. Let $I = wS^1$ be the right ideal of S generated by w and let $J = x \cdot \mathbb{Z} + xR_0$ be the right ideal of R_0 generated by x. The subring R of A consisting of all elements $\sum_{s\in S} a_s \cdot s \in A$ such that $a_s \in J$ for any $s \in S \setminus I$ is the S-graded ring we desire. Fix $d = x \cdot u - x \cdot v \in R$. As $x^2 = 0$ and $uw = vw$, we have $dR = 0$. This implies that $d \in \mathfrak{Z}_\ell(R) \subseteq \mathfrak{F}(R)$ and $d \neq 0$. Now Remark 5 shows that $\mathfrak{F}(R)$ is not homogeneous. □

Recall that as mentioned in Example 4 the ideal function $\mathcal{T}$ denotes the upper radical determined by the class $\mathcal{E}$ of all matrix rings over division rings. Hence, if R is a ring, then $\mathcal{T}(R) = \bigcap_{I \trianglelefteq R,\, R/I \in \mathcal{E}} I$.

Corollary 7. *Let S be a semigroup and let $\mathfrak{F}$ be an ideal function such that for every S-graded ring R the ideal $\mathfrak{F}(R)$ is homogeneous.*

1. *If for every ring R we have $\mathfrak{Z}_\ell(R) \subseteq \mathfrak{F}(R) \subseteq \mathcal{T}(R)$, then S is right cancelative.*
2. *If for every ring R we have $\mathfrak{Z}_r(R) \subseteq \mathfrak{F}(R) \subseteq \mathcal{T}(R)$, then S is left cancelative.*
3. *If for every ring R we have $\mathfrak{Z}(R) \subseteq \mathfrak{F}(R) \subseteq \mathcal{T}(R)$, then S is cancelative.*

Proof. For (1), as $\mathbb{M}_2(D)$ always contains a nonzero nilpotent element whenever D is a division ring (or even a nonzero ring), we can take $\mathcal{C}$ to be the collection $\mathcal{E}$ of matrix rings over division rings and then apply Theorem 6. Parts (2) and (3) follow by symmetry considerations. □

An interesting consequence of this corollary is that the (right, left) cancelativity of a semigroup can be characterized in terms of the homogeneity of $\mathfrak{Z}$ (respectively, $\mathfrak{Z}_\ell$, $\mathfrak{Z}_r$).

Corollary 8. *Let S be a semigroup.*

1. *For every S-graded ring R the ideal $\mathfrak{Z}_\ell(R)$ is homogeneous if and only if S is right cancelative.*
2. *For every S-graded ring R the ideal $\mathfrak{Z}_r(R)$ is homogeneous if and only if S is left cancelative.*
3. *For every S-graded ring R the ideal $\mathfrak{Z}(R)$ is homogeneous if and only if S is cancelative.*

Proof. (1) $\Rightarrow$: Since for any ring R we have $\mathfrak{Z}_\ell(R) \subseteq \mathcal{T}(R)$, it suffices to apply Corollary 7(1) with $\mathfrak{F} = \mathfrak{Z}_\ell$.

$\Leftarrow$: Assume S is right cancelative and let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring. Let $a = a_{s_1} + a_{s_2} + \cdots + a_{s_n} \in \mathfrak{Z}_\ell(R)$ with $s_1, s_2, \ldots, s_n \in S$ distinct elements and $a_{s_i} \in R_{s_i}$ for each i. Then for any homogeneous element $r_s \in R$ we have $0 = ar_s = a_{s_1}r_s + a_{s_2}r_s + \cdots + a_{s_n}r_s$. Since S is right cancelative, the products $a_{s_1}r_s, a_{s_2}r_s, \ldots, a_{s_n}r_s$ lie in distinct components and thus $a_{s_i}r_s = 0$ for each i. As this holds for each homogeneous element $r_s \in R$ it follows that $a_{s_i}R = 0$, and hence $a_{s_i} \in \mathfrak{Z}_\ell(R)$.

(2) The proof of (2) is analogous to that of (1).

(3) $\Rightarrow$: This follows from Corollary 7(3).

$\Leftarrow$: Assume S is cancelative and let R be an S-graded ring. By (1) and (2) the ideals $\mathfrak{Z}_\ell(R)$ and $\mathfrak{Z}_r(R)$ are homogeneous and thus so is their sum $\mathfrak{Z}_\ell(R) + \mathfrak{Z}_r(R) = \mathfrak{Z}(R)$. □

We finish our discussion of cancelativity by noting two important special cases of Corollary 7, when $\mathfrak{F}$ is either the Jacobson radical or the upper nilradical.

Corollary 9 (cf. [5, Theorem 9])**.** *If S is a semigroup for which the Jacobson radical $J(R)$ is homogeneous whenever a ring R is S-graded, then S is cancelative.*

Corollary 10. *If S is a semigroup for which the upper nilradical* $\mathrm{Nil}^*(R)$ *is homogeneous whenever a ring R is S-graded, then S is cancelative.*

1.3. **Torsion-free semigroups.** Recall that a semigroup S is *torsion-free* if whenever $s^n = t^n$ for commuting elements $s, t \in S$ and some positive integer n, then we have $s = t$. (This definition is weaker than that utilized by some authors who do not assume s and t commute, but their stronger statement is most often applied when S is a commutative semigroup. We prefer the definition we have given, as it agrees with the usual definition of "torsion-free" when S is a group.) This condition is fundamentally important with regards to homogeneity in S-graded rings for a large class of sandwiched ideal functions.

Proposition 11. *Let S be a semigroup and let $\mathfrak{F}$ be a $\mathfrak{W}$-$\mathcal{C}$-sandwiched ideal function, where $\mathcal{C}$ is an isomorphically closed class of rings such that for every prime p the class $\mathcal{C}$ contains a nonzero commutative ring of characteristic p. If for every S-graded ring R the ideal $\mathfrak{F}(R)$ is homogeneous, then the semigroup S is torsion-free.*

Proof. Assume that there exist commuting elements $s, t \in S$ with $s^n = t^n$ for some $n \geq 1$, and let T be the subsemigroup of S generated by s and t. To show that $s = t$ it suffices to consider the case when $n = p$ is prime. By assumption there exists a nonzero commutative ring $R_p \in \mathcal{C}$ of characteristic p. Let $r \in R_p$ with $r \neq 0$. Then $(r \cdot s - r \cdot t)^p = 0$ in $R_p[T]$, and as $R_p[T]$ is a commutative ring we have that $d := r \cdot s - r \cdot t$ generates a nilpotent ideal. Therefore, $d \in \mathfrak{W}(R_p[T]) \subseteq \mathfrak{F}(R_p[T])$. By assumption $\mathfrak{F}(R_p[T])$ is homogeneous since $R_p[T]$ is an S-graded ring, hence by Remark 5 we must have $\mathfrak{F}(R_p[T]) = 0$, which implies $s = t$. □

Corollary 12. *Let S be a semigroup and let $\mathfrak{F}$ be an ideal function such that for every ring R we have $\mathfrak{W}(R) \subseteq \mathfrak{F}(R) \subseteq \mathcal{F}(R)$. If for every S-graded ring R the ideal $\mathfrak{F}(R)$ is homogeneous, then the semigroup S is torsion-free.*

Proof. The class of division rings contains $\mathbb{F}_p$, for every prime p, and so Proposition 11 applies. □

We can now classify those *commutative* semigroups S that answer Question A.

Theorem 13. *Let S be a commutative semigroup. The upper nilradical* $\mathrm{Nil}^*(R)$ *is homogeneous whenever a ring R is an S-graded ring if and only if S is cancelative and torsion-free.*

Proof. $\Rightarrow$: This implication follows from Corollaries 10 and 12.

$\Leftarrow$: Let S be a commutative, cancelative, torsion-free semigroup. Let $R = \bigoplus_{s \in S} R_s$ be an S-graded ring and assume $\mathrm{Nil}^*(R)$ is not homogeneous. Let $I \trianglelefteq R$ be the largest homogeneous ideal contained in $\mathrm{Nil}^*(R)$. As $\mathrm{Nil}^*(R/I) = \mathrm{Nil}^*(R)/I$, after passing to a factor ring we may as well assume that $\mathrm{Nil}^*(R)$ contains no nonzero homogeneous elements. There then exists an element $a = a_{s_1} + a_{s_2} + \cdots + a_{s_k} \in \mathrm{Nil}^*(R)$ with $s_1, \ldots, s_k \in S$ distinct elements, $a_{s_i} \in R_{s_i} \setminus \mathrm{Nil}^*(R)$ for each $i \geq 1$, and $k \geq 2$ minimal.

Given any homogeneous element $r_s \in R_s$ we find that $ar_s a_{s_k} - a_{s_k} r_s a \in \mathrm{Nil}^*(R)$ has fewer homogeneous components. (Note that it is essential to assume that S is commutative for this to be true, for we are using $s_i s s_k = s_k s s_i$.) From the minimality of k we must have $ar_s a_{s_k} = a_{s_k} r_s a$. As this holds for all homogeneous elements r_s, we also have $ara_{s_k} = a_{s_k} ra$ for all $r \in R$.

Since $a_{s_k} \notin \mathrm{Nil}^*(R)$ there exist elements $p_i, q_i \in R$ (for $1 \leq i \leq n$, for some integer $n \geq 1$) such that $\alpha := \sum_{i=1}^n p_i a_{s_k} q_i$ is not nilpotent. On the other hand, $a \in \mathrm{Nil}^*(R)$ and thus there exists some $m \geq 1$ such that $(\sum_{i=1}^n p_i a q_i)^m = 0$. We then compute

$$(\alpha^m RaR)^m \subseteq \alpha^m (RaR)^m = \left(\sum_{i=1}^n p_i a_{s_k} q_i \right)^m (RaR)^m \subseteq \left(\sum_{i=1}^n p_i a q_i \right)^m (Ra_{s_k} R)^m = 0.$$

Hence, if P is a minimal prime ideal then either $\alpha^m \in P$ or $a \in P$. Since S is a totally ordered semigroup by [7, Corollary 3.4], it follows from [9, Theorem 1.2] that P is homogeneous and thus if $a \in P$ then $a_{s_k} \in P$ and consequently $\alpha^m \in P$. Hence α^m is in all minimal prime ideals of R, thus α^m belongs to the prime radical and therefore must be nilpotent. Hence α must be nilpotent too, giving us the needed contradiction. □

1.4. **The bounded nilradical.** The bounded nilradical $\mathfrak{B}$ is a function defined on rings by the rule

$$\mathfrak{B}(R) = \{a \in R : aR \text{ has bounded index of nilpotence}\}.$$

It is easy to prove that this definition is left-right symmetric. Surprisingly $\mathfrak{B}(R)$ is an ideal in R, which was first proved by Amitsur (see [15, Theorem 2.6.27]), and so $\mathfrak{B}$ is an ideal function. The bounded nilradical contains the Wedderburn radical and is contained in the prime radical; for more basic information see the paper [8]. In [13] it was proven that the bounded nilradical is homogeneous for $\mathbb{Z}$-gradings. Using the techniques developed above, we can improve on that result by proving an analogue of Theorem 13 for $\mathfrak{B}$.

First, we need to introduce a few auxiliary tools. Let S be a commutative semigroup. Fix some total ordering $\prec$ on S, which is not assumed to be a semigroup ordering. We use this ordering simply to provide an easy way to differentiate elements of S. Recall that a partition λ of an integer n is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We will often write $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ for such a partition, and we call k then *length* of the partition.

Given a partition λ of length k, we let $P_\lambda(x_1, x_2, \ldots, x_k)$ denote the sum of all monomials in the noncommuting variables $x_1, x_2, \ldots, x_k$ such that x_i occurs exactly λ_i times in a monomial.

Lemma 14. *Let S be a commutative semigroup, let $R = \bigoplus_{s\in S} R_s$ be an S-graded ring, and let $n \geq 1$ be an integer. If $r = \sum_{s\in S} r_s \in R$ where $r_s \in R_s$ for each $s \in S$, then*

$$r^n = \sum_{t\in S}\left(\sum_{\text{partitions } \lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_k\} \text{ of } n}\left(\sum_{\substack{\{s_1,s_2,\ldots,s_k\}\subseteq \mathrm{supp}(r):\\ s_1^{\lambda_1}s_2^{\lambda_2}\cdots s_k^{\lambda_k}=t, \text{ if } \lambda_p=\lambda_{p+1} \text{ then } s_p\prec s_{p+1}}} P_\lambda(r_{s_1}, r_{s_2}, \ldots, r_{s_k})\right)\right)$$

Proof. The proof is analogous to [13, Lemma 3]. The equality of the lemma is just the statement of what happens when terms in the product r^n are grouped together according to how often each r_{s_i} occurs. Note that the condition "if $\lambda_p = \lambda_{p+1}$ then $s_p \prec s_{p+1}$" prevents the over-counting that could occur by switching r_{s_p} and $r_{s_{p+1}}$ when $\lambda_p = \lambda_{p+1}$. Also note that $P_\lambda(r_{s_1}, r_{s_2}, \ldots, r_{s_k}) \in R_t$ is homogeneous, since S is commutative. □

We need to recall an alternate characterization of the bounded nilradical. Namely,

$$\mathfrak{B}(R) = \{a \in R : \exists\, n \geq 1, \forall \text{ partitions } \lambda = \{\lambda_1, \ldots, \lambda_k\} \text{ of } n, aR \text{ satisfies } P_\lambda(x_1, x_2, \ldots, x_k)\}, \tag{15}$$

where by "aR satisfies $P_\lambda(x_1, x_2, \ldots, x_k)$" we mean that if we replace the variables $x_1, x_2, \ldots, x_k$ by elements of aR, then the polynomial expression is zero. This is [13, Theorem 5]. (Technically, the theorem proven there was done for rings with 1, but the same proofs work for non-unital rings as well.) We are now ready to prove the analog of Theorem 13 for the bounded nilradical.

Theorem 16. *Let S be a commutative semigroup. The bounded nilradical $\mathfrak{B}(R)$ is homogeneous whenever R is an S-graded ring if and only if S is cancelative and torsion-free.*

Proof. $\Rightarrow$: This implication follows from Corollaries 7 and 12, since the bounded nilradical contains the Wedderburn radical and is contained in the prime radical.

$\Leftarrow$: Write $R = \bigoplus_{s\in S} R_s$ as usual. As S is commutative, cancelative, and torsion-free, there exists a total, strict semigroup ordering $\leq$ on S. (A semigroup ordering $\leq$ on a semigroup S is *strict* if whenever

$a, b \in S$ satisfy $a < b$, then for every $x \in S$ we also have $xa < xb$ and $ax < bx$.) Fix $a \in \mathfrak{B}(R)$, say $a = a_{s_1} + a_{s_2} + \cdots + a_{s_m}$ where $s_1 < s_2 < \ldots < s_m$ and $a_{s_i} \in R_{s_i}$ is homogeneous. It suffices to show that $a_{s_m} \in \mathfrak{B}(R)$, as then $a - a_{s_m} \in \mathfrak{B}(R)$ has fewer terms.

As $a \in \mathfrak{B}(R)$, by (15) we can fix some integer $n \geq 1$ so that for every integer $k \geq 0$ and any partition λ of n of length k, the set aR satisfies $P_\lambda(x_1, x_2, \ldots, x_k)$. Let r be an arbitrary element of R, and write $r = \sum_{u \in S} r_u \in R$ with $r_u \in R_u$. By Lemma 14, we have

$$(a_{s_m} r)^n \in \sum_{\substack{\text{partitions } \lambda \text{ of } n \text{ of length } k,\\ \{u_1, u_2, \ldots, u_k\} \subseteq S}} P_\lambda(a_{s_m} r_{u_1}, a_{s_m} r_{u_2}, \ldots, a_{s_m} r_{u_k}) R^1.$$

If we can show that each summand is zero, we then have that $a_{s_m} \in \mathfrak{B}(R)$ as desired. Recall that aR satisfies P_λ, and so

$$P_\lambda(ar_{u_1}, ar_{u_2}, \ldots, ar_{u_k}) = 0. \tag{17}$$

The term of largest grade on the left-hand side of (17) is exactly $P_\lambda(a_{s_m} r_{u_1}, \ldots, a_{s_m} r_{u_k})$, and hence this term must be zero. □

1.5. **Strict, total orderings.** While commutativity of S plays a central role in Theorem 13, for radicals other than the upper nilradical this assumption can often be weakened. Our next result does just that for the Wedderburn radical.

Proposition 18. *If S is a totally, strictly ordered semigroup and R is an S-graded ring, then the Wedderburn radical of R is homogeneous.*

Proof. Assume R is S-graded, say $R = \bigoplus_{s \in S} R_s$. Fix $a \in \mathfrak{W}(R)$, and write $a = a_{s_1} + a_{s_2} + \cdots + a_{s_n}$ with $a_{s_i} \in R_{s_i}$ and $s_1 < s_2 < \ldots < s_n$. It suffices to show that $a_{s_n} \in \mathfrak{W}(R)$.

Fix an integer $m \geq 1$ such that $(aR)^m = 0$. Given $r_{t_i} \in R_{t_i}$ (for $1 \leq i \leq m$) we have

$$ar_{t_1} ar_{t_2} \cdots ar_{t_m} = 0.$$

The term on the left-hand side occurring in the largest homogeneous component is

$$a_{s_n} r_{t_1} a_{s_n} r_{t_2} \cdots a_{s_n} r_{t_m} = 0.$$

As this equality holds for arbitrary homogeneous elements $r_{t_1}, \ldots, r_{t_m} \in R$ this implies that $(a_{s_n} R)^m = 0$. Hence $a_{s_n} \in \mathfrak{W}(R)$ as desired. □

2. When are homogeneous subrings of Jacobson radical rings also Jacobson radical?

In this section we focus on those semigroups S for which every S-graded Jacobson radical ring has all homogeneous subrings Jacobson radical. There is one immediate restriction that one obtains on such semigroups; they have no idempotents. Indeed, let S be a semigroup with an idempotent $s_0 \in S$ and let R be a Jacobson radical ring containing a subring R_0 that is not Jacobson radical. (It is well known that such a ring R exists; e.g. take a Jacobson radical ring that is not nil.) We grade R by S, setting $R_s = 0$ for $s \neq s_0$, and putting $R_{s_0} = R$. Obviously R_0 is a homogeneous subring of R but not Jacobson radical. This shows that any subgroup answering Question B must be idempotent-free.

2.1. **A power series construction.** Smoktunowicz in [16, Proposition 0.1] and independently Lee and Puczyłowski in [10, Theorem 5.9] proved that Question B has a positive answer for the semigroup of positive integers. Smoktunowicz's proof relies on the construction of a special ring of formal series, and this construction exists in greater generality. Let S be a semigroup, and let $R = \bigoplus_{s \in S} R_s$ be any S-graded ring. Consider the set of formal series

$$P := \left\{ r = \sum_{s \in S} r_s \;:\; r_s \in R_s,\ \operatorname{supp}(r) \text{ is contained in a finitely generated subsemigroup of } S \right\}$$

where $\operatorname{supp}(r)$ denotes the *support* of r, that is $\operatorname{supp}(r) := \{s \in S : r_s \neq 0\}$. The set P is an abelian group under component-wise addition, and it contains R. We would like for P to be a ring under the multiplication rule

$$rr' = \left(\sum_{s\in S} r_s\right)\left(\sum_{t\in S} r'_t\right) = \sum_{u\in S}\left(\sum_{s,t\,:\,st=u} r_s r'_t\right), \tag{19}$$

and then R will sit as a subring inside P. As $r, r' \in P$, there exist finitely generated subsemigroups $S_r, S_{r'} \subseteq S$ containing the supports of r and r' respectively. Let $U \subseteq S$ be the finitely generated subsemigroup of S generated by $S_r \cup S_{r'}$. The only nonzero entries on the right hand side of (19) arise when $u \in U$, and therefore the support of $rr' \in R$ belongs to the finitely generated semigroup U.

Of course, we still need to know that for each $u \in U$ the summation $\sum_{s,t\,:\,st=u} r_s r'_t$ makes sense. It suffices to guarantee that the set $X_u := \{(s,t) \in S_r \times S_{r'} : st = u\}$ is finite (for each $u \in U$). As the following lemma shows, for this it is enough to assume that $\bigcap_{n\geq 1} T^n = \emptyset$ for every finitely generated subsemigroup T of S.

Lemma 20. *Let T be a finitely generated semigroup such that $\bigcap_{n\geq 1} T^n = \emptyset$. For any $u \in T$ the set $X_u = \{(t_1, t_2) \in T \times T : t_1 t_2 = u\}$ is finite.*

Proof. Let $u \in T$. Since $\bigcap_{n\geq 1} T^n = \emptyset$, there exists a maximal positive integer m with $u \in T^m$. By assumption $T = \langle C\rangle$ for some finite set C, and so the maximality of m implies that $u \notin C^{m+1} \cup C^{m+2} \cup C^{m+3} \cup \cdots$. Hence if $t_1, t_2 \in T$ and $t_1 t_2 = u$, then $t_1, t_2 \notin C^k$ for all $k > m$. Thus, t_1 and t_2 belong to the set $C \cup C^2 \cup \cdots \cup C^m$, which is finite. Therefore X_u is finite as well. $\square$

2.2. **A large class of examples.** With this generalized power series construction in hand, we can describe a large class of semigroups which give a positive answer to Question B. As usual, $J(R)$ denotes the Jacobson radical of a ring R, and if $J(R) = R$ we say that the ring R is Jacobson radical.

Theorem 21. *Let S be a semigroup such that every finitely generated subsemigroup T of S satisfies $\bigcap_{n\geq 1} T^n = \emptyset$. If A is a homogeneous subring of an S-graded ring R, then $J(R) \cap A \subseteq J(A)$. In particular, if R is Jacobson radical, then so is A.*

Proof. Suppose R is an S-graded ring and write $R = \bigoplus_{s\in S} R_s$ for the grading. As in the discussion above, we define the ring of formal series

$$P = \left\{ r = \sum_{s\in S} r_s \;:\; r_s \in R_s,\ \operatorname{supp}(r) \text{ is contained in a finitely generated subsemigroup of } S \right\}.$$

Recall that an element x belongs to the Jacobson radical of a ring R if and only if for every $r \in xR$ there exists an element $r' \in R$ such that $r + r' + rr' = r + r' + r'r = 0$. The element r', if it exists, is unique and called the *quasi-inverse* of r. The ring P is Jacobson radical, because any element $p \in P$ has a quasi-inverse $p' = \sum_{i=1}^{\infty}(-p)^i \in P$. Note that p' is well-defined from the assumption on S.

Let A be a homogeneous subring of R. Given $a \in J(R) \cap A$ write $a = a_{s_1} + a_{s_2} + \cdots + a_{s_k}$ with each $a_{s_i} \in R_{s_i}$. Notice that each $a_{s_i} \in A$ since A is homogeneous. As $a \in J(R)$, there exists a quasi-inverse $a' \in R$ for a. Write $a' = r_{t_1} + r_{t_2} + \cdots + r_{t_\ell}$ where each $r_{t_j} \in R_{t_j}$. Let $a'' = \sum_{i=1}^{\infty}(-a)^i$ be the quasi-inverse for a from P. As $a' \in P$ and quasi-inverses are unique, we must have $a' = a''$. Thus, each homogeneous component in a' comes from the corresponding homogeneous components in a''. But the homogeneous components in a'' belong to the subring generated by the homogeneous components of a. Thus $a' \in A$, and so $J(R) \cap A \subseteq J(A)$. The final statement is clear. $\square$

2.3. **More examples.** We obtain [10, Theorem 5.9] and [16, Proposition 0.1] as corollaries to Theorem 21, since the positive integers satisfy the condition on S in the theorem. Surprisingly, the semigroups considered in Theorem 21 are not the most general structures for which the conclusion is true. Indeed, we now describe a general procedure for creating new semigroups which yield a positive answer to Question B.

Lemma 22. *Let R, R_1, and R_2 be rings such that $R = R_1 \oplus R_2$ (as abelian groups) and R_2 is an ideal of R. Let $\mathfrak{K}$ be a hereditary Kurosh-Amitsur radical (such as the Jacobson radical, upper nilradical, or prime radical). We have $R \in \mathfrak{K}$ if and only if $R_1, R_2 \in \mathfrak{K}$.*

Proof. $\Rightarrow$: As $R \in \mathfrak{K}$ and $\mathfrak{K}$ is hereditary, we have $R_2 \in \mathfrak{K}$. As $R_1 \cong R/R_2$ is a homomorphic image of a ring in $\mathfrak{K}$, we have $R_1 \in \mathfrak{K}$.

$\Leftarrow$: Assuming $R_1, R_2 \in \mathfrak{K}$, we have $\mathfrak{K}(R) \supseteq R_2$, and so $R/\mathfrak{K}(R)$ is both a quotient of R_1 (hence in $\mathfrak{K}$) and $\mathfrak{K}$-semisimple. Hence $R = \mathfrak{K}(R)$. □

We will apply the previous lemma in the special case of the Jacobson radical, but the reader should be aware that the proposition to follow holds in much greater generality.

Proposition 23. *Let S be a semigroup which is the disjoint union of two subsemigroups S_1, S_2, where S_2 is an ideal in S. Let R be a ring graded by S, say $R = \bigoplus_{s\in S} R_s$. Set $R_1 = \bigoplus_{s\in S_1} R_s$ and $R_2 = \bigoplus_{s\in S_2} R_s$. The ring R has the property that every homogeneous subring is Jacobson radical if and only if the same is true of both R_1 and R_2.*

Proof. $\Rightarrow$: Let A_i be a homogeneous subring of R_i. Clearly A_i is also a homogeneous subring of R, and thus is Jacobson radical.

$\Leftarrow$: Let A be a homogeneous subring of R. We can write $A = A_1 \oplus A_2$ for homogeneous subrings $A_i \subseteq R_i$. By hypothesis, both A_1 and A_2 are Jacobson radical and it is easy to check that A_2 is an ideal of A. By the previous lemma, A is Jacobson radical. □

Example 24. There exists a finitely generated, cancelative semigroup S for which $\bigcap_{n\geq 1} S^n \neq \emptyset$, but if R is a Jacobson radical ring graded by S then every homogenous subring is still Jacobson radical.

Construction. Define S to be the semigroup generated by three letters t_1, t_2, t_3, subject to relation $t_2t_1t_3 = t_1$. Clearly $\bigcap_{n\geq 1} S^n \neq \emptyset$. We leave the easy proof that S is cancelative to the reader.

When speaking of the *degree* of a monomial we will mean the total number of times t_1 occurs in a word. Note that the relation $t_2t_1t_3 = t_1$ preserves the number of times t_1 appears in a word, so this is a well-defined notion. Let $S_2 = \bigcap_{n\geq 1} S^n$ and let $S_1 = S \setminus S_2$. It is easy to see that S_2 is the ideal of S consisting of those monomials of positive degree, and $S_1 = \langle t_2, t_3\rangle$ is the subsemigroup consisting of monomials of degree 0.

Let R be a Jacobson ring graded by S, and let R_1 and R_2 be the subrings generated by the homogeneous components in S_1 and S_2 respectively. We know that R_1 and R_2 are Jacobson radical rings by Lemma 22. Theorem 21 implies that homogeneous subrings of R_i (for $i = 1, 2$) are Jacobson radical, since $\bigcap_{n\geq 1} S_i^n = \emptyset$. (When $i = 1$ we are in a free semigroup, and when $i = 2$ degree considerations suffice.) Applying Proposition 23, we see that R has the stated property. □

Example 25. Any Jacobson radical ring graded by the additive semigroup

$$S = \{\alpha \text{ is an ordinal} : 1 \leq \alpha < \omega^2\}$$

has all homogeneous subrings Jacobson radical. Indeed, let $S_2 = \bigcap_{n\geq 1} S^n = \{\alpha \text{ is an ordinal} : \omega \leq \alpha < \omega^2\}$ and $S_1 = S \setminus S_2 = \mathbb{N}^+$. We have $\bigcap_{n\geq 1} S_i^n = \emptyset$ for $i = 1, 2$. The same reasoning as in the previous example gives the desired conclusion.

2.4. **Characterizing semigroup properties.** The main results of this section relate the semigroup condition utilized in Theorem 21 to other well known properties. Recall that ACCPL is shorthand for the assumption that the semigroup has the ascending chain condition on principal left ideals, and ACCPR is defined similarly for principal right ideals. The ACCPL condition is the subject of recent papers related to power series constructions and semidirect products; for example, see [12] and [17].

Theorem 26. *Let S be a semigroup. Consider the following conditions:*

(1) *If $S = \langle T\rangle$, then for every $s \in S$ the set $\{n \in \mathbb{N}^+ : s \in T^n\}$ is finite.*
(2) $\bigcap_{n\geq 1} S^n = \emptyset$.
(3) *For every countable sequence $s_1, s_2, \ldots \in S$, we have $\bigcap_{n\geq 1} s_1 s_2 \cdots s_n S = \emptyset$.*
(4) *S has ACCPL and no idempotents.*

We have (1)⇔(2)⇒(3)⇒(4)*. If S is finitely generated then* (3)⇒(1)*. If S is cancelative then* (4)⇒(3)*. No other implications hold in general.*

Proof. **(1)⇒(2)**: Taking $T = S$, this is a tautological weakening.

(2)⇒(1): We prove the contrapositive. Fix $s \in S = \langle T\rangle$ and suppose $s \in T^{n_1} \cap T^{n_2} \cap \cdots$ for some positive integers $n_1 < n_2 < \ldots$. Given $n \geq 1$, there exists some k such that $n_k > n$, and so for some $t_1, t_2, \ldots, t_{n_k} \in T$ we have

$$s = t_1 t_2 \cdots t_{n_k} = t_1 t_2 \cdots t_{n-1}(t_n \cdots t_{n_k}) \in S^n.$$

As $n \geq 1$ is arbitrary, $s \in \bigcap_{n\geq 1} S^n$.

(2)⇒(3): This is a tautological weakening.

(3) and S is finitely generated⇒(1): By hypothesis $S = \langle T\rangle$ for some finite set T. Working contrapositively, assume S does not satisfy (1). Fix $s \in S$ such that the set $\{n \in \mathbb{N}^+ : s \in T^n\}$ is infinite. Hence, we have $s \in T^{n_1} \cap T^{n_2} \cap \cdots$ for some sequence of positive integers $n_1 < n_2 < \ldots$. We can then write

$$s = t_{1,1} t_{1,2} \cdots t_{1,n_1} = t_{2,1} t_{2,2} \cdots t_{2,n_2} = \cdots$$

for elements $t_{i,j} \in T$. As T is finite, there is some element $s_1 \in T$ that occurs infinitely often as $t_{i,1}$. Similarly, there is some element $s_2 \in T$ such that $s_1 s_2$ occurs infinitely often as $t_{i,1} t_{i,2}$. Repeating this process, we have elements $s_1, s_2, \ldots \in T$ and $x_1, x_2, \ldots \in S$ such that

$$s = s_1 x_1 = s_1 s_2 x_2 = \cdots .$$

Hence $s \in \bigcap_{n\geq 1} s_1 s_2 \cdots s_n S$, so (3) does not hold.

(3)⇒(4): Clearly (3) implies that there are no idempotents. For the rest of the proof of this implication we will work contrapositively. Let $S^1 a_1 \subsetneq S^1 a_2 \subsetneq \ldots$ be a strictly increasing chain of principal left ideals in S. For each $n \geq 1$, fix $s_n \in S^1$ so that $a_n = s_n a_{n+1}$. Note that $s_n \neq 1$ since the chain of left ideals is strictly increasing. Thus

$$a_1 = s_1 a_2 = s_1 s_2 a_3 = \cdots \in \bigcap_{n\geq 1} s_1 s_2 \cdots s_n S.$$

(4) and S is cancelative⇒(3): Working contrapositively, assume there are no idempotents and there is a sequence $s_1, s_2, \ldots \in S$ such that $\bigcap_{n\geq 1} s_1 s_2 \cdots s_n S \neq \emptyset$. Write $x = s_1 x_1 = s_1 s_2 x_2 = \cdots$. Now, by (left) cancelativity, we have $x_n = s_{n+1} x_{n+1}$ for each $n \geq 1$. This gives us the chain

$$S^1 x_1 \subseteq S x_2 \subsetneq S^1 x_2 \subseteq S x_3 \subsetneq S^1 x_3 \ldots .$$

Notice that $S x_n$ is proper in $S^1 x_n$ since otherwise we would have $x_n = y x_n$ for some y, hence $y x_n = y^2 x_n$, and by (right) cancelativity this would give us an idempotent $y = y^2$.

(3)⇏(2): Let

$$S = \langle x_{i,j}\ (i, j \in \mathbb{N}^+,\ j \leq i) : x_{1,1} = x_{2,1} x_{2,2} = x_{3,1} x_{3,2} x_{3,3} = \cdots \rangle.$$

This semigroup clearly fails (2). To show that (3) holds, define the *minimal degree* of an element $s \in S$ as the shortest length (when written in the generators $x_{i,j}$ of S) of any element in sS^1, which we denote $\mathrm{mdeg}(s)$. For example $\mathrm{mdeg}(x_{1,1}) = 1$ and $\mathrm{mdeg}(x_{3,3}x_{4,1}x_{4,2}x_{4,3}) = 2$. For any sequence $s_1, s_2, \ldots \in S$, the sequence of products $s_1, s_1s_2, s_1s_2s_3, \ldots$ has minimal degree growing to infinity, and thus (3) must hold. While not necessary, we note in passing that this semigroup is cancelative, and so (4) does not imply (1) even if S is cancelative. (This can also be seen from the fact that there exists a cancelative semigroup with ACCPR but not ACCPL [12, Example 2.6], but (1) is left-right symmetric and so implies ACCPR.)

(4)$\not\Rightarrow$(3): Let

$$S = \{\alpha \text{ is an ordinal} : 1 \leq \alpha < \omega^2\}$$

which is a finitely generated, additive semigroup. This semigroup fails (3) since

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} + \omega = \omega.$$

Obviously S has no idempotents, and it suffices to show that S has ACCPL. For any $\alpha \in S$, let (α) denote the left ideal of S generated by α. Suppose there exists a strictly increasing chain of principal left ideals $(\alpha_1) \subsetneq (\alpha_2) \subsetneq (\alpha_3) \subsetneq \ldots$. Then for any i we have $\alpha_i = \beta_i + \alpha_{i+1}$ for some $\beta_i \geq 1$ and thus $\alpha_1 > \alpha_2 > \alpha_3 > \ldots$, contradicting the fact that ordinals are well-ordered. It is interesting to note that S is left cancelative, but not right cancelative. $\square$

A semigroup ordering $\leq$ on a semigroup S is called a *positive ordering* if for every $s, t \in S$ it happens that $s \leq st$ and $t \leq st$. A semigroup S for which such an ordering exists is said to be *positively orderable*. The semigroup $\mathbb{N}^+$ is the prototypical example of a positively orderable semigroup. The last proposition of this paper relates positive orderings to the conditions considered previously.

Proposition 27. *Let S be a finitely generated, cancelative semigroup. If S is positively orderable, then S satisfies properties* (1)-(4) *above. The converse is not true in general.*

Proof. The forward direction is [11, Proposition 1.3].

To show the converse is not always true, first notice that if $\leq$ is a positive ordering on a semigroup S, then given $x, y, z \in S$ we must have $x \leq xy$ and hence $xz \leq xyz$. Let $S = \langle a_1, a_2, b_1, b_2, x : a_1xa_2 = b_1b_2,\ b_1xb_2 = a_1a_2\rangle$. This semigroup cannot be positively ordered, because we would then have $a_1a_2 \leq a_1xa_2 = b_1b_2 \leq b_1xb_2 = a_1a_2$, but $a_1a_2 \neq b_1b_2$.

By a *reduced word*, we will mean a monomial written in the shortest form (thus, replacing all instances of a_1xa_2 by b_1b_2, and all instances of b_1xb_2 by a_1a_2). That reduced words are unique is an easy consequence of Bergman's Diamond Lemma [3]. If k is the number of instances where a_1a_2 or b_1b_2 occur as subwords of a reduced word m, then m is a product of the generators of S in at most 2^k ways. Thus property (1) holds. It remains to show that S is left cancelative, and by symmetry we will obtain right cancelativity.

Write $m_1m_2 = m_1m_3$ for some monomials $m_1, m_2, m_3 \in S$, each written as reduced words. The two products $m_1m_2 = m_1m_3$ must reduce to the same reduced word. If no reductions need to be made in either product, then clearly $m_2 = m_3$. So, without loss of generality we may suppose that the concatenation of m_1 and m_2 is not reduced. We may then write $m_1 = m_1'n_1$ and $m_2 = n_2m_2'$ where $m_1', m_2' \in S^1$ are reduced words (since m_1 and m_2 are reduced) and $(n_1, n_2) \in \{(a_1, xa_2), (a_1x, a_2), (b_1, xb_2), (b_1x, b_2)\}$. In each of these four cases, we find that for m_1m_3 to have the same reduced form as m_1m_2, we must have $m_2 = m_3$. For example, if $n_1 = a_1$ and $n_2 = xa_2$ then m_1m_2 reduces to $m_1'b_1b_2m_2'$. For m_1m_3 to match this, we must have $m_3 = xa_2m_2' = m_2$. The other three cases are similar. $\square$

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