NON-SELF-INJECTIVE INJECTIVE HULLS WITH COMPATIBLE MULTIPLICATION

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Abstract. Let R be a ring, and let S be the injective hull of the right regular module R_R. Suppose that S can be made into a ring with multiplication compatible with that of R. Osofsky, in 1964, asked if S_S is necessarily injective. We construct examples giving a negative answer to this question, and even construct an infinite chain of such rings.

To Carl Faith on his eightieth birthday.

1. Introduction

Let R be a ring and let $E = E(R)$ be the injective hull of the right R-module R_R. The module E_R can often be made into a ring with multiplication compatible with that of R. For example, this is the case if R is right nonsingular, in which case E is the maximal right ring of quotients of R, see [9, Johnson's Theorem 13.36], [7, Corollary 12C]. For an extensive discussion of quotient rings and their properties (with proofs) and numerous examples, see [9, Chapter 4]. We also note the fact that if the injective hull of R_R is a rational extension of R then $E(R)$ has a unique ring structure [9, Theorem 13.11]. We recommend [6] for a very interesting view of the early development of the subject.

Osofsky in 1964 gave an amazing example of a ring R with 32 elements for which no compatible ring structure could be put on E (see [11], or for a modern reconstruction see [9, Example 3.45]). In that same paper, Osofsky asked if whenever E has a ring structure compatible with the multiplication of R, is the ring E injective over itself. This question was asked as an open problem recently in [2], and was brought to the attention of the first author by Gary Birkenmeier.

This question naturally divides into two parts. Many of the examples constructed investigating this problem are finite and sometimes even algebras over fields. Because injective envelopes are created as submodules of dual spaces, which will remain finite dimensional, the injective envelopes of these examples will be finite. Therefore it is a compelling question indeed if there is a finite example.

In fact one may already exist without having been identified. Birkenmeier, Park, and Rizvi have made an extensive study of quotient rings in great generality, and in particular of the problem of putting a compatible ring structure on these objects. In a forthcoming series of long and informative papers they compute many examples of ring hulls. In fact, they give examples in which many different compatible ring structures can be put on E. These papers and preprints are highly recommended reading. See [2], [3], [4], and [5]. Also, for an accessible presentation of the basics of injective modules and injective hulls consult [1].

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For the case of infinite rings, we present a large class of examples showing that Osofsky's question has a negative answer. These examples are motivated by [10, Example 2.6].

2. The Construction

Throughout the paper we let I be an indexing set with infinite cardinality. We also take K to be a field. Let V be a vector space over K with K-dimension $|I|$. In other words $V_K \cong K^{(I)}$. Let B be a non-degenerate, symmetric bilinear form on V. This bilinear form will allow us to define a multiplication on V. We construct a ring $R = K \oplus V \oplus P$, where $P = Kx$ is a K-vector space of dimension 1 with basis vector x, by letting addition in the ring be component wise, and multiplication be given by

$$(k + v + p)(k' + v' + p') = kk' + (kv' + k'v) + (kp' + k'p + B(v, v')x).$$

Alternatively, R is isomorphic to the ring of 3×3 upper-triangular matrices of the form

$$\left\{ \begin{pmatrix} k & v & p \\ 0 & k & v \\ 0 & 0 & k \end{pmatrix} \middle| \, k \in K, \, v \in V, \, p \in P \right\}$$

where the entries along any fixed diagonal are constant. While in general upper-triangular matrix rings are non-commutative, this ring is clearly commutative. If we drop the hypothesis that B is symmetric we lose commutativity, but this construction still results in an associative ring with 1.

Another way to represent R is as a polynomial ring in $I + 1$ variables, modulo an ideal of relations. Fixing a basis $\{v_i\}_{i \in I}$ for V_K, we have

$$R \cong K[\{x_i\}_{i \in I}, x]/(x_i x, \, x^2, \, x_i x_j - B(v_i, v_j)x)$$

Definition 1. We denote the ring we constructed above by $Gr(V, B)$, since the ring is graded as a K-module. When there is no need to emphasize the choice of V and B in the construction, we denote this ring by R.

Remark 2. Notice that $\mathrm{rad}(R) = V \oplus P$, $\mathrm{rad}(R)^2 = P$, and $\mathrm{rad}(R)^3 = 0$. The equality $\mathrm{rad}(R)^2 = P$ follows from the fact that B is non-trivial.

Given a K-vector space A, we let $\widehat{A} = \mathrm{Hom}_K(A_K, K_K)$ denote the dual. Consider the set $\widehat{R} = \mathrm{Hom}_K(R, K)$. This set has a natural right R-module structure given by the rule $(fr)(r') = f(rr')$ for each $f \in \widehat{R}$ and $r, r' \in R$. As mentioned previously, the ring R is a graded K-module, with grading given by $R = K \oplus V \oplus P$. Similarly, $\widehat{R}$ is a graded K-module, with grading $\widehat{R} = \widehat{P} \oplus \widehat{V} \oplus \widehat{K}$ (the graded pieces reverse order). Let $\varphi \in \widehat{P}$ denote the identity projection $kx \mapsto k$. We can define an R-module homomorphism $\Phi : R_R \to \widehat{R}_R$ by sending $1 \mapsto \varphi$, and then $r \mapsto \varphi \cdot r$.

Proposition 3. *Under the notation above, the map $\Phi : R_R \to \widehat{R}_R$ is an R-module monomorphism.*

Proof. The fact that the map is an R-module homomorphism is clear since R_R is free. To show that Φ is a monomorphism, it suffices to show that given $r \in R$ with $r \neq 0$, the map $\varphi \cdot r$ is not identically zero. Equivalently, we need to show that there is some element $r' \in R$ for which $rr' \in P$, $rr' \neq 0$.

Write $r = k + v + p$, with $k \in K$, $v \in V$, and $p \in P$. If $k \neq 0$, just take $r' = x \in P$. If $k = 0 = v$, then take $r' = 1$. We now have only to consider the case when $k = 0$ but $v \neq 0$. Since B is a non-degenerate bilinear form, there exists some $v' \in V$ with $B(v, v') \neq 0$. In particular, we can take $r' = v'$ and then $rr' = vv' + pv' = B(v, v')x \neq 0$ with $rr' \in P$. □

Lemma 4. *The socle of $\widehat{R}_R$ is essential and simple, and equals $\widehat{K}$. In particular, $\Phi(R_R)$ is essential in $\widehat{R}_R$.*

Proof. Clearly $\widehat{K}$ is a simple R-module and K-module. An essential, simple submodule must equal the socle, so to prove the first claim it suffices that show that $\widehat{K}$ is essential in $\widehat{R}$. Fix a nonzero element $\widehat{r} \in \widehat{R}$, and write $\widehat{r} = \widehat{p} + \widehat{v} + \widehat{k}$ with $\widehat{k} \in \widehat{K}$, $\widehat{v} \in \widehat{V}$, and $\widehat{p} \in \widehat{P}$.

We want to find an element $r \in R$ for which $\widehat{r} \cdot r \in \widehat{K}$. If $\widehat{p} \neq 0$, then $r = x \in P$ suffices. If $\widehat{p} = \widehat{v} = 0$, then $r = 1$ works. Thus, we reduce to the case $\widehat{p} = 0$ and $\widehat{v} \neq 0$. Since $\widehat{v} \neq 0$, there is some $v \in V$ for which $\widehat{v}(v) \neq 0$. We claim that one may take $r = v$. In fact, given $r' \in R$ write it as $r' = k' + v' + p'$ (with these elements in the appropriate places). One computes that

$$(\widehat{v} \cdot r)(r') = \widehat{v}(rr') = \widehat{v}(vk' + vv') = \widehat{v}(v)k'$$

and hence $\widehat{v} \cdot v$ is a non-zero function in $\widehat{K}$.

Finally, notice that $\Phi(R_R) \cap \mathrm{Soc}(\widehat{R}_R) \neq 0$ since the socle is essential. However, the socle is also simple, and hence must live inside $\Phi(R_R)$. In particular, $\Phi(R_R)$ is essential in $\widehat{R}_R$. □

Lemma 5 ([8, Theorem 2, p. 247])**.** *If K is a field and I is an infinite set then $\dim_K(K^I) = |K|^{|I|} > |I|$.*

Lemma 6. *The module $\widehat{R}_R$ is injective, while R_R is not injective.*

Proof. The fact that $\widehat{R}_R$ is injective follows from two general principles. First, K_K is injective. Second, the dual into an injective module is always injective by [9, Injective Producing Lemma 3.5].

Since the module $\Phi(R_R) \cong R_R$ has $\widehat{R}_R$ as an essential extension, the rest of the lemma follows once we can establish $\Phi(R)$ is a proper submodule of $\widehat{R}$. This is an easy consequence of Lemma 5, since $\dim_K(R_K) = |I|$ while $\dim_K(\widehat{R}_K) = |K|^{|I|} > |I|$. □

All that remains in answering Osofsky's question is to define a multiplication on $\widehat{R}$ compatible with the R-module structure, since the ring R then embeds *as a ring* via Φ. Amazingly, this can almost be done arbitrarily! First, it is helpful to find the image of R in $\widehat{R}$.

Suppose we fix $v \in V$. Given $r' \in R$ write $r' = k' + v' + p'$ as before. We compute

$$(\varphi \cdot v)(r') = \varphi(vr') = \varphi(B(v, v')x) = B(v, v').$$

Since r' is arbitrary, we see that V embeds into $\widehat{V}$ via the map $v \mapsto B(v, -)$. On the other hand, clearly $\Phi(R) \supset \widehat{P} = K\varphi$, and $\Phi(R) \supset \widehat{K} = \mathrm{Soc}(\widehat{R}_R)$ by Lemma 4. Thus, $\widehat{R}_K = \Phi(R)_K \oplus V'_K$ where V' is any vector space complement to $\mathrm{Span}\{B(v, -) \in \widehat{V} \mid v \in V\}$ inside $\widehat{V}$.

Fix, once and for all, such a complement V', and also fix a basis $\mathcal{C}'$ for V'. The embedding of V into $\widehat{V}$, along with the R-module structure on $\widehat{V}$, allows one to partially define a bilinear form $\widehat{B}$ by "extending" B. In other words, given $v \in V$ and $\widehat{v} \in \widehat{V}$, we define $\widehat{B}(\widehat{v}, B(v,-)) = (\widehat{v} \cdot v)(1) = \widehat{v}(v)$. In particular, note that

$$\widehat{B}(\Phi(v_1), \Phi(v_2)) = \widehat{B}(B(v_1,-), B(v_2,-)) = B(v_1, v_2) \tag{1}$$

for all $v_1, v_2 \in V$ (which is why we say $\widehat{B}$ extends B). To make $\widehat{B}$ symmetric, we set $\widehat{B}(B(v,-), \widehat{v}) = \widehat{B}(\widehat{v}, B(v,-))$. Since a bilinear form is uniquely determined by its values on a basis, we can define $\widehat{B}(v_1, v_2) = \widehat{B}(v_2, v_1)$ arbitrarily, for each pair of vectors $v_1, v_2 \in \mathcal{C}'$. Fix such an extension $\widehat{B}$ of B.

Lemma 7. *In the notation above, the bilinear form $\widehat{B}$ is symmetric and non-degenerate.*

Proof. Symmetricity is clear. Given $0 \neq \widehat{v} \in \widehat{V}$, there is some $v \in V$ so that $\widehat{v}(v) \neq 0$. Thus

$$\widehat{B}(\widehat{v}, B(v,-))(1) = (\widehat{v} \cdot v)(1) = \widehat{v}(v) \neq 0.$$

Hence $\widehat{B}(\widehat{v}, \Phi(V)) \neq 0$, and in particular $\widehat{B}(\widehat{v}, \widehat{V}) \neq 0$. □

We are nearly ready to present our example. For notational ease, let the ring $S = Gr(\widehat{V}, \widehat{B})$ be constructed just as we constructed $R = Gr(V, B)$ as in Definition 1. The ring S is graded as a K-algebra, and this allows us to identify S with $\widehat{R}$ by identifying the graded pieces. We can embed R (as a set) into S via Φ. The form $\widehat{B}$ was defined so that the embedding $\Phi : R \hookrightarrow S$ is a unital ring homomorphism. (In other words, $\widehat{B}$ was defined exactly to make the embedding of R into S compatible with the ring structure on R; see Equation (1).) In particular, $S_R = \widehat{R}_R$.

Theorem 8. *In the notations above, S_R is the injective hull of R_R. However, S_S is not injective.*

Proof. We have $S_R = \widehat{R}_R$ is the injective hull of R_R by Lemma 6. Since S is a ring of the same form as R, S_S is not injective for exactly the same reason R_R is not self-injective, by Lemma 6. In other words, we can repeat the construction and define a ring $T = \widehat{S}$ containing (an embedding of) S properly, but for which T_S is the injective hull of (this embedding of) S. Hence S_S is not injective. □

Corollary 9. *There exists an infinite chain of rings $R_1 \subsetneq R_2 \subsetneq \ldots$ with $(R_{j+1})_{R_j}$ the injective hull of $(R_j)_{R_j}$ for each $j \in \mathbb{Z}_+$.*

Proof. Just repeat the construction of S from R, as above, over and over; noting that S is a ring of the same form as R. □

Notice that if we have an ascending chain of rings of the form $Gr(V_i, B_i)$, then the union is also a ring of the same form, since the union of non-degenerate forms is still non-degenerate. So the previous corollary not only provides a countable chain of counter-examples, but by a straightforward use of transfinite induction a chain of any ordinal length.

3. Compatible Multiplications

In the previous section, we embedded R into S via the monomorphism Φ of Proposition 3. In this section we are going to consider what happens if we make distinct choices for $\widehat{B}$, and so we have to be a little more careful with our construction.

First, to simplify notation we will identify V with its image inside $\widehat{V}$, and view B as a bilinear form on this subspace of $\widehat{V}$. Since $\widehat{R}$ has an R-module structure, we can naturally extend B to a symmetric form $\widetilde{B}$ defined on pairs $\widehat{v} \in \widehat{V}$ and $v \in V$, by the rule $\widetilde{B}(\widehat{v}, v) = (\widehat{v}v)(1)$. As before, we can further extend $\widetilde{B}$ to a non-degenerate bilinear form $\widehat{B}$ on all of $\widehat{V}$ by arbitrarily fixing $\widehat{B}(v_1, v_2)$ for pairs $v_1, v_2 \in \mathcal{C}'$, forcing $\widehat{B}$ to be symmetric if we need it to be.

Thus, any symmetric extension $\widehat{B}$ of B is determined by the choice of values for $\widehat{B}(v_1, v_2)$ with $v_1, v_2 \in \mathcal{C}'$. If B_1 and B_2 are two such extensions of B, let $S_1 = Gr(\widehat{V}, B_1)$ and $S_2 = Gr(\widehat{V}, B_2)$ be the rings constructed from these bilinear forms. We think of R as a subring of both S_1 and S_2, and thus these rings are R-algebras.

Proposition 10. *In the notations above, if λ is an R-algebra isomorphisms between S_1 and S_2, then $B_1 = B_2$.*

Proof. The embeddings of R into S_1 and S_2 are K-linear, and hence the isomorphism λ is also K-linear. We also know λ preserves radicals, and so given $\widehat{v} \in \widehat{V} \subset \mathrm{rad}(S_1)$ we have $\lambda(\widehat{v}) = \widehat{v}' + p$ for some $\widehat{v}' \in \widehat{V}$ and $p \in P$. Write $\cdot$ for multiplication in S_1 and $*$ for multiplication in S_2. For each $v \in V \subset R$ we compute

$$\lambda(\widehat{v} \cdot v) = \lambda(\widehat{v}) * v = (\widehat{v}' + p) * v = \widetilde{B}(\widehat{v}', v)x$$

where $P = Kx$. On the other hand

$$\lambda(\widehat{v} \cdot v) = \lambda(\widetilde{B}(\widehat{v}, v)x) = \widetilde{B}(\widehat{v}, v)\lambda(x) = \widetilde{B}(\widehat{v}, v)x$$

since $x \in P \subset R$. Therefore $\widetilde{B}(\widehat{v}, v) = \widetilde{B}(\widehat{v}', v)$ for all $v \in V$. In other words $\widehat{v}(v) = \widehat{v}'(v)$ for all $v \in V$, so $\widehat{v} = \widehat{v}'$ as functions. We have thus shown $\lambda(\widehat{v}) = \widehat{v} + p$ for some $p \in P$.

Given $v_1, v_2 \in \mathcal{C}'$ we find

$$\lambda(v_1 \cdot v_2) = \lambda(B_1(v_1, v_2)x) = B_1(v_1, v_2)\lambda(x) = B_1(v_1, v_2)x$$

but also

$$\lambda(v_1 \cdot v_2) = \lambda(v_1) * \lambda(v_2) = (v_1 + p_1) * (v_2 + p_2) = B_2(v_1, v_2)x.$$

Therefore $B_1(v_1, v_2) = B_2(v_1, v_2)$ for all $v_1, v_2 \in \mathcal{C}'$, whence $B_1 = B_2$. □

Theorem 11. *There are at least $2^{|K|^{|I|}}$ distinct R-algebra structures on the injective hull $E(R_R)$, such that under each such structure $E(R)$ is not self-injective.*

Proof. By Theorem 8 and Proposition 10, it suffices to show that there are $2^{|K|^{|I|}}$ distinct symmetric extensions $\widehat{B}$ of B. There are $|K|^{|I|}$ vectors in $\mathcal{C}'$, and there are $|K| \geqslant 2$ choices for the value of $\widehat{B}(v, v')$ for any given pair of vectors $v, v' \in \mathcal{C}'$. Since $\widehat{B}$ is uniquely determined by such choices we obtain $|K|^{|K|^{|I|}} = 2^{|K|^{|I|}}$ symmetric extensions. □

Remark 12. In [4], the authors construct an artinian ring whose injective envelope has distinct R-algebra structures. Our construction of course completely breaks down in that case because then we start with a QF ring.

In the introduction, we gave numerous examples from the literature where it is proven that any two ring structures on the injective hull of a module must be isomorphic (as rings). It turns out that we can do a little better than Theorem 11 by actually constructing ring structures on the injective hull of R which are non-isomorphic *as rings*.

Theorem 13. *Let R be as above. The injective hull of R has rings structures which are non-isomorphic as rings.*

Proof. Let B_1 be the extension of B satisfying $B_1(v_1, v_2) = 0$ for all $v_1, v_2 \in \mathcal{C}'$. Let B_2 be an extension of B which is not symmetric. Clearly S_1 is not isomorphic to S_2, since the latter ring is not even commutative. □

4. A Ring Characterization

At this point, it might be helpful to consider an explicit example. Our construction was motivated by the ring

$$R = K[x_1, x_2, \ldots]/(x_i x_j - \delta_{ij} x_1^2)$$

(where we identify x_i with its image modulo the relations). Playing with the relations, it is easy to prove that any monomial of degree 3 is zero in this ring, and the space of degree 2 monomials is generated by x_1^2. This ring can alternately be described in terms of a bilinear form. Set $V = \mathrm{Span}(x_1, x_2, \ldots)$, $P = Kx_1^2$, and let the bilinear form on V just be given by $B(x_i, x_j) = \delta_{ij}$. It is straightforward to check that $R \cong Gr(V, B)$.

A quotient of the formal power series ring $S = K[\![x_1, x_2, \ldots]\!]/I$ turns out to be the injective hull of R, as long as I is chosen correctly. One must make sure all power series which consist of monomials of degree more than 2 belong to I, the space of power series consisting of monomials of exact degree 2 is 1 dimensional and essential as an R-module, and finally that the multiplication is compatible with that on R. We leave it to the enthusiastic reader to work out an explicit choice for the ideal I satisfying these conditions.

The following may be of independent interest. Recall the following standard definition: A field is said to be *formally real* if -1 is not the sum of squares. For example $\mathbb{Q}$ (and in fact any extension of $\mathbb{Q}$ inside $\mathbb{R}$) is formally real. Surprisingly, a field is formally real if and only if it has a total ordering. We define $|k| = k$ if $k \geqslant 0$ and $|k| = -k$ otherwise.

Theorem 14. *Let K be a formally real field such that $|k|$ has a square-root for each $k \in K$. Let R_1 be a commutative ring with K-algebra structure $K \oplus M$, with $M^2 \neq 0$ but $M^3 = 0$, with $\mathrm{Soc}(R_1)$ simple and essential in R_1, and with $\dim_K(R_1)$ countable. Let R_2 be the ring $K[x_1, x_2, \ldots]$ in countably many variable with relations given by $x_i x_j = \delta_{ij} x_1^2$. The ring R_1 is isomorphic to R_2 if and only if all elements of R_1 whose squares are zero are contained in the socle of R_1.*

Proof. We first observe that R_2 has the property that all elements which square to zero are contained in the socle. In fact, if $r \in R_2$ and

$$0 = r^2 = \left(\sum_{i \in \mathbb{Z}_+} k_i x_i + c x_1^2 \right)^2 = \sum_{i \in \mathbb{Z}_+} k_i^2 x_1^2$$

then $\sum_{i \in Z_+} k_i^2 = 0$. But since K is formally real, we have $k_i = 0$ for all i, and thus $r = cx_1^2$ is in the socle of R_2. Therefore, if R_1 is isomorphic to R_2 then R_1 must also have the property that elements which square to zero are in the socle.

Conversely, suppose that the socle of R_1 contains the elements whose squares are zero. This implies $0 \neq M^2 \subseteq \mathrm{Soc}(R_1)$, and since the socle is simple $M^2 = \mathrm{Soc}(R_1)$. Let $\{y_1, y_2, \ldots\}$ be a K-basis for M/M^2 and for each i fix a lift $z_i \in M$ of y_i. Note that $z_i^2 \neq 0$ since the converse would imply $z_i \in \mathrm{Soc}(R_1) = M^2$.

We will inductively construct elements $w_i \in R_1$ such that $w_i^2 = w_j^2 \neq 0$, $w_i w_j = 0$ when $i \neq j$, and $\mathrm{Span}\{w_1 + M^2, \ldots, w_n + M^2\} = \mathrm{Span}\{y_1, \ldots, y_n\}$ for each n. In particular, note that this last condition implies that $\{w_i + M^2\}_{i \in \mathbb{Z}_+}$ is a K-basis for M/M^2 and the map $w_i \mapsto x_i$ induces an isomorphism from R_1 to R_2.

To start, we just set $w_1 = z_1$, and notice $z_1^2 \neq 0$. Suppose now that we have constructed $w_1, \ldots, w_n$ satisfying the necessary properties. Since M^2 is simple, for each $i \leqslant n$ we have $z_{n+1} w_i = \lambda_i w_1^2$ for some $\lambda_i \in K$. Putting $u_{n+1} = z_{n+1} - \sum_{i=1}^n \lambda_i w_i$, we see that $u_{n+1} w_i = 0$ for all $i \leqslant n$. Further, because $u_{n+1} + M^2$ has non-zero support for y_{n+1}, we know $u_{n+1} \notin M^2$. The simplicity of M^2 then yields

$$u_{n+1}^2 = \alpha w_1^2 \neq 0.$$

If α has a square-root, then the element $w_{n+1} = (1/\sqrt{\alpha}) u_{n+1}$ satisfies the induction conditions. If α does not have a square-root then $-\alpha$ does and we find

$$(u_{n+1} + \sqrt{-\alpha} w_1)^2 = u_{n+1}^2 + 2\sqrt{-\alpha} u_{n+1} w_1 - \alpha w_1^2 = \alpha w_1^2 - \alpha w_1^2 = 0.$$

Hence, $u_{n+1} + \sqrt{-\alpha} w_1 \in \mathrm{Soc}(R_1) = M^2$, a contradiction (since $u_{n+1} + \sqrt{-\alpha} w_1$ modulo M^2 has non-zero support for the element y_{n+1}). Thus, we can construct w_{n+1} satisfying the properties, and we arrive at our isomorphism. □

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