SIMPLIFYING SMOKTUNOWICZ'S EXTRAORDINARY EXAMPLE

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Abstract. At the turn of the 21st century Agata Smoktunowicz constructed the first example of a nil algebra over a countable field such that the polynomial ring over the algebra is not nil. This answered an old question of Amitsur. We present a simplification of the example.

Introduction

This monograph began as the author's earnest attempt to understand the beautiful example constructed in [1], and was originally meant only to serve as a set of personal notes. However, the author believes the simplifications offered will draw more people to understand and appreciate Smoktunowicz's example. Further, as of the writing of this paper the example has been used in more than twenty other research papers, and continues to be generalized and expanded. It is hoped that this note will aid in this process and simplify future constructions. Almost all of the original research comes from Smoktunowicz's work in [1], and many of the lemmas, theorems, and corollaries of this note are direct generalizations of results from that paper. The present paper reworks and streamlines the bounds and numerical computations of that paper. Further, following the methods of [2], we answer the implicit question left open at the end of the paper, showing that the main construction technique needs only one polynomial variable (and not two) to reach the desired example.

Rather than construct each of the components of the example separately and then put them together at the end, we take a motivational approach where we give the underlying reason for each piece before it is constructed. It is hoped that this will aid the reader in keeping track of the important information as it is needed. As the paper is in essence a single example, any quantities defined outside of lemmas, theorems, or corollaries will continue to be defined throughout the paper and retain their meanings when invoked in proofs or in the statement of propositions.

1. Nilpotence over polynomial rings

Let K be a countable field. Our goal is to construct a nil K-algebra R such that the polynomial ring $R[X]$ is not nil. The first half of our goal is simple. We let $A = K\langle x_1, x_2, x_3\rangle$ be the (unital) free algebra in the three noncommuting variables x_1, x_2, and x_3. We also let A' denote the (nonunital) K-subalgebra of A consisting of the elements with zero constant term. We fix an indexing $\{f_1, f_2, f_3, \ldots\}$ of A' by the positive integers. This is the one and only place where we use the countability of K. For each $i \geq 1$ we let e_i be a positive integer and we let I be an ideal of A containing $f_i^{e_i}$ for every $i \geq 1$. We set $R = A'/I$ and see that R is nil.

As arbitrary ideals are difficult to work with, we take the simplifying position that I is a *homogeneous* ideal. Thus, we assume:

(1) The ideal I is generated by the homogeneous components of $\{f_i^{e_i} : i \geq 1\}$.

One might ask: Why use three noncommuting variables to form A? It turns out that in fact two variables suffice (which will be shown at the end of this section), but with three the proof is significantly

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simplified. In particular, condition (1) is compatible with the 3-generator construction but not with the corresponding 2-generator construction.

The difficulty in effectuating the second half of our goal is in controlling the exponents e_i enough so that $R[X]$ is not nil. Intuitively, if the exponents grow quickly then that should suffice. The remainder of the paper will make formal what constitutes "quickly." As might be expected, we demonstrate that $R[X]$ is not nil by exhibiting an element of the form $a + bX \in R[X]$ which is not nilpotent. There is a characterization of when $a + bX$ is nilpotent which we can translate into information on the exponents e_i. We begin with the following easy exercise:

Lemma 1. *Given a ring S and two elements $a, b \in S$, then $a + bX \in S[X]$ is nilpotent if and only if $1 - (aX + bY) \in S[X, Y]$ is a unit in $S[X, Y]$.*

Proof. We first introduce some notation. Given integers $i, j \geq 0$, let $w(i, j)$ denote the coefficient of $X^i Y^j$ in $(aX + bY)^{i+j}$. Notice that $f(X, Y) = 1 - (aX + bY)$ is always a unit in the power series ring $S[\![X, Y]\!]$, with inverse given by

$$g(X, Y) = \sum_{n \geq 0} (aX + bY)^n = \sum_{i,j \geq 0} w(i, j) X^i Y^j.$$

Thus $f(X, Y)$ is a unit in $S[X, Y]$ if and only if $g(X, Y)$ is a (finite) polynomial. But $g(X, Y)$ is a polynomial if and only if $w(i, j) = 0$ whenever $i + j$ is large enough.

On the other hand, $a + bX$ is nilpotent if and only if there exists an integer $N \geq 1$ so that for all integers $n \geq N$, we have

$$0 = (a + bX)^n = \sum_{j=0}^{n} w(n - j, j) X^j.$$

Or equivalently, $w(n - j, j) = 0$ for all integers $n \geq N$ and all integers $j \geq 0$. This is exactly the same condition as at the end of the first paragraph. $\square$

Set $\overline{x}_i = x_i + I \in R$ and consider the element $z = \overline{x}_1 + \overline{x}_2 X + \overline{x}_3 Y \in R[X, Y]$. Recall that as R is nil this entails that $\overline{x}_1 \in R$ is nilpotent, say of index $k \geq 1$. Let $y = (1 - \overline{x}_1)^{-1} = 1 + \overline{x}_1 + \overline{x}_1^2 + \cdots + \overline{x}_1^{k-1} \in R$. We compute

$$1 - z = (1 - \overline{x}_1)(1 - y\overline{x}_2 X + y\overline{x}_3 Y).$$

Thus, by the previous lemma, $y\overline{x}_2 + y\overline{x}_3 X \in R[X]$ is nilpotent if and only if z is a unit in $R[X, Y]$. Our job now is to:

(2) Choose the exponents e_i so that $1 - (x_1 + x_2 X + x_3 Y)$ is not a unit modulo I in $R[X, Y]$.

Once this is done we will know R (or even the subalgebra of R generated by the two elements $a = y\overline{x}_2$ and $b = y\overline{x}_3$) gives us the example we are after.

2. Homogeneous polynomials related to $x_1 + x_2 X + x_3 Y$

We introduce some notation and terminology which will be central throughout the rest of the paper. By $\mathbb{N}$ we will mean the set of non-negative integers. Let M_n denote the set of monomials in x_1, x_2, x_3 of total degree $n \in \mathbb{N}$, and let H_n denote the set of homogeneous polynomial of degree n. Note that $|M_n| = 3^n$, and these monomials form a basis for the K-vector space H_n. Let $w = y_1 y_2 \cdots y_n \in M_n$, where $y_i \in M_1$ for each $i \geq 1$. Given a permutation $\sigma \in S_n$, we define $w^\sigma = y_{\sigma(1)} y_{\sigma(2)} \cdots y_{\sigma(n)} \in M_n$. We extend this action K-linearly to H_n. Given $f \in A$ we let $\deg(f)$ denote the largest total degree of a monomial appearing with nonzero support in f, and let $d_{x_i}(w)$ denote the degree of the monomial w in the variable x_i.

For each $n \in \mathbb{N}$ we let $T_n = \{(n_1, n_2, n_3) \in \mathbb{N}^3 : n_1 + n_2 + n_3 = n\}$. Given $r = (n_1, n_2, n_3) \in T_n$ we define

$$w(r) = w(n_1, n_2, n_3) = \sum \{w \in M_{n_1+n_2+n_3} : d_{x_1}(w) = n_1,\ d_{x_2}(w) = n_2, d_{x_3}(w) = n_3\}.$$

In other words, this is the sum of all monomials w of total degree $n = n_1 + n_2 + n_3$ whose degree in x_i is n_i. We call these the *nice* homogeneous polynomials.

There is an interesting connection between these nice polynomials and $x_1 + x_2X + x_3Y \in A[X, Y]$. Namely, given $n_1, n_2, n_3 \in \mathbb{N}$ the coefficient in $(x_1 + x_2X + x_3Y)^n$ of $X^{n_2}Y^{n_3}$ is $w(n_1, n_2, n_3)$ if $n_1 + n_2 + n_3 = n$. There is another connection, which is related to what happens in the proof of Lemma 1.

Lemma 2. *The element $1 - (x_1 + x_2X + x_3Y) \in A[X, Y]$ is a unit modulo I in $R[X, Y]$ if and only if there exists an integer $N \geq 1$ such that for each $n \geq N$ and each $r \in T_n$, $w(r) \in I$.*

Proof. Recall that from (1), x_1 is nilpotent modulo I, say of index $k \geq 1$. We see that if $r = (n_1, n_2, n_3) \in \mathbb{N}^3$ is a triple with $n_1 > (k - 1)(n_2 + n_3 + 1)$ then any monomial in $w(r)$ has a string of k successive x_1's, and thus belongs to I. In particular, $1 - (x_1 + x_2X + x_3Y)$ has an inverse modulo I in the power series ring $R[\![X, Y]\!]$, given by

$$\sum_{n \in \mathbb{N}} (x_1 + x_2X + x_3Y)^n + I[\![X, Y]\!] = \sum_{n_2, n_3 \in \mathbb{N}} \left(\sum_{0 \leq n_1 \leq (k-1)(n_2+n_3+1)} w(n_1, n_2, n_3) \right) X^{n_2}Y^{n_3} + I[\![X, Y]\!].$$

Thus $1 - (x_1 + x_2X + x_3Y)$ is a unit modulo I in $R[X, Y]$ if and only if the power series above is a finite polynomial. As I is homogeneous, the power series is a polynomial if and only if $w(r) \in I$ for all $r \in T_n$, for all sufficiently large integers n. $\square$

This lemma lets us rephrase our goal (2) in the following form:

(3) Choose the e_i and an increasing sequence of integers $1 \leq m_1 < m_2 < m_3 < \ldots$ so that for each $j \geq 1$ there is some triple $r \in T_{m_j}$ with $w(r) \notin I$.

This will guarantee the noninvertibility of $1 - (x_1 + x_2X + x_3Y)$ modulo I in $R[X, Y]$.

It is difficult turn information about the exponents e_i and the sequence of integers m_i into a proof that $w(r) \notin I$ for some $r \in T_{m_i}$. Our way around this difficulty is to construct a map (one for each $i \geq 1$) which is zero on $I \cap H_{m_i}$ but nonzero on at least one nice polynomial in H_{m_i}. We begin this process by first investigating how nice polynomials behave under K-linear mappings in §3. Then in §4 we study the behavior of the homogeneous components of $f_i^{e_i}$. Finally, we put our knowledge together to actually construct the needed maps in §5.

3. Mappings on the nice polynomials

We start with two easy lemmas. The proofs are left to the reader.

Lemma 3. *The following hold:*

(a) *For each $n \in \mathbb{N}$, the set $S = \{w(r) : r \in T_n\}$ is right linearly independent over A.*
(b) *Given $a, b \in \mathbb{N}$, if $t \in T_{a+b}$ then*

$$w(t) = \sum_{r \in T_a,\ s \in T_b,\ r+s=t} w(r)w(s).$$

Let $\prec$ denote the lexicographic ordering on $\mathbb{Z}^3$. So $(m_1, m_2, m_3) \prec (n_1, n_2, n_3)$ holds if and only if (i) $m_1 < n_1$, or (ii) $m_1 = n_1$ and $m_2 < n_2$, or (iii) $m_1 = n_1$, $m_2 = n_2$, and $m_3 < n_3$.

Lemma 4. *Let $r, r', s \in \mathbb{Z}^3$. If $r \prec r'$ then $r + s \prec r' + s$.*

Assume $a, b \in \mathbb{N}$, and let $r_0 \in T_a$ and $s_0 \in T_b$. Putting Lemma 3(b) and Lemma 4 together we have

$$w(r_0 + s_0) = w(r_0)w(s_0) + \sum_{\substack{r \in T_a,\ s \in T_b,\ r+s=r_0+s_0,\\ r \prec r_0 \text{ or } s \prec s_0}} w(r)w(s). \tag{4}$$

The next lemma tells us that a multiplicative K-linear function cannot be zero on nice polynomials, unless one of the component functions is zero on nice polynomials.

Lemma 5. *Let $a, b \in \mathbb{N}$. Suppose we have three K-linear maps $\varphi : H_{a+b} \to A$, $\gamma : H_a \to A$, and $\delta : H_b \to A$ satisfying $\varphi(uv) = \gamma(u)\delta(v)$ for each $u \in H_a$ and each $v \in H_b$. If $\gamma(w(r)) \neq 0$ for some $r \in T_a$ and $\delta(w(s)) \neq 0$ for some $s \in T_b$ then $\varphi(w(t)) \neq 0$ for some $t \in T_{a+b}$.*

Proof. Let $r_0 = (a_1, a_2, a_3) \in T_a$ and $s_0 = (b_1, b_2, b_3) \in T_b$ be minimal with respect to the lexicographical ordering such that $\gamma(w(r_0)) \neq 0$ and $\delta(w(s_0)) \neq 0$. By equation (4), and the multiplicative nature of φ, we have

$$\begin{aligned}\varphi(w(r_0 + s_0)) &= \varphi(w(r_0)w(s_0)) + \sum_{\substack{r \in T_a,\ s \in T_b,\ r+s=r_0+s_0,\\ r \prec r_0 \text{ or } s \prec s_0}} \varphi(w(r)w(s)) \\ &= \gamma(w(r_0))\delta(w(s_0)) + \sum_{\substack{r \in T_a,\ s \in T_b,\ r+s=r_0+s_0,\\ r \prec r_0 \text{ or } s \prec s_0}} \gamma(w(r))\delta(w(s)) = \gamma(w(r_0))\delta(w(s_0)) \neq 0.\end{aligned}$$

□

It may be of interest to note that the previous lemma is true more generally if each of φ, γ, and δ map to a common K-algebra without zero-divisors.

The previous lemma told us that if φ is multiplicative and the components γ and δ are not zero on all nice polynomials, then φ is not zero on all nice polynomials. If one of the component functions is the identity then we have even more information available.

Theorem 6. *Let $a, b \in \mathbb{N}$. Suppose $\varphi : H_{a+b} \to A$ and $\delta : H_b \to A$ are K-linear maps, with $\varphi(uv) = u\delta(v)$ for each $u \in H_a$ and each $v \in H_b$. If $\delta(w(s)) \neq 0$ for some $s \in T_b$ then the dimension of the K-subspace of A spanned by $\{\varphi(w(t)) : t \in T_{a+b}\}$ is $\geq (a+1)(a+2)/2$.*

Proof. Let $s_0 \in T_b$ be the unique element of T_b which is minimal with respect to the lexicographical ordering and subject to the condition $\delta(w(s_0)) \neq 0$. Fix $r_0 = (a_1, a_2, a_3) \in T_a$. Notice that there are exactly $(a+1)(a+2)/2 = |T_a|$ choices for such an r_0.

By Lemma 3(a) we know that

$$\sum_{r \in T_a} w(r)A \tag{5}$$

is a direct sum. Applying φ to (4), we have

$$\begin{aligned}\varphi(w(r_0 + s_0)) &= \varphi(w(r_0)w(s_0)) + \sum_{\substack{r \in T_a,\ s \in T_b,\ r+s=r_0+s_0,\\ r \prec r_0 \text{ or } s \prec s_0}} \varphi(w(r)w(s)) \\ &= w(r_0)\delta(w(s_0)) + \sum_{\substack{r \in T_a,\ s \in T_b,\ r+s=r_0+s_0,\\ r \prec r_0 \text{ or } s \prec s_0}} w(r)\delta(w(s)) \\ &= w(r_0)\delta(w(s_0)) + \sum_{\substack{r \in T_a,\ s \in T_b,\ r+s=r_0+s_0,\\ r \prec r_0}} w(r)\delta(w(s)).\end{aligned}$$

Thus, we see that the largest $r \in T_a$ for which $\varphi(w(r_0 + s_0))$ has a nonzero summand occurring in the decomposition (5) is r_0. As r_0 ranges over T_a, the elements $\varphi(w(r_0 + s_0))$ have their highest nonzero components in different summands in (5), hence the set $\{\varphi(w(r_0+s_0)) : r_0 \in T_a\}$ is K-linearly independent and has cardinality $(a+1)(a+2)/2$. $\square$

4. The homogeneous components of powers of elements

We now understand how the nice polynomials behave under K-linear mappings. What we need next is an understanding of how the homogeneous components of $f_i^{e_i}$ behave. In particular, as our goal is to construct maps which are zero on $I \cap H_{m_i}$ we need to understand how "large" this set is. The following lemma and theorem address what happens for a single element $f \in A'$.

Lemma 7. *Let $f \in A'$ have degree $t \geq 1$. For each $n \geq t$ there exists a set $G(f,n) \subseteq H_n$ with*

$$|G(f,n)| \leq \frac{2(t-1)(3^t-1)n - 3^t(t-3)(2t-1) - 5t + 3}{4t} \tag{6}$$

such that, for every integer $m \geq n$, f^m belongs to the right ideal of A generated by polynomials of the form hgf^{m-n}, where $h \in G(f,n)$ and $\deg(g) < t$.

Proof. Write $f = \sum_{i=1}^t f_i$ where $f_i \in H_i$. Thus f^m is a sum of terms of the form $f_{i_1} f_{i_2} \cdots f_{i_k} f_{i_{k+1}} \cdots f_{i_m}$, where k denotes the largest subscript such that $i_1 + i_2 + \cdots + i_k \leq n$. We then have the decomposition:

$$\begin{aligned} f^m &= \sum_{k \leq n} \left(\sum_{i_1+\cdots+i_k=n} f_{i_1} \cdots f_{i_k} f^{m-k} + \sum_{j=1}^{t-1} \left(\sum_{i_1+\cdots+i_k=n-j,\, i_{k+1}>j} (f_{i_1} \cdots f_{i_k}) f_{i_{k+1}} f^{m-k-1} \right) \right) \\ &= \sum_{k \leq n} v_{0,k} f^{m-k} + \sum_{k \leq n-1} \left(\sum_{j=1}^{t-1} v_{j,k}(f_{j+1} + f_{j+2} + \cdots + f_t) f^{m-k-1} \right). \end{aligned}$$

where $v_{j,k}$ is the homogeneous component of degree $n-j$ in f^k.

Taking $G(f,n) = \bigcup_{0 \leq j \leq t-1} \bigcup_k v_{j,k} M_j$, the representation above has the desired properties. To see this, note that in the second summation each term $v_{j,k} f_{j+i} f^{m-k-1}$ lies in the span of all members of $v_{j,k} M_{j+i} f^{m-n} f^{n-k-1} \subseteq v_{j,k} M_j M_i f^{m-n} A$. As elements of $v_{j,k} M_j$ lie in $G(f,n)$ and elements of M_i have degree $< t$, the product lives in the desired right ideal. The same is still more easily seen for terms of the first summation.

Notice that if we want $v_{j,k}$ to be nonzero then we necessarily must have $(n-j)/t \leq k \leq n-j$. We compute

$$|G(f,n)| \leq \sum_{j=0}^{t-1} \sum_{k=\lceil (n-j)/t \rceil}^{n-j} 3^j \leq \frac{2(t-1)(3^t-1)n - 3^t(t-3)(2t-1) - 5t + 3}{4t}.$$

$\square$

Given an integer $n \geq 1$ and a subset $F \subseteq H_m$ we let

$$B_n(F) = \sum_{i=0}^{\infty} M_{ni} F A, \tag{7}$$

which is a right ideal. Intuitively, $B_n(F)$ is nearly the two-sided ideal in A generated by the subset F except that on the left we only assume closure under multiplication by monomials of degree divisible by n.

Theorem 8. *Let $f \in A'$ have degree $t \geq 1$. For each $r \geq 2t-1$ there exists a set $F \subseteq H_r$ such that*

$$|F| \leq (3^t - 1) \frac{(t-1)(3^t-1)r - 3^t(2t^2 - 6t + 3) - 4t + 3}{4t} < 3^{2t-1} r \tag{8}$$

and, for every integer $s \geq r$, the ideal $\langle f^{3s} \rangle$ is contained in $B_s(F)$.

Proof. Let $F = \bigcup_{0 \leq i \leq t-1} M_i G(f, r-i)$. The first inequality in (8) now follows from the previous lemma. One checks the second inequality for $t = 1, 2$ directly, and for $t \geq 3$ it is straightforward.

To prove the remainder of the lemma, it suffices to show that $uf^{3s} \in B_s(F)$, for each monomial u of degree d with $0 < d < s$. Clearly $t < 2s - d < 3s$ so we apply the previous lemma to f with $n = 2s - d$ and $m = 3s$. Thus f^{3s} belongs to the right ideal of A generated by elements of the form $hgf^{3s-2s+d} = hgf^{s+d}$, where $h \in G(f, 2s-d)$ and $\deg(g) < t$. As $uh \in H_{2s}$ it suffices to show that $gf^{s+d} \in B_s(F)$, where we may assume g is a monomial of degree $\ell < t$. As $t \leq r - \ell < s + d$ we again apply the previous lemma to f, but this time with $n = r - \ell$ and $m = s + d$. It follows that f^{s+d} belongs to the right ideal of A generated by expressions of the form $h'g'f^{s+d-r+\ell}$, where $h' \in G(f, r-\ell)$ and $\deg(g') < t$. By definition, $gh' \in F$, and thus $gf^{s+d} \in FA \subseteq B_s(F)$ as wanted. $\square$

Remarks: (a) In the case that $t = 1$ we have $F = \{f^r\}$ and one can prove that $s + r - 1$ works in place of $3s$.

(b) As $B_s(F)$ is a homogeneous right ideal, it contains not only the ideal $\langle f^{3s} \rangle$ but also the homogeneous components of this ideal.

We wish to rephrase Theorem 8 in terms of all elements of A' simultaneously. To do this we make a simplifying assumption about the indexing $\{f_1, f_2, \ldots\} = A'$. We choose this indexing so that

$$f_1 = x_1 \text{ and } \deg(f_i) \leq i/2 \text{ when } i > 1. \tag{9}$$

Corollary 9. *Let $1 \leq m_1 < m_2 < m_3 < \ldots$ be an increasing sequence of positive integers. There exist subsets $F_i \subseteq H_{m_i}$ with*

$$|F_i| \leq 3^{i-1} m_i \tag{10}$$

such that the two-sided ideal of A generated by the homogeneous components of $\{f_i^{3m_{i+1}}\}_{i=1}^{\infty}$ is contained in the K-subspace (and right ideal) $\sum_{i=1}^{\infty} B_{m_{i+1}}(F_i)$.

Given a sequence $1 \leq m_1 < m_2 < m_3 < \ldots$ fix subsets F_i (once and for all) satisfying this corollary. The corollary tells us how to choose our exponents:

$$\text{For each integer } i \geq 1, \text{ fix } e_i = 3m_{i+1}. \tag{11}$$

The sequence $\{m_i\}_{i \geq 1}$ will be determined later. Note that with this choice of exponents we have $I \subseteq \sum_{i=1}^{\infty} B_{m_{i+1}}(F_i)$. We thus restate our ultimate goal (3) as:

(12) Choose an increasing sequence of integers $1 \leq m_1 < m_2 < m_3 < \ldots$ so that for each $i \geq 1$ there is some triple $r \in T_{m_i}$ with $w(r) \notin \sum_{i=1}^{\infty} B_{m_{i+1}}(F_i)$.

5. Linear maps

There is another simplifying assumption we enforce on the sequence $1 \leq m_1 < m_2 < m_3 < \ldots$, namely:

$$\text{For each } i \geq 1,\ m_i | m_{i+1}. \tag{13}$$

This divisibility property will be essential for what happens next.

For each subset $S \subseteq H_n$, fix (once and for all) a K-vector space decomposition $\text{span}(S) \oplus S^{\perp} = H_n$. We let $\psi_S : H_n \to H_n$ be the projection map which is the identity on $S^{\perp}$ and zero on S. We are now ready to define K-linear maps which will be zero on homogeneous components of $\sum_{i=1}^{\infty} B_{m_{i+1}}(F_i)$. First, define $\varphi_1 : H_{m_1} \to H_{m_1}$ to be the identity map. Now we use condition (13); given a monomial $w \in M_{m_2}$ we can write $w = w_1 w_2 \cdots w_{m_2/m_1}$ where $w_j \in M_{m_1}$ for each $j \geq 1$. We define $\varphi_2 : M_{m_2} \to M_{m_2}$ by the rule

$$\varphi_2(w) = \psi_{\varphi_1(F_1)}(\varphi_1(w_1)) \prod_{j=2}^{m_2/m_1} \varphi_1(w_j).$$

In other words, we repeat φ_1 a total of m_2/m_1 times, except that on the first instance we post-compose with a projection map. Thus φ_2 is simply the identity, except that on the first m_1 variables we kill anything that lived in the span of $\varphi_1(F_1)$, and project to a complement. This map is defined on all of H_{m_2} by extending K-linearly.

We repeat this process and define φ_{i+1} inductively by the rule

$$\varphi_{i+1}(w) = \psi_{\varphi_i(F_i)}(\varphi_i(w_1)) \prod_{j=2}^{m_{i+1}/m_i} \varphi_i(w_j) \tag{14}$$

where $w = w_1 w_2 \cdots w_{m_{i+1}/m_i} \in M_{m_{i+1}}$ and $w_j \in M_{m_i}$ for each $j \geq 1$.

Theorem 10. *For each $\ell \geq 1$ we have*

$$H_{m_{\ell+1}} \cap I \subseteq H_{m_{\ell+1}} \bigcap \left(\sum_{i=1}^{\ell} B_{m_{i+1}}(F_i) \right) \subseteq \ker(\varphi_{\ell+1}). \tag{15}$$

Proof. As $\varphi_{\ell+1}$ is K-linear it suffices to show that $\varphi_{\ell+1}(x) = 0$ for each $x \in H_{m_{\ell+1}} \cap B_{m_{i+1}}(F_i)$, for each i in the range $1 \leq i \leq \ell$. From (7) we know x is a sum of terms of the form gfh where $f \in F_i \subseteq H_{m_i}$, $g \in M_{tm_{i+1}}$ for some $t \in \mathbb{N}$, and $h \in M_{m_{\ell+1}-tm_{i+1}-m_i}$. Without loss of generality we may assume $x = gfh$.

We want to break $x = gfh$ into a product of $m_{\ell+1}/m_\ell$ homogeneous pieces each of degree m_ℓ. Write $g = x_1 x_2 \cdots x_{k-1} g'$ where $x_j \in M_{m_\ell}$ for each $j \geq 1$ and $\deg(g') < m_\ell$. As $m_i | m_{i+1}$ and $m_i | m_\ell | m_{\ell+1}$ we have $\deg(g'f) \leq m_\ell$, and so we can also write $h = h'h''$ with $\deg(g'fh') = m_\ell$. We fix $x_k = g'fh'$. As $\deg(x) = m_{\ell+1} = (m_{\ell+1}/m_\ell)m_\ell$ we can further decompose h'' as $h'' = x_{k+1}x_{k+2} \cdots x_{m_{\ell+1}/m_\ell}$ with $x_j \in M_{m_\ell}$ for each $j \geq k+1$, and in particular $x = x_1 x_2 \cdots x_{m_{\ell+1}/m_\ell}$. The inductive definition of $\varphi_{\ell+1}$ given by (14) yields

$$\varphi_{\ell+1}(x) = \psi_{\varphi_\ell(F_\ell)}(\varphi_\ell(x_1)) \prod_{j=2}^{m_{\ell+1}/m_\ell} \varphi_\ell(x_j). \tag{16}$$

We establish $\varphi_{\ell+1}(x) = 0$ by induction on $\ell \geq 1$.

Case 1: Assume $i = \ell$. Notice that this includes the base case when $\ell = 1$. We have $g = 1$, $k = 1$ and $x_1 = f \in F_\ell$. Thus the term $\psi_{\varphi_\ell(F_\ell)}(\varphi_\ell(x_1))$ in equation (16) is zero.

Case 2: Assume $\ell > 1$ and $i < \ell$. We have $m_{i+1} | m_\ell$ and so $m_{i+1} | \deg(g')$. Thus $x_k = g'fh' \in H_{m_\ell} \cap B_{m_{i+1}}(F_i)$, and so our inductive hypothesis implies $\varphi_\ell(x_k) = 0$ and hence (16) is still zero. $\square$

We have now accomplished the first half of what we set out to do. We have defined maps φ_i which are zero on $H_{m_i} \cap I$, for each $i \geq 2$. All that remains is to show that they do not annihilate all nice polynomials in H_{m_i}. To do this we will need an auxiliary sequence of integers which measures how much φ_i acts as the identity map (in a sense to be made formal shortly). Define

$$p_1 = m_1 \text{ and } p_{i+1} = p_i \left(\frac{m_{i+1}}{m_i} - 1 \right) \text{ for } i \geq 1. \tag{17}$$

Note that (13) implies that this new sequence also consists of integers satisfying $p_i | p_{i+1}$.

Lemma 11. *Given an integer $c \geq 2$, if for every $i \geq 1$ we have $m_i = c^{i-1}$ then $p_i = (c-1)^{i-1}$.*

The sequence of numbers p_i is defined so that (in retrospect) the proof of the following theorem flows smoothly. Before we begin the proof, we introduce one piece of notation. Given a K-linear map $\varphi : H_n \to H_n$ and a permutation $\sigma \in S_n$ we define a new linear map $\varphi^\sigma : H_n \to H_n$ by

$$\varphi^\sigma(w) = (\varphi(w^\sigma))^{\sigma^{-1}}.$$

Theorem 12. *For each integer $i \geq 1$ there exists $\sigma_i \in S_{m_i}$ and a K-linear map $\delta_i : H_{m_i-p_i} \to H_{m_i-p_i}$ satisfying, for each $u \in H_{p_i}$ and each $v \in H_{m_i-p_i}$, the formula*

$$\varphi_i^{\sigma_i}(uv) = u\delta_i(v). \tag{18}$$

Proof. To prove the theorem we induct on $i \geq 1$. The case $i = 1$ is trivial.

Given $\sigma \in S_k$ and any integer $r \geq 1$, define $\overline{\sigma} \in S_{rk}$ by taking a monomial $w_1w_2\cdots w_r$ (where $w_j \in M_k$) to $w_1^\sigma w_2^\sigma \cdots w_r^\sigma$. In other words, we just repeat σ on the appropriate length pieces.

Each monomial $u \in M_{p_i}$ is uniquely written as $u = u_1u_2\cdots u_{(m_i/m_{i-1})-1}$ with each $u_j \in M_{p_{i-1}}$. Similarly, $v \in M_{m_i-p_i}$ can be written uniquely as $v = wv_1v_2\cdots v_{(m_i/m_{i-1})-1}$ where $v_j \in M_{m_{i-1}-p_{i-1}}$ and $w \in M_{m_{i-1}}$. Define a permutation $\rho_i \in S_{m_i}$ by

$$(uv)^{\rho_i} = wu_1v_1u_2v_2\cdots u_{(m_i/m_{i-1})-1}v_{(m_i/m_{i-1})-1}.$$

Recursively, we set $\sigma_i = \overline{\sigma}_{i-1} \circ \rho_i$. Working out what this does to the left-hand side of (18), we compute

$$\begin{aligned}
\varphi_i^{\sigma_i}(uv) &= (\varphi_i((uv)^{\sigma_i}))^{\sigma_i^{-1}} = \left(\varphi_i(w^{\sigma_{i-1}}(u_1v_1)^{\sigma_{i-1}}(u_2v_2)^{\sigma_{i-1}}\cdots(u_{(m_i/m_{i-1})-1}v_{(m_i/m_{i-1})-1})^{\sigma_{i-1}})\right)^{\sigma_i^{-1}}\\
&= \left((\psi_{\varphi_{i-1}(F_{i-1})}\circ\varphi_{i-1})^{\sigma_{i-1}}(w)\varphi_{i-1}^{\sigma_{i-1}}(u_1v_1)\varphi_{i-1}^{\sigma_{i-1}}(u_2v_2)\cdots\varphi_{i-1}^{\sigma_{i-1}}(u_{(m_i/m_{i-1})-1}v_{(m_i/m_{i-1})-1})\right)^{\rho_i^{-1}}\\
&= \left((\psi_{\varphi_{i-1}(F_{i-1})}\circ\varphi_{i-1})^{\sigma_{i-1}}(w)u_1\delta_{i-1}(v_1)u_2\delta_{i-1}(v_2)\cdots u_{(m_i/m_{i-1})-1}\delta_{i-1}(v_{(m_i/m_{i-1})-1})\right)^{\rho_i^{-1}}\\
&= u\left((\psi_{\varphi_{i-1}(F_{i-1})}\circ\varphi_{i-1})^{\sigma_{i-1}}(w)\delta_{i-1}(v_1)\delta_{i-1}(v_2)\cdots\delta_{i-1}(v_{(m_i/m_{i-1})-1})\right).
\end{aligned}$$

Thus, we may define

$$\delta_i(v) = (\psi_{\varphi_{i-1}(F_{i-1})}\circ\varphi_{i-1})^{\sigma_{i-1}}(w)\left(\prod_{j=1}^{(m_i/m_{i-1})-1}\delta_{i-1}(v_j)\right).$$

The map δ_i is extended K-linearly to all of $H_{m_i-p_i}$. □

Hereafter, we suppose that σ_i and δ_i have been chosen satisfying the previous theorem. Notice that the theorem says that $\varphi_i^{\sigma_i}$ behaves as a multiplicative map like the map φ in Theorem 6. In point of fact, we now combine Theorem 12 with Lemma 5 and Theorem 6 to finish our goal (12). In particular, we are ready to set the m_i to be specific integers, and hence by (11) also fix the exponents e_i.

Theorem 13. *For every $i \geq 1$ set $m_i = 6^{i-1}$. For every $i \geq 1$ there exists a triple $r_i \in T_{m_i}$ with $\varphi_i(w(r_i)) \neq 0$.*

Proof. We work by induction on $i \geq 1$, the case when $i = 1$ being trivial. So assume that $r_i \in T_{m_i}$ has been chosen so that $\varphi_i(w(r_i)) \neq 0$.

We first recall some information. By Lemma 11, $p_i = 5^{i-1}$, so the sets F_i of Corollary 9 satisfy

$$|F_i| \leq 3^{i-1}m_i \leq 18^{i-1} < (p_i+1)(p_i+2)/2. \tag{19}$$

We also have $w(r_i)^{\sigma_i} = w(r_i)$. Thus $\varphi_i^{\sigma_i}(w(r_i)) \neq 0$, and in particular $\delta_i(w(t_i)) \neq 0$ for some $t_i \in T_{m_i-p_i}$ (by Lemma 3(b)). By Theorem 6, with $\varphi = \varphi_i^{\sigma_i}$, $\delta = \delta_i$, $a = p_i$ and $b = m_i - p_i$, we conclude that the set $\{\varphi_i(w(r)) : r \in T_{m_i}\}$ spans a K-vector space of dimension at least $(p_i+1)(p_i+2)/2$. In particular, by (19) the span cannot belong to $\ker(\psi_{\varphi_i(F_i)})$. This proves that $\psi_{\varphi_i(F_i)}(\varphi_i(w(r_i'))) \neq 0$ for some $r_i' \in T_{m_i}$.

From the inductive definition of φ_{i+1} given by (14) and by Lemma 5, we must have $\varphi_{i+1}(w(r_{i+1})) \neq 0$ for some $r_{i+1} \in T_{m_{i+1}}$. □

This finishes our construction.

6. Summary

Theorem 14. *Let K be a countable field, let $A = K\langle x_1, x_2, x_3\rangle$, and let $A' \subset A$ be the K-subalgebra of polynomials with zero constant term. Let $\{f_1, f_2, \ldots\} = A'$ be an indexing by the positive integers, with $f_1 = x_1$ and $\deg(f_i) \leq i/2$ for each $i \geq 2$. Let I be the ideal of A generated by the homogeneous components of the set $\{f_i^{3\cdot 6^i}\}$. The ring $R = A'/I$ is nil but $R[X]$ is not nil. In fact, the subring of R generated by $yx_2 + I$ and $yx_3 + I$ (where $y = (1 - x_1 + I)^{-1}$) has the same properties.*

We end with three interesting observations.

1. Let I be an ideal in the 2-generated algebra $B = K\langle a, b\rangle$ and let B' be the subalgebra of polynomials with zero constant term. It is impossible to choose I with B'/I nil and yet guarantee $w(n_1, n_2) \notin I$ (where $w(n_1, n_2)$ is as defined in Lemma 1), if I is *homogeneous* in terms of a and b. This is because $1 - (a + bX) = (1 - a)(1 - (1 - a)^{-1}bX)$ always has a polynomial inverse when a and $(1 - a)^{-1}b$ are nilpotent. However, taking $a = yx_2 + I$ and $b = yx_3 + I$ as above, our construction does guarantee that for each $n \in \mathbb{N}$ there is some pair $(n_1, n_2) \in \mathbb{N}^2$ with $n_1 + n_2 = n$ and $w(n_1, n_2) \notin I$, even when I is homogeneous. This is possible since I is not homogeneous in the letters a and b.

2. If we are a little more careful, we can take $m_1 = 1$ and $m_2 = 2$ in the construction above, although we must then make the rest of the m_i increase very quickly. Using the first remark after Theorem 8, this means we can take $x_1^2 \in I$ and still achieve our example.

3. In Theorem 13, if we take $m_i = 5^{i-1}$ then the inequalities work out for $i \geq 12$. We can make the inequalities work when $i < 12$ if we choose the ordering of $A' = \{f_1, f_2, \ldots\}$ a little more carefully. Or even more trivially, we may pad our indexing of the set A' with extra copies of 0 and make the growth rate of the index of nilpotence as slow as we like. The essential point to take away is that if we make only the mild assumptions on the indexing of our polynomials given in (9), all growth rate considerations are subsumed in Corollary 9 and in the choice of the m_i so that (19) holds.

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