The Exchange Property for Modules and Rings

by

Pace Peterson Nielsen

B.S. (Brigham Young University, Utah) 2001

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

GRADUATE DIVISION
of the
UNIVERSITY of CALIFORNIA, BERKELEY

Committee in charge:

Professor Tsit Yuen Lam, Chair
Professor George Bergman
Professor Deborah Nolan

Spring 2006

The dissertation of Pace Peterson Nielsen is approved:

Chair Date

Date

Date

University of California, Berkeley

Spring 2006

The Exchange Property for Modules and Rings

Abstract

The Exchange Property for Modules and Rings

by

Pace Peterson Nielsen
Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Tsit Yuen Lam, Chair

The exchange property was introduced in 1964 by Crawley and Jónsson. Our work focuses on the problem they posed of when the finite exchange property implies the full exchange property. The methods of this work are two-fold. The first chapter presents a ring-theoretical construction showing that modules whose endomorphism rings are strongly π-regular or strongly clean have $\aleph_0$-exchange. These methods are further refined to show that modules with Dedekind-finite, regular endomorphism rings have full exchange.

The second chapter focuses on direct sum decompositions for modules. We generalize a theorem of Stock, which shows that projective modules with $\aleph_0$-exchange have full exchange. We further show that one may "mod out" by the radical, when considering the exchange property for projective modules.

Professor Tsit Yuen Lam
Dissertation Committee Chair

To my wife, Sarah, for her patience and love.

Contents

Acknowledgements

I want to thank Tsit Yuen Lam and George Bergman, for carefully reading my work and suggesting many interesting avenues of future research. I also wish to thank Alex Dugas for reading early manuscripts and for his kind remarks.

1 Notations and Conventions

By *ring* we will mean an associative ring with 1. By *module* we will mean a unital module over a ring, usually with the ring not made explicit and acting on the right. Let M be a module and I an indexing set. We write $M^{(I)}$ for the direct sum of I copies of M, and M^I for the direct product of I copies. Endomorphisms of right modules will be written on the left. If a submodule $N \subseteq M$ is a direct summand we write $N \subseteq^{\oplus} M$. Some introductory texts on rings and modules include [7] and [8].

Suppose we have two modules $N \subseteq M$. We say N is *essential* in M if for every non-zero submodule $A \subseteq M$ we have $N \cap A \neq (0)$. Dually, we say N is *small* in M if the equation $M = K + N$ implies $K = M$. In the first case we write $N \subseteq_e M$, and in the latter case we write $N \subseteq_s M$.

A submodule $N \subset M$ is said to be *maximal* if $N \neq M$ and $N \subset X \subset M$ with $X \neq N$ implies $X = M$. The *radical* of M is defined to be the intersection of all maximal submodules of M, written as $\mathrm{rad}(M)$. If there are no maximal submodules then this is the intersection of the empty family and we put $\mathrm{rad}(M) = M$. It is well known that $\mathrm{rad}(M)$ is also equal to the union of all small submodules of M.

Let R be a ring. We write $J(R)$ for the Jacobson radical of R. In other words $J(R)$ is the intersection of all the maximal left (or right) ideals of R.

In the Introduction and throughout Chapter 1, all modules will be right k-modules, for an arbitrary ring k, unless specified otherwise. We will often write $E = \mathrm{End}(M_k)$ and we let R be an arbitrary ring (possibly not associated to any module). In Chapter 2 we depart from this convention and treat all modules as right R-modules. This is due to the fact that Chapter 2 focuses on projective modules, and the ring which acts on such a module plays a central role.

2 Introduction

Introduced in 1964 by Crawley and Jónsson [3] for general algebras, the exchange property has since become an important module-theoretic and ring-theoretic tool. The definition is as follows:

Definition 2.1. Let $\aleph > 0$ be a cardinal, let k be a ring, and let M_k be a right k-module. We say that M has the $\aleph$-*exchange property* if whenever we have right k-module decompositions, $A = M \oplus N = \bigoplus_{i \in I} A_i$, for some indexing set I with $|I| \leqslant \aleph$, then there are submodules $A_i' \subseteq A_i$ with $A = M \oplus \left(\bigoplus_{i \in I} A_i'\right)$.

We say that M has *finite exchange* if M has $\aleph$-exchange for all finite cardinals. We say M has *full exchange* if M has $\aleph$-exchange for all cardinals.

Example 2.2. Let k be a division ring and let M be a vector space over k, contained in another vector space A. Suppose further that A has a direct sum decomposition $A = \bigoplus_{i \in I} A_i$. It is straightforward to check that M has full exchange by extending a basis for M to a basis for A, by using only vectors in the submodules A_i.

In the definition of $\aleph$-exchange, note that due to the modularity law we have $A_i' \subseteq^{\oplus} A_i$ for each $i \in I$. Also note that the exchange property passes to isomorphic modules. If M has $\aleph$-exchange then M has $\aleph'$-exchange for all $\aleph' \leqslant \aleph$. The exchange property behaves well in terms of direct summands and finite direct sums. To prove this we first need the following result:

Lemma 2.3 ([3, p. 812])**.** *If M has $|I|$-exchange and $A = M \oplus N \oplus X = \left(\bigoplus_{i \in I} A_i\right) \oplus X$ then there are submodules $A_i' \subseteq A_i$ with $A = M \oplus \left(\bigoplus_{i \in I} A_i'\right) \oplus X$.*

Proof. Let $p : A \to M \oplus N$ be the natural projection from A onto $M \oplus N$, with kernel X. Fix $B_i = p(A_i)$. Since $\ker(p) = X$ we have $p|_{A_i} : A_i \to B_i$ is an isomorphism for each $i \in I$, and $M \oplus N = \bigoplus_{i \in I} B_i$. By the $|I|$-exchange property, there exist $B_i' \subseteq^{\oplus} B_i$ for each $i \in I$, with $M \oplus N = M \oplus \left(\bigoplus_{i \in I} B_i'\right)$. Now set

$A'_i = p|_{A_i}^{-1}(B'_i)$, and so $\left(\bigoplus_{i\in I} A'_i\right) \oplus X = \left(\bigoplus_{i\in I} B'_i\right) \oplus X$. Then it is clear that

$$A = M \oplus N \oplus X = M \oplus \left(\bigoplus_{i\in I} B'_i\right) \oplus X = M \oplus \left(\bigoplus_{i\in I} A'_i\right) \oplus X.$$

□

Lemma 2.4. *If M and M' have the $\aleph$-exchange property, then $M \oplus M'$ has the $\aleph$-exchange property. Further, if $P \subseteq^{\oplus} M$ then P also has the $\aleph$-exchange property.*

Proof. We identify M and M' with their images in the (outer) direct sum $M \oplus M'$, and hence consider this as an inner direct sum. Suppose $A = (M \oplus M') \oplus N = \bigoplus_{i\in I} A_i$ with $|I| \leqslant \aleph$. Then, using the $\aleph$-exchange property on M, we have submodules $A'_i \subseteq A_i$ with $A = M \oplus \left(\bigoplus_{i\in I} A'_i\right)$. But then, using the $\aleph$-exchange property on M', and Lemma 2.3 with $X = M$, we have submodules $A''_i \subseteq A'_i$ with

$$A = M \oplus M' \oplus \left(\bigoplus_{i\in I} A''_i\right).$$

For the second half of the lemma, write $M = P \oplus Q$, and suppose $A = P \oplus N = \bigoplus_{i\in I} A_i$ with $|I| \leqslant \aleph$. We may assume, without loss of generality, $Q \cap A = (0)$. Now

$$A \oplus Q = M \oplus N = Q \oplus \left(\bigoplus_{i\in I} A_i\right)$$

and so, using the $\aleph$-exchange property of M, there are submodules $A'_i \subseteq A_i$, and $Q' \subseteq Q$, with

$$A \oplus Q = M \oplus N = M \oplus Q' \oplus \left(\bigoplus_{i\in I} A'_i\right) = P \oplus Q \oplus Q' \oplus \left(\bigoplus_{i\in I} A'_i\right).$$

Consequently we see $Q' = 0$. Since $P \oplus \left(\bigoplus_{i\in I} A'_i\right) \subseteq A$, the above string of equalities implies $P \oplus \left(\bigoplus_{i\in I} A'_i\right) = A$. □

Crawley and Jónsson asked the following fundamental question that is still open:

Question 2.5. If a module has 2-exchange, does it have full exchange?

It turns out that 2-exchange is equivalent to finite exchange. We will give a straightforward proof in a later section. The question of whether finite exchange implies full exchange, or even $\aleph_0$-exchange, can be answered affirmatively for large classes of modules. One nice way to find such classes is by expressing the $\aleph$-exchange property in terms of equations in the endomorphism ring of the module. For this we need the following notion:

Definition 2.6. Let M be a module and set $E = \mathrm{End}(M)$. Fix a family $\{x_i\}_{i\in I}$ of elements $x_i \in E$ (not necessarily distinct). We say that this family is *summable* if for each $m \in M$ the set $F_m = \{i \in I \mid x_i(m) \neq 0\}$ is finite. Thus, in this situation we have a well-defined element $\sum_{i\in I} x_i \in E$ with action

$$\left(\sum_{i\in I} x_i\right)(m) = \sum_{i\in F_m} x_i(m).$$

With this notion in hand, we are ready to prove the following:

Proposition 2.7 ([23, Proposition 3])**.** *Let M be a module. For any given cardinal, $\aleph$, the following are equivalent:*

(1) The module M has the $\aleph$-exchange property.

(2) If we have

$$A = M \oplus N = \bigoplus_{i\in I} A_i$$

with $A_i \cong M$ for all $i \in I$, $|I| \leqslant \aleph$, then there are submodules $A_i' \subseteq A_i$ such that

$$A = M \oplus \bigoplus_{i\in I} A_i'.$$

(3) For every summable family $\{x_i\}_{i\in I}$ of elements in $E = \mathrm{End}(M)$, where the sum is $\sum_{i\in I} x_i = 1$, and $|I| \leqslant \aleph$, there are orthogonal idempotents $e_i \in Ex_i$ with $\sum_{i\in I} e_i = 1$.

Proof. Since this proposition is used repeatedly, we include the proof for completeness.

(1) $\Rightarrow$ (2) : Tautology.

(2) $\Rightarrow$ (3) : Let I be an indexing set with $|I| \leqslant \aleph$. Let $\{x_i\}_{i \in I}$ be a summable family in E, summing to 1. Put $A = \bigoplus_{i \in I} A_i$ with $A_i = M$ for each $i \in I$. Also, put $M' = \{a \in A \,|\, a = (x_i(m))_{i \in I}$ for some $m \in M\}$. Define $x : M \to M'$ by $m \mapsto (x_i(m))_{i \in I}$ and define $y : A \to M$ via $(a_i)_{i \in I} \mapsto \sum_{i \in I} a_i$. We see that the map x is an isomorphism, and the inclusion $M' \subseteq A$ splits via $x \circ y$. So we have

$$A = M' \oplus N = \bigoplus_{i \in I} A_i$$

and hence by (2) there are submodules $A_i' \subseteq^{\oplus} A_i$, for each $i \in I$, such that

$$A = M' \oplus \left(\bigoplus_{i \in I} A_i' \right).$$

Write $A_i = A_i' \oplus A_i''$, $A' = \bigoplus_{i \in I} A_i'$, and $A'' = \bigoplus_{i \in I} A_i''$. Let $\widetilde{\varphi}$ be the projection from A to A'' with kernel A', and put $\varphi = \widetilde{\varphi}|_{M'} : M' \to A''$, which is an isomorphism. Let $p_i : A'' \to A_i''$ be the natural projections, for each $i \in I$. Set $e_i = y\varphi^{-1} p_i \varphi x$ for each $i \in I$. We see that

$$e_i e_j = y\varphi^{-1} p_i \varphi x y \varphi^{-1} p_j \varphi x = y\varphi^{-1} p_i p_j \varphi x = \delta_{i,j} e_i.$$

Therefore, these are orthogonal idempotents. Further, letting $z_i : A_i \to A_i''$ be the natural projection, with kernel A_i', we have $p_i \varphi x = z_i x_i$, and so $e_i \in E x_i$. One easily checks that the family $\{e_i\}_{i \in I}$ is summable, and $\sum_{i \in I} e_i = 1$.

(3) $\Rightarrow$ (1) : Suppose we have

$$A = M \oplus N = \bigoplus_{i \in I} A_i$$

with $|I| \leqslant \aleph$. Let $p : A \to M$ and $\pi_i : A \to A_i$ be the natural projections corresponding to the respective decomposition. Setting $x_i = p\pi_i|_M$ for each $i \in I$, we see $\{x_i\}_{i \in I}$ is a summable family, and sums to 1. By (3) we have orthogonal idempotents $e_i \in Ex_i$, for each $i \in I$, which are summable and sum to 1. Fix $r_i \in E$ with $e_i = r_i x_i$.

Set $\varphi_i = e_i r_i p \pi_i : A \to M$, for each $i \in I$. We see that $\varphi_i|_M = e_i$ and hence $\varphi_i \varphi_j = \delta_{i,j} \varphi_i$. Next notice that $\{\varphi_i\}_{i \in I}$ is summable, so put $\varphi = \sum_{i \in I} \varphi_i$. We see that φ is an idempotent, since it is a sum of orthogonal idempotents. Now

$$\varphi|_M = \sum_{i \in I} \varphi_i|_M = \sum_{i \in I} e_i = 1.$$

One easily checks that $a \in \ker(\varphi)$ if and only if $\pi_i(a) \in \ker(\varphi_i)$ for each $i \in I$. So

$$A = \operatorname{im}(\varphi) \oplus \ker(\varphi) = M \oplus \ker(\varphi) = M \oplus \left(\bigoplus_{i \in I} (A_i \cap \ker(\varphi_i)) \right).$$

□

Corollary 2.8 (cf. [13, Theorem 2.1])**.** *Let M be a module. The following are equivalent:*

(1) The module M has the 2-exchange property.

(2) Given an equation $x_1 + x_2 = 1$ in $E = \operatorname{End}(M)$, there are (orthogonal) idempotents $e_1 \in Rx_1$ and $e_2 \in Rx_2$ with $e_1 + e_2 = 1$.

Definition 2.9. Following Warfield [22], we say a ring R is an *exchange ring* when the right R-module R_R has the 2-exchange property.

From the previous corollary, since $\operatorname{End}(R_R) \cong R$, we see that saying R is an exchange ring is equivalent to saying that for each equation $x_1 + x_2 = 1$ we can find idempotents $e_i \in Rx_i$ with $e_1 + e_2 = 1$. The notion of an exchange ring is a left-right symmetric notion, as proved in [13] and [22]; that is, R_R has 2-exchange if and only if ${}_RR$ does also. If $J(R)$ is the Jacobson radical of R, Warfield [22] proved that R is an exchange ring if and only if $R/J(R)$ is an exchange ring and idempotents lift modulo $J(R)$. Nicholson [13] proved that a ring is an exchange ring if and only if idempotents lift modulo all left (respectively right) ideals.

Remark 2.10 ([13, Proposition 1.9])**.** Let R be an exchange ring. Given $x \in R$, we know that there is an idempotent $e_r \in Rrx$ with $1 - e_r \in R(1 - rx)$, for each $r \in R$. Suppose x is such that we can take $e_r = 0$ for all $r \in R$. Then $(1 - rx)$ is left invertible for all $r \in R$, and hence $x \in J(R)$. From the contrapositive of what we've shown, if I is a (left) ideal of R with $I \not\subseteq J(R)$ then there is some non-zero idempotent $e \in I$. Conversely, for any ring R the only idempotent in $J(R)$ is 0. Thus, *if R is an exchange ring then $J(R)$ is the largest (left) ideal of R without a non-zero idempotent.*

Example 2.11. A ring R is *(von Neumann) regular* if for each $x \in R$ there exists $y \in R$ so that $xyx = x$. A ring R is *unit-regular* if for each $x \in R$ there is a unit $u \in U(R)$ so that $xux = x$. A ring R is *strongly-regular* if for each $x \in R$ there exists $y \in R$ so that $x = x^2 y$. These definitions are standard, and the following are well known, non-reversible implications, found in [8] as exercises:

$$\text{strongly-regular} \;\Rightarrow\; \text{unit-regular} \;\Rightarrow\; \text{Dedekind-finite and regular} \;\Rightarrow\; \text{regular}.$$

Idempotents in regular rings abound (for example, for x and y as in the definition of regularity, both xy and yx are idempotents). All regular rings are exchange rings. In fact, given $x \in R$, fix y so that $xyx = x$. Then set $f = yx$ which is an idempotent such that $x = xf$, and put $e = f + (1 - f)x \in Rx$. One can check that e is also an idempotent and $(1 - e) = (1 - f)(1 - x) \in R(1 - x)$.

A ring is *π-regular* if for each $x \in R$ there is some $y \in R$ and $n \in \mathbb{Z}_+$ so that $x^n y x^n = x^n$. In other words, some power of x is regular. The above proof, slightly modified, shows that π-regular rings are exchange rings.

A ring is *strongly π-regular* if for each $x \in R$ the chain $xR \supseteq x^2R \supseteq x^3R \supseteq \ldots$ stabilizes. As proved in [4], this is a left-right symmetric notion. Further, strongly π-regular rings are π-regular, and hence exchange rings.

Proposition 2.7 gives us two different avenues which we can follow to explore the exchange property. We can either focus on the module-theoretic properties of direct sums of M, or we can study rings where elements which sum to 1 can be left multiplied into orthogonal idempotents that sum to 1. We will follow both of these paths, with very different results.

Chapter 1

From a Ring-theoretic Point of View

The material in this chapter is taken from my papers [15] and [16], reproduced with permission. A few small changes have been made to make the presentation accessible to a wider audience, and to include a few proofs and examples that were not included in the previously published versions.

3 $\aleph$-Exchange Rings

Using Proposition 2.7, we expressed the 2-exchange property entirely in terms of ring equations in Corollary 2.8, which then allowed us to define exchange rings. The $\aleph$-exchange property, however, required a notion of summability. When the sums are infinite this is not expressible in terms of properties of the ring without an endomorphism ring structure. However, we can overcome this obstacle if we introduce a ring topology.

Definition 3.1. Let R be a ring. A *ring topology* is a topology on the set R, for which addition and multiplication are continuous maps from $R \times R \to R$ (where the set on the left is given the product topology). A nice introduction to such topologies is [1]. A *(left) linear Hausdorff topology* on R is a ring topology on R with a basis of neighborhoods of zero, $\mathfrak{U}$, consisting of left ideals, with $\bigcap_{U \in \mathfrak{U}} U = (0)$.

Suppose R has a given linear Hausdorff topology. Let $\{x_i\}_{i \in I}$ be a collection of (not necessarily distinct) elements of R. We say this family is *summable* to $x \in R$ if for each $U \in \mathfrak{U}$ there exists a finite set $F_U \subseteq I$ (depending on U) such that $x - \sum_{i \in F} x_i \in U$ for each finite set $F \supseteq F_U$. Equivalently, $x - \sum_{i \in F_U} x_i \in U$ and $x_j \in U$ for $j \notin F_U$.

Again, let R be a ring with a linear Hausdorff topology. Let $\mathfrak{U}$ be the guaranteed basis of neighborhoods of zero. We define limits as follows. Given any directed set I, and any I-indexed family of elements $\{x_i\}_{i \in I}$, we put $\lim_{i \in I} x_i = x \in R$ if for each $U \in \mathfrak{U}$ there is some $i_U \in I$ so that for each $j \geqslant i_U$ we have $x_j - x \in U$. Since the topology is Hausdorff a limit (if it exists) is unique. A family $\{x_i\}_{i \in I}$ of elements of R is said to be *Cauchy* if for each $U \in \mathfrak{U}$ there is some $i_U \in I$ so that for each $j, \ell \geqslant i_U$ we have $x_j - x_\ell \in U$. (This is the usual definition of Cauchy in any ring topology.) As usual, every convergent sequence in a ring with a linear Hausdorff topology is Cauchy. A topology is *complete* if every Cauchy sequence converges.

There are quite a number of nice properties that a linear Hausdorff topology has.

Lemma 3.2. *Let R have a linear Hausdorff topology. Then:*

(1) The ideal $J(R)$ is closed (i.e. the complement is open) if R is an exchange ring.

(2) The quotient topology on the ring $R/J(R)$ is a linear Hausdorff topology if R is an exchange ring.

(3) Let $\{x_i\}_{i\in I}$ be a family in R. Let $\mathbb{F}(I)$ be the set of finite subsets of I, ordered by inclusion, and directed by union. The family $\{x_i\}_{i\in I}$ sums to $x \in R$ if and only if $\lim_{F\in\mathbb{F}(I)} \sum_{i\in F} x_i = x$.

Proof. **Part (1):** For endomorphism rings, this is [12, Lemma 11], although the proof works in this more general context. We give the proof for completeness. By Remark 2.10, $J = J(R)$ is the largest left ideal without non-zero idempotents. Let $\overline{J}$ be the closure of J, and take $e^2 = e \in \overline{J}$. Note that $\overline{J}$ is a left ideal, so if we can prove $e = 0$ then we must have $\overline{J} = J$. Given $U \in \mathfrak{U}$, the set $J + U$ is closed because its complement is a union of cosets of U, which are each open since U is. Therefore $e \in J + U$ for each $U \in \mathfrak{U}$.

Fix some $U \in \mathfrak{U}$. We know $0 \cdot e = 0$, so since multiplication is continuous in the first variable, there is some open left ideal $V \in \mathfrak{U}$ satisfying $Ve \subseteq U$. By the last sentence of the previous paragraph, we may write $e = j + v$ for $j \in J$ and $v \in V$. We have

$$(1 - j)e = (1 - e + v)e = ve \in Ve \subseteq U,$$

whence $e = (1 - j)^{-1}(1 - j)e \in U$. Since U is arbitrary, $e \in \bigcap_{U\in\mathfrak{U}} U = (0)$.

Part (2): As a general fact, we note that if a set, S, is closed in a linear topology then $\bigcap_{U\in\mathfrak{U}}(S + U) = S$. The proof is straightforward, once one notices that $S + U$ is closed, as in part (1) above. In the quotient topology on $R/J(R)$, we have a basis of neighborhoods of zero given by $\{U + J(R)\}_{U \in \mathfrak{U}}$. But since U is a left ideal, so is $U + J(R)$. Further, by part (a) we know $J(R)$ is closed and so

$$\bigcap_{U \in \mathfrak{U}} (U + J(R)) = J(R)$$

which is the zero ideal in $R/J(R)$. Therefore, the topology is Hausdorff.

Part (3): Direct application of the definitions, left to the reader. $\square$

Definition 3.3. Let M be a right k-module, and set $E = \mathrm{End}(M_k)$. There exists a natural linear Hausdorff topology on E, where a basis of neighborhoods of zero is given by annihilators in E of finite subsets of M. This topology on E is called the *finite topology.* In this topology E is complete. Further, with this topology on $\mathrm{End}(M)$, our new notion of summability agrees with the one we gave in Definition 2.6 for endomorphisms.

Proposition 2.7 then motivates the following definitions:

Definition 3.4. Let R be a ring with a linear Hausdorff topology, and let $\aleph$ be a cardinal. We say R is an $\aleph$-*exchange (topological) ring* if for each indexing set I with $|I| \leqslant \aleph$, and for each summable family $\{x_i\}_{i \in I}$, where $\sum_{i \in I} x_i = 1$, there are summable, orthogonal idempotents $e_i \in Rx_i$ with $\sum_{i \in I} e_i = 1$. If R is an $\aleph$-exchange ring for all $\aleph$ then we say R is a *full exchange (topological) ring.* Our definitions vary slightly from those given in [12], but after personal communications with the authors of that paper it became clear that the definitions given here are what the authors meant to communicate.

Note that the definition of "exchange ring" has no reference to any topology, but an exchange ring with a linear Hausdorff topology is a 2-exchange topological ring. When the context is clear, we will drop the word "topological." The reader should not confuse an exchange ring with a full exchange ring (the latter requiring a topology, the former only requiring the exchange property on R_R). Note also that a module M has the $\aleph$-exchange property if and only if $\mathrm{End}(M)$ (in the finite topology) is an $\aleph$-exchange topological ring, by Proposition 2.7.

In the definition of an $\aleph$-exchange ring, we need the family $\{e_i\}_{i\in I}$ to be summable. In an endomorphism ring, this is automatic as long as each e_i is a left multiple of x_i. Likewise, we want to put a condition on a topological ring which will ensure that such a family of idempotents can be summed.

Definition 3.5. Let R be a ring with a linear Hausdorff topology. We say a summable family $\{x_i\}_{i\in I}$ in R is *left multiple summable* if for any choice of $r_i \in R$, for each $i \in I$, the new family $\{r_i x_i\}_{i\in I}$ is still summable. If every summable family in R is left multiple summable we say that R has a *left multiple summable topology.*

Remark 3.6. Let $\{x_i\}_{i\in I}$ be a summable family, summing to x, in a ring with a linear Hausdorff topology. For any $r \in R$ the families $\{rx_i\}_{i\in I}$ and $\{x_i r\}_{i\in I}$ are automatically summable, summing to rx and xr respectively. The proof is straightforward and left to the reader.

Although the motivation for defining "left multiple summability" is clear, the definition still may seem unnatural. There is a much more natural condition on summable families which is equivalent. It was pointed out earlier that convergent limits are always Cauchy. It turns out that summable families of elements satisfy a "Cauchy-like" condition (sometimes even called "Cauchy," as in [12]; see also [2]). Because this condition isn't the usual Cauchy condition, we give it a new name.

Definition 3.7. Let R be a ring with a linear Hausdorff topology. Let $\{x_i\}_{i\in I}$ be a family of elements of R. We say this family is Σ-*Cauchy* (short for summable-Cauchy) if for each $U \in \mathfrak{U}$ there exists a finite set $F_U \subseteq I$ so that $x_i \in U$ when $i \notin F_U$. One may check that every summable family is Σ-Cauchy. If the converse holds (that is, every Σ-Cauchy family is summable) we say R is Σ-*complete.*

We now have the following surprising result (based on an idea of George Bergman):

Proposition 3.8. *Let R be a ring with a linear Hausdorff topology. The topology is Σ-complete if and only if it is left multiple summable.*

Proof. The forward direction is easy since $\mathfrak{U}$ is made up of left ideals, so left multiples of Σ-Cauchy families are still Σ-Cauchy. For the other direction let $X = \{x_i\}_{i\in I}$ be a Σ-Cauchy family. For each $U \in \mathfrak{U}$ fix a finite set $F_U \subseteq I$ so that $x_i \in U$ for each $i \notin F_U$. Let I' be an copy of I, disjoint from I. For each subset $A \subseteq I$, let $A' \subseteq I'$ be the corresponding subset under the identification of I with I'. Let $T = \{x_i, -x_{i'}\}_{i\in I, i'\in I'}$, so T is the disjoint union of X and $-X$. Then we claim that T is summable and sums to 0. In fact, for each finite set $F \supseteq F_U \cup F'_U$ we see that $\sum_{i\in F\cap I} x_i + \sum_{i'\in F\cap I'} -x_{i'} \in U$. Thus T satisfies the definition of summability.

By left multiple summability, we can multiply the elements of $T \cap X$ by 1 and the elements of $T \cap (-X)$ by 0, and we still have a summable family. But this new family is just X (along with extra copies of 0), which proves the claim. $\square$

We end this section by investigating the relationship between completeness and Σ-completeness.

Proposition 3.9. *Let R be a ring with a linear Hausdorff topology. If R is complete then it is Σ-complete.*

Proof. Let $\{x_i\}_{i\in I}$ be a Σ-Cauchy family. Fix a basis $\mathfrak{U}$ of left ideal neighborhoods of zero. For each $U \in \mathfrak{U}$ fix a finite set $F_U \subseteq I$ so that $x_i \in U$ whenever $i \notin F_U$.

We want to show that our family is summable. Let $\mathbb{F}(I)$ be the set of finite subsets of I, ordered by inclusion. By Lemma 3.2 is suffices to show that $\lim_{F\in\mathbb{F}(I)} \sum_{i\in F} x_i$ exists. By completeness, it suffices to show that the family $\{\sum_{i\in F} x_i\}_{F\in\mathbb{F}(I)}$ is Cauchy. But for $F, F' \in \mathbb{F}(I)$ with $F, F' \supseteq F_U$ we have

$$\sum_{i\in F} x_i - \sum_{i\in F'} x_i = \sum_{i\in F\setminus F_U} x_i - \sum_{i\in F'\setminus F_U} x_i \in U$$

by definition of F_U. Hence, the family is Cauchy, and the proposition follows. $\square$

The following example, also kindly suggested by G. Bergman, shows that the converse of Proposition 3.9 does not hold.

Example 3.10. Let k be a ring and let ω_1 be the first uncountable ordinal. Let R be the subring of k^{ω_1} consisting of sequences that eventually become constant. For each ordinal $\kappa < \omega_1$, let U_κ be the set of sequences that are 0 before the κth coordinate. Clearly $\bigcap_{\kappa<\omega_1} U_\kappa = (0)$, and each U_κ is a left (and right) ideal. Topologizing R with respect to these open sets gives R a linear Hausdorff topology.

For each $\kappa < \omega_1$ define x^κ by deciding its ith coordinate by the rule

$$x_i^\kappa = \begin{cases} 1 & \text{if } i < \kappa \text{ and } i \text{ is a limit ordinal} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\lim_{\kappa\to\omega_1} x^\kappa$ doesn't exist, since (pointwise) the sequence converges to a series that never becomes constant. However, the family $\{x^\kappa\}_{\kappa<\omega_1}$ is Cauchy. Therefore R is not complete in the given topology.

Now, let $\{x_i\}_{i\in I}$ be a Σ-Cauchy family. For each $\kappa < \omega_1$ there is a finite set $F_\kappa \subseteq I$ so that $x_i \in U_\kappa$ for $i \notin F_\kappa$. We shall prove that all but finitely many x_i are 0, which will immediately imply summability, and hence Σ-completeness of R. Suppose, by way of contradiction, that (after renumbering the indices) we have $x_i \neq 0$ for all $i < \omega_0$. In particular, let κ_i be the first non-zero entry of x_i. Since there are only countably many of these ordinals, we have $\lambda = \sup_{i<\omega_0} \kappa_i < \omega_1$. But then $x_i \notin U_\lambda$ for each $i < \omega_0$, a contradiction. As stated, this proves Σ-completeness for R.

4 From 2-Exchange to Finite Exchange

Let M_k be a right k-module, and let $E = \mathrm{End}(M_k)$. We saw above that for $n \in \mathbb{N}$, the n-exchange property for M is equivalent to the statement that for every equation

$$x_1 + \cdots + x_n = 1$$

over E we can find orthogonal idempotents $e_i \in Ex_i$ with

$$e_1 + \cdots + e_n = 1.$$

The idea is to use the 2-exchange property, and find an inductive process that will yield n-exchange. The important case to consider here is going from 2-exchange to 3-exchange. Before we begin, we first need a nice equivalence relation on idempotents.

Definition 4.1. Let R be a ring and let $e, e' \in R$ be idempotents. We say that these idempotents are *left associate* if $e'e = e'$ and $ee' = e$. We write $e' \sim_\ell e$ for this equivalence relation. *Right associate* idempotents are defined similarly, and we write $e' \sim_r e$. These definitions are equivalent to left and right associateness in the usual sense, due to the following lemma.

Lemma 4.2 (cf. [16, Lemma 2.2])**.** *Let R be a ring, and let $e', e \in R$ be idempotents. The following are equivalent:*

(1) $e' \sim_\ell e$.

(2) $Re' = Re$.

(3) $e' = e + (1-e)re$ *for some* $r \in R$.

(4) $e' = ue$ *for some* $u \in U(R)$.

(5) $e' = ue$ *for some* $u \in U(R)$*, with* $u(1-e) = (1-e)$.

(6) $e' = ue$ *for some* $u \in U(R)$*, with* $u = 1 + (1-e)re$ *for some* $r \in R$.

(7) $(1-e') \sim_r (1-e)$.

Furthermore, if $R = \mathrm{End}(M_k)$ *for some module* M_k*, then the following properties are also equivalent to the ones above:*

(8) $\ker(e') = \ker(e)$.

(9) $(1-e')M = (1-e)M$.

Proof. The implications (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (2) are clear.

(1) $\Leftrightarrow$ (7): The equations $e'e = e'$ and $ee' = e$ are equivalent to the equations $(1-e')(1-e) = (1-e)$ and $(1-e)(1-e') = (1-e')$ (respectively).

(2) $\Rightarrow$ (1): Since $Re' = Re$ we have $e' = re$ for some $r \in R$. In particular $e'e = ree = re = e'$. Similarly, $ee' = e$.

(1) $\Rightarrow$ (3): We know every element of R, in particular e', can be written in the form $e' = er_1e + (1-e)r_2e + er_3(1-e) + (1-e)r_4(1-e)$. But since $e'e = e'$, we can take $r_3 = r_4 = 0$. Also, since $ee' = e$, we find $er_1e = e$, and so we may take $r_1 = 1$. Therefore $e' = e + (1-e)r_2e$ as claimed.

(3) $\Rightarrow$ (6): Set $u = 1 + (1-e)re$. It is clear that $e' = ue$, and u is a unit with inverse $u^{-1} = 1 - (1-e)re$.

For the furthermore statement, it is easy to see (2) $\Rightarrow$ (8) $\Leftrightarrow$ (9) $\Rightarrow$ (7). $\square$

The characterization in the Lemma we are most interested in is (5). However, it should be pointed out that using the unit in item (6) one can arrive at more equations, such as $e'u = e'$. Two idempotents e' and e are said to be *isomorphic* if $Re' \cong Re$ (or equivalently, if $e'R \cong eR$). Thus, left (or right) associate idempotents are isomorphic. On the other hand, if two idempotents are both left and right associate we have $e' = e'e = e$.

Remark 4.3. We can rephrase properties (3) and (6) of Lemma 4.2. The set of idempotents left associate to e is precisely $e + (1-e)Re$, and the set of units u such that ue is still an idempotent is $1 + (1-e)Re$.

We are now ready to create the inductive process that goes from 2-exchange to 3-exchange, and a little more.

Lemma 4.4. *Let R be an exchange ring, and let $x_1 + x_2 + x_3 = 1$ be an equation in R. If x_1 is an idempotent, then there are orthogonal idempotents $e_1 \in Rx_1$, $e_2 \in Rx_2$, and $e_3 \in Rx_3$, such that $e_1 + e_2 + e_3 = 1$ and $e_1 \sim_\ell x_1$.*

Proof. Let $f = 1 - x_1$, and multiply by f on the left and right of $x_1 + x_2 + x_3 = 1$ to obtain $fx_2f + fx_3f = f$. Since corner rings of exchange rings are exchange rings by Lemma 2.4, fRf is an exchange ring. Hence, there are orthogonal idempotents $f_2 \in fRf(fx_2f)$ and $f_3 \in fRf(fx_3f)$ summing to f, the identity in fRf. Write $f_2 = fr_2fx_2f$ and $f_3 = fr_3fx_3f$ for some $r_2, r_3 \in R$.

Let $e_2 = f_2r_2fx_2 \in Rx_2$ and let $e_3 = f_3r_3fx_3 \in Rx_3$. By an easy calculation we see that e_2 and e_3 are orthogonal idempotents. Let $e_1 = 1 - e_2 - e_3$, so e_1 is orthogonal to e_2 and e_3, with $e_1 + e_2 + e_3 = 1$. Then

$$\begin{aligned} e_1x_1 &= (1 - e_2 - e_3)(1 - f) = 1 - e_2 - e_3 - f + e_2f + e_3f \\ &= e_1 - f + f_2 + f_3 = e_1 - f + f = e_1, \end{aligned}$$

so $e_1 \in Rx_1$. Finally, since $fe_2 = e_2$ and $fe_3 = e_3$, we see

$$x_1e_1 = x_1(1 - e_2 - e_3) = x_1.$$

□

To make use of Lemma 4.4 to its fullest, we need to see how changing one idempotent to a left associate idempotent affects internal decompositions.

Lemma 4.5. *Assume R has a linear Hausdorff topology. Also assume that $e = \sum_{i \in I} g_i$ where $\{g_i\}_{i \in I}$ is a summable family of orthogonal idempotents. Then e is an idempotent and:*

(1) If f is an idempotent, orthogonal to e, then f is orthogonal to each g_i.

If e' is another idempotent, with $e' \sim_\ell e$, then:

(2) The family $\{e'g_i\}_{i\in I}$ consists of summable, orthogonal idempotents, summing to the idempotent e'.

(3) We have $g_i \sim_\ell e'g_i$ for every $i \in I$.

(4) If $e' = ue$ then $e'g_i = ug_i$ for every $i \in I$.

Proof. We will use the equalities $g_i e = g_i = eg_i$ and $ee' = e$. The fact that e is an idempotent is easy, and part (1) is obtained by the calculation $fg_i = feg_i = 0 = g_i ef = g_i f$. The summability statement of part (2) follows from Remark 3.6. Orthogonality and idempotence result from the equations

$$(e'g_i)(e'g_j) = e'(g_i e)e'g_j = e'g_i(ee')g_j = e'g_i eg_j = e'g_i g_j = \delta_{i,j}e'g_i.$$

Part (3) is expressed in the equations $g_i(e'g_i) = (g_i e)(e'g_i) = g_i eg_i = g_i$ and $(e'g_i)g_i = e'g_i$. For part (4), if $e' = ue$ then $e'g_i = ueg_i = ug_i$. □

Proposition 4.6. *A 2-exchange ring is an n-exchange ring for all integers $n \geqslant 2$.*

Proof. We work by induction. Assuming that R is an n-exchange ring for some fixed integer $n \geqslant 2$, we want to show R is an $(n+1)$-exchange ring.

Suppose we have an equation $x_1 + \cdots + x_n + x_{n+1} = 1$ in R. By n-exchange we can find orthogonal idempotents $g_1 \in Rx_1, \ldots, g_{n-1} \in Rx_{n-1}$, $g_n \in R(x_n + x_{n+1})$, summing to 1. Setting $g = \sum_{i=1}^{n-1} g_i$, and writing $g_n = r(x_n + x_{n+1})$, we have

$$g + rx_n + rx_{n+1} = 1.$$

By Lemma 4.4, there are orthogonal idempotents $h_1 \in Rg$, $h_2 \in Rrx_n \subseteq Rx_n$, and $h_3 \in Rrx_{n+1} \subseteq Rx_{n+1}$, summing to 1, with $h_1 \sim_\ell g$. Put $e_i = h_1 g_i$ for $i < n$, $e_n = h_2$ and $e_{n+1} = h_3$. By Lemma 4.5 we have

$$e_1 + \cdots + e_n + e_{n+1} = 1,$$

the summands are orthogonal idempotents, and $e_i \in Rx_i$ for each i. □

Note that in the proof above, when we use Lemma 4.4 we are required to replace previously constructed idempotents with left associate ones. In some settings we want to work with our original idempotents and not the left associates. What we lose by doing this is the orthogonality. However, we do retain one half of the orthogonality relations. In fact, if e and f are orthogonal, and $e' \sim_\ell e$, we still have $e'f = e'ef = 0$. The following results tell us when we can modify a sum of "almost" orthogonal idempotents into a sum of actually orthogonal idempotents.

Lemma 4.7 ([12, Lemma 2])**.** *Let e_i for $i \in I$ be a family of idempotents in a ring R, where I is a totally ordered set. If $e_ie_j \in J(R)$ whenever $i < j$ then $\sum_{i \in F} e_iR$ is direct, and a direct summand of R, for each finite subset $F \subseteq I$.*

Proposition 4.8 (cf. [12, Lemma 8])**.** *Let $\{e_i\}_{i \in I}$ be a summable family of idempotents in a ring, R, with a linear Hausdorff topology, and assume I is totally ordered. Suppose that $e_ie_j \in J(R)$ whenever $i < j$, and that $\sum_{i \in I} e_i = \varphi$ is right invertible. Fix ψ so that $\varphi\psi = 1$. Then $\{e_i\psi\}_{i \in I}$ and $\{\psi e_i\}_{i \in I}$ are summable families of orthogonal idempotents, summing to $\varphi\psi = 1$ and $\psi\varphi$ respectively.*

Proof. The only part of the proposition that is not immediate is the statement that the families consist of orthogonal idempotents. We have

$$e_j = \varphi\psi e_j = \sum_{i \in I} e_i \psi e_j.$$

Let $\mathfrak{U}$ be a basis of left ideal neighborhoods of zero. Fix $U \in \mathfrak{U}$ and a finite set $F_U \subseteq I$ such that $e_i\psi e_j \in U$ for $i \notin F_U$. In particular, from summability we have

$$e_j - \sum_{i \in F_U} e_i \psi e_j \in U. \tag{4.1}$$

Since $\sum_{i \in F_U \cup \{j\}} e_iR$ is direct and a summand of R by the previous lemma, there are idempotents f_i (for each $i \in F_U \cup \{j\}$) so that $f_ie_k = \delta_{i,k}e_k$ for each $k \in F_U \cup \{j\}$. In particular, since U is a left ideal, we can multiply equation (4.1) by f_i, finding $e_i\psi e_j \in U$ for all $i \neq j$, and $e_j - e_j\psi e_j \in U$ also. But U is arbitrary, and so

$$e_i\psi e_j = \delta_{i,j} e_j$$

for all $i, j \in I$. The orthogonality of $\{e_i\psi\}_{i\in I}$ and $\{\psi e_i\}_{i\in I}$ now follows. $\square$

We will see later this proposition does not hold true if φ is left invertible rather than right invertible, and thus the fact that we have a *left* linear topology rather than a right linear topology is important.

5 From Finite Exchange to Countable Exchange

We can try to use the same idea as in Proposition 4.6 to prove that any 2-exchange ring is an $\aleph_0$-exchange ring. There is one technical condition that arises in this process. However, a large class of rings meet this condition.

Theorem 5.1. *Let R be an exchange ring with a linear Hausdorff Σ-complete topology. If the condition $(*)$ (which is defined and boxed below) holds then R is an $\aleph_0$-exchange ring.*

Proof. Let $\{x_i\}_{i\in\mathbb{Z}_+}$ be a summable family of elements in R, with $\sum_{i=1}^{\infty} x_i = 1$. For notational ease, set $y_j = \sum_{i>j} x_i$. For each $j \in \mathbb{Z}_+$ we will construct elements $e_{i,j} \in Rx_i$ (for $i \leqslant j$), $f_j \in Ry_j$, and $v_j \in U(R)$ such that the following conditions hold:

(A) The family $\{e_{1,j}, e_{2,j}, \dots, e_{j,j}, f_j\}$ consists of orthogonal idempotents that sum to 1.

(B) For all $i \leqslant j$, $v_j e_{i,i} = e_{i,j}$ and $v_j f_j = f_j$.

Set $v_1 = 1$. Since R is an exchange ring, the equation $x_1 + y_1 = 1$ implies that there are orthogonal idempotents $e_{1,1} \in Rx_1$ and $f_1 \in Ry_1$ with $e_{1,1} + f_1 = 1$. It is easy to check that condition (A) holds for $j = 1$, and condition (B) holds trivially in this case. This finishes the base case. Suppose, by induction, we have fixed elements $e_{i,j} \in Rx_i$ (for all $i \leqslant j$), $f_j \in Ry_j$, and $v_j \in U(R)$ satisfying the conditions above, for $j = n$. Writing $f_n = r_n y_n$ for some $r_n \in R$, we have

$$1 = e_{1,n} + \cdots + e_{n,n} + f_n = (e_{1,n} + \cdots + e_{n,n}) + r_n x_{n+1} + r_n y_{n+1}.$$

Lemma 4.4 allows us to pick pair-wise orthogonal idempotents

$$h_1 \in R(e_{1,n} + \cdots + e_{n,n}), \qquad h_2 \in Rr_n x_{n+1}, \qquad h_3 \in Rr_n y_{n+1}$$

with $h_1 + h_2 + h_3 = 1$ and $h_1 \sim_\ell \sum_{i=1}^{n} e_{i,n}$. By Lemma 4.2, property (5), there exists $u_{n+1} \in U(R)$ such that $u_{n+1}(e_{1,n} + \cdots + e_{n,n}) = h_1$ and $u_{n+1} f_n = f_n$. Putting

$e_{i,n+1} = u_{n+1}e_{i,n} \in Rx_i$ (for $i \leqslant n$), $e_{n+1,n+1} = h_2 \in Rx_{n+1}$, and $f_{n+1} = h_3 \in Ry_{n+1}$, Lemma 4.5 shows that condition (A) above holds.

By Lemma 4.2, property (6), $(e_{n+1,n+1} + f_{n+1})$ is right associate to f_n, hence $f_n e_{n+1,n+1} = e_{n+1,n+1}$ and $f_n f_{n+1} = f_{n+1}$. Putting $v_{n+1} = u_{n+1}v_n$, and remembering $u_{n+1}f_n = f_n$, we calculate

$$\begin{aligned} v_{n+1}f_{n+1} &= (u_{n+1}v_n)(f_n f_{n+1}) = u_{n+1}v_n f_n f_{n+1} \\ &= u_{n+1}f_n f_{n+1} = f_n f_{n+1} = f_{n+1} \end{aligned}$$

and similarly $v_{n+1}e_{n+1,n+1} = e_{n+1,n+1}$. Finally, for $i < n+1$,

$$v_{n+1}e_{i,i} = u_{n+1}v_n e_{i,i} = u_{n+1}e_{i,n} = e_{i,n+1}.$$

Therefore, condition (B) holds. This finishes the inductive step.

So we have constructed elements $e_{i,j}$ (for $i \leqslant j$), f_j, and v_j satisfying the properties above, for all $j \in \mathbb{Z}_+$. Since $\{x_i\}_{i\in\mathbb{Z}_+}$ is summable, and the topology is left multiple summable, the family $\{e_{i,i}\}_{i\in\mathbb{Z}_+}$ is also summable. We put $\varphi = \sum_{i\in\mathbb{Z}_+} e_{i,i}$ and assume that the following condition holds:

$$\boxed{(*) \quad \varphi \text{ is a unit.}}$$

Now, for $i < j$ we have $e_{i,i}e_{j,j} = v_j^{-1}v_j e_{i,i}e_{j,j} = v_j^{-1}e_{i,j}e_{j,j} = 0 \in J(R)$. So, by Proposition 4.8, $\{\varphi^{-1}e_{i,i}\}_{i\in\mathbb{Z}_+}$ is a summable, orthogonal set of idempotents, summing to 1. Finally, $e_i = \varphi^{-1}e_{i,i} \in Rx_i$, so R satisfies the definition of an $\aleph_0$-exchange ring. □

Note that the element φ is not uniquely determined, and hence condition $(*)$ needs some quantification. When we say that $(*)$ holds we mean that φ is a unit, for all choices of φ that could possibly arise as in the construction in the previous theorem. When we speak of *the* element φ, we merely mean some choice of φ arising as in the construction. It turns out that, in general, the element φ of this theorem can be a non-unit.

Example 5.2. Let k be a division ring and let M_k be a countably infinite-dimensional vector space over k, say with basis $\{m_i\}_{i\in\mathbb{Z}_+}$. Let $R = E = \mathrm{End}(M_k)$ with the finite topology, which is an exchange ring. Define $x_i \in R$, for $i \in \mathbb{Z}_+$, as follows:

$$x_i(m_j) = \begin{cases} -m_i & \text{if } j = i-1 \\ m_i + m_{i+1} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\sum_{i=1}^{\infty} x_i = 1$. Set $y_i = \sum_{j>i} x_j$. Then one calculates easily

$$y_i(m_j) = \begin{cases} 0 & \text{if } j < i \\ -m_{i+1} & \text{if } j = i \\ m_j & \text{if } j > i. \end{cases}$$

Since x_1 is already an idempotent we can take $e_{1,1} = x_1$ and $f_1 = y_1$. We leave the calculations to the reader, and claim that we may take $e_{i,i} = f_{i-1}x_i f_{i-1}x_i$ and $f_i = f_{i-1}y_i f_{i-1}y_i$. If we make these choices it turns out that

$$e_{i,i}(m_j) = \begin{cases} -m_i - m_{i+1} & \text{if } j = i-1 \\ m_i + m_{i+1} & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_i(m_j) = \begin{cases} 0 & \text{if } j < i \\ -m_{i+1} & \text{if } j = i \\ m_j & \text{if } j > i. \end{cases}$$

Setting $\varphi = \sum_{i=1}^{\infty} e_{i,i}$, we calculate $\varphi(m_j) = m_j - m_{j+2}$. Thus we see that φ is not a unit since it isn't surjective; in fact $\mathrm{im}(\varphi)$ is contained in the subspace where the coefficients of the m_j sum to 0. However, M has full exchange by Example 2.2 and so E is a full exchange ring in the finite topology. Therefore, this shows that the converse of Theorem 5.1 is not true. Modifying the example slightly, we see that condition $(*)$ also fails when M is a direct sum of an infinite number of copies of any non-zero module.

While it's true that φ is not a unit in this example, it is clear that φ is left-invertible. Let ψ be such that $\psi\varphi = 1$. From our earlier computation $\varphi(m_j) =$

$m_j - m_{j+2}$ so we know $\psi(m_{j+2}) = -m_j + \psi(m_j)$. Hence ψ is uniquely determined once we know $\psi(m_1)$ and $\psi(m_2)$. Further, these values are arbitrary. Suppose we take $\psi(m_1) = -m_1$ and $\psi(m_2) = 0$. Then clearly $\psi e_{1,1}$ is not an idempotent, and so $\{\psi e_{i,i}\}_{i\in\mathbb{Z}_+}$ will not be a family of idempotents.

On the other hand, if we take $\psi(m_1) = \psi(m_2) = 0$, one finds in general $\psi(m_j) = -m_{j-2} - m_{j-4} - \cdots$, and we leave it to the reader to show that $\{\psi e_{i,i}\}_{i\in\mathbb{Z}_+}$ is a family of orthogonal idempotents, summing to 1. So while we cannot generalize Proposition 4.8 to work for sums which are only left-invertible, it appears that in some cases we may be able to choose the left inverse correctly and still end up with orthogonal idempotents.

While we've shown φ does not, in general, have to equal a unit, there are certain properties which it always possesses. We know $\lim_{n\to\infty} y_n = 0$, and the topology is linear, so we have $\lim_{n\to\infty} f_n = 0$. Thus,

$$\varphi = \sum_{i=1}^{\infty} e_{i,i} = \lim_{n\to\infty}\left(\sum_{i=1}^{n} e_{i,i} + f_n\right) = \lim_{n\to\infty} v_n^{-1}\left(\sum_{i=1}^{n} e_{i,n} + f_n\right) = \lim_{n\to\infty} v_n^{-1}.$$

Therefore, φ is a limit of units.

In an endomorphism ring, with the finite topology, a limit of injective maps must be an injective map because nothing is introduced into the kernel in the limit process. More generally, a limit of left-invertible elements in a ring with a linear Hausdorff topology is a left non-zero-divisor. To prove this, let $w = \lim_{i\in I} w_i$ with each w_i left-invertible (with left inverse v_i), and with I directed. Let $U \in \mathfrak{U}$ be an arbitrary, open, left ideal. If $wr = 0$ then $\lim_{i\in I} w_i r = 0$ and hence for a large index N we have $w_N r \in U$. But U is a left ideal, whence $r = v_N w_N r \in U$. Therefore $r \in \bigcap_{U\in\mathfrak{U}} U = (0)$, or $r = 0$.

We can use the fact that φ is a left non-zero-divisor to show that many classes of rings are $\aleph_0$-exchange rings, as long as the ring has the correct type of topology.

Definition 5.3. Let M_k be a module. Following Lam [8], we say M is *hopfian* if every surjective endomorphism is injective. We say M is *cohopfian* if every injective endomorphism is surjective.

Lemma 5.4. *Let R be a ring. Every left non-zero-divisor of R is a unit if and only if R_R is cohopfian.*

Proof. We have a natural isomorphism $\mathrm{End}(R_R) \cong R$, and injective endomorphisms correspond to left non-zero-divisors. $\square$

Corollary 5.5. *Let R be a ring with a linear Hausdorff Σ-complete topology. If R is Dedekind-finite and π-regular then R is an $\aleph_0$-exchange ring.*

Proof. Recall, a ring is Dedekind-finite when $xy = 1$ implies $yx = 1$. We saw earlier that all π-regular rings are exchange rings. So we can apply Theorem 5.1 once we show R_R is cohopfian. Let x be a left non-zero-divisor in R. Fix $y \in R$ and $n \in \mathbb{Z}_+$ so that $x^n y x^n = x^n$. Then $x^n(1 - yx^n) = 0$, and so $1 = yx^n$. Since R is Dedekind-finite, yx^{n-1} is an inverse for x. $\square$

Definition 5.6. Following Nicholson, we say R is a *semi-regular* (respectively *semi-π-regular*) ring if $R/J(R)$ is regular (respectively π-regular) and idempotents lift modulo $J(R)$. Such rings are exchange rings from what we said in the paragraph after Definition 2.9.

Note that $\varphi \in R$ is a unit if and only if $\overline{\varphi}$ is a unit in $R/J(R)$. Further, by Lemma 3.2 part (2), we have

$$\overline{\varphi} = \sum_{i \in \mathbb{Z}_+} \overline{e}_{i,i} = \lim_{i \in \mathbb{Z}_+} \overline{v}_i^{\,-1}$$

and so by the same reasoning as in the previous corollary we see that any Dedekind-finite, semi-π-regular ring with a linear Hausdorff Σ-complete topology is an $\aleph_0$-exchange ring.

We've used the fact that φ is a limit of units. There is another nice property that φ exhibits, arising from the fact it is a sum of "almost" orthogonal idempotents. For this we need another lemma.

Lemma 5.7. *Let $\{e_i\}_{i \in I}$ be a summable family of idempotents in a ring, R, with a linear Hausdorff topology, and assume I is well-ordered. Suppose that $e_i e_j = 0$*

whenever $i < j$, and set $\varphi = \sum_{i \in I} e_i$. Let $K \subseteq R$ be a subset such that $(1-\varphi)K \supseteq K$. Then $\varphi K = (0)$.

Proof. It suffices to show that $e_i K = (0)$ for each $i \in I$. We work by induction. Let $r \in K$. Then $r = (1-\varphi)r'$ for some $r' \in K$. Hence $e_1 r = e_1(1-\varphi)r' = 0$, where 1 is the first element of I. Since $r \in K$ is arbitrary, $e_1 K = (0)$. Suppose now, by induction, that $e_i K = (0)$ for all $i < \beta$. Then

$$e_\beta r = e_\beta(1-\varphi)r' = e_\beta r' - \sum_{i \leqslant \beta} e_\beta e_i r' = -\sum_{i<\beta} e_\beta e_i r' = 0.$$

Hence $e_\beta K = (0)$. This finishes our induction. □

If we let φ be the element constructed in Theorem 5.1, we already proved it is a left non-zero-divisor, and so in the context of the previous lemma, since $\varphi K = (0)$ we have $K = (0)$. So, if we want φ to be a unit we just need some ring theoretic condition which forces the existence of a non-zero subset $K \subseteq R$ such that $(1-\varphi)K \supseteq K$ whenever φ isn't a unit.

Definition 5.8. Following Nicholson, we call an element $r \in R$ *clean* when we can write $r = u + e$, where $u \in U(R)$ and $e^2 = e \in R$. The element r is *strongly clean* if we can choose u and e as above so that $r = u + e$ and $ue = eu$. A ring is *clean* (respectively, *strongly clean*) when every element is clean (respectively, strongly clean). Nicholson proved that all clean rings are exchange rings [13, Proposition 1.8], and every strongly π-regular ring is strongly clean [14, Theorem 1].

Corollary 5.9. *Let R be a ring with a linear Hausdorff Σ-complete topology. If R is strongly clean then R is an $\aleph_0$-exchange ring.*

Proof. It suffices to show that $\varphi = \sum_{i=1}^{\infty} e_{i,i}$ (as constructed in Theorem 5.1) is a unit. Since φ is strongly clean, we can write $\varphi = u + e$ with $u \in U(R)$, $e^2 = e \in R$, and $ue = eu$. We compute $(1-\varphi)e = (1-e-u)e = -ue$. Hence $(1-\varphi)$ acts as the unit $-u$ on eR. Further, since u and e commute, this means $(1-\varphi)$ is an automorphism of eR. By Lemma 5.7, $\varphi eR = (0)$ and in particular $\varphi e = 0$.

But φ is a limit of units, and hence a left non-zero-divisor. This means $e = 0$, so $\varphi = u + e = u$ is a unit. $\square$

Note that we can weaken the condition that R is strongly clean to the condition that $R/J(R)$ is strongly clean and idempotents lift modulo $J(R)$, just as we did when weakening π-regularity to semi-π-regularity. This is a nontrivial generalization due to an example in [20].

Although we stated Lemma 5.7 for a general index set (so that we can use the result in later sections) in this section we are most interested in the case when $I = \mathbb{Z}_+$. For this choice of I, one can improve Lemma 5.7 in a straightforward manner to show that $\varphi K = 0$ where $K = \bigcap_{i=1}^{\infty}(1-\varphi)^i R$. In particular, if φ is a left non-zero-divisor we must have $K = 0$. This yields:

Proposition 5.10. *Let R be an exchange ring with a linear Hausdorff Σ-complete topology. If R is such that, for $x \in R$, $\bigcap_{i=1}^{\infty}(1-x)^i R = 0$ implies $x \in U(R)$, then R is an $\aleph_0$-exchange ring.*

6 From Finite Exchange to Full Exchange

When trying to push the proof of Theorem 5.1 up to full exchange one runs into problems when passing through limit ordinals. However, we can get around these roadblocks by assuming a few more conditions. Just as in the proof of Theorem 5.1, we will define these conditions in the body of the proof. But first we need a couple easy lemmas.

Lemma 6.1. *Let $\{e_i\}_{i\in I}$ be a summable family of idempotents in a topological ring, R, with a linear Hausdorff topology, and assume I is totally ordered. Put $e = \sum_{i\in I} e_i$ and suppose that $e_ie_j = 0$ whenever $i < j$. Then each e_i belongs to the closure of the left ideal Re^2, and in particular so does e.*

Proof. Let $\mathfrak{U}$ be a basis of left ideal neighborhoods. We can see from the proof of Lemma 3.2 that the closure of a set, S, is $\bigcap_{U\in\mathfrak{U}}(S + U)$ and so the closure of Re^2 is $\bigcap_{U\in\mathfrak{U}}(Re^2 + U)$. Hence, it suffices to prove that $e_i \in Re^2 + U$ for each $i \in I$. By summability of the families $\{e_i\}_{i\in I}$ and $\{e_ie_j\}_{i,j\in I}$, there is a finite set F_U such that $e_i \in Re^2 + U$ whenever $i \notin F_U$, and $e_ie_j \in Re^2 + U$ whenever either $i \notin F_U$ or $j \notin F_U$. Further, we may suppose F_U is the smallest set with these properties.

For each subset $J \subseteq I$, set $e(J) = \sum_{i\in J} e_i$. Since $Re^2 + U$ is a union of cosets of U, it is an open left ideal, and hence also a closed set. Therefore, $e(I \setminus F_U) \in Re^2+U$, and hence $e\cdot e(I\setminus F_U) \in Re^2+U$. On the other hand $e\cdot e(I) = e^2 \in Re^2+U$, and so

$$e \cdot e(F_U) = e \cdot e(I) - e \cdot e(I \setminus F_U) \in Re^2 + U.$$

By a similar argument $e(F_U)^2 \in Re^2 + F_U$.

Suppose F_U is not empty, and write $F_U = \{i_1, \ldots, i_n\}$, where the elements are written with respect to their total order. Then, since $e_ie_j = 0$ whenever $i < j$, we find

$$e_{i_1} = e_{i_1}\left(\sum_{k=1}^{n} e_{i_k}\right) = e_{i_1}e(F_U).$$

Hence $e_{i_1} = e_{i_1}e(F_U) = e_{i_1}e(F_U)^2 \in Re^2 + U$, and so $e_\ell e_{i_1} \in Re^2 + U$ for all $\ell \in I$. This is easily seen to contradict the minimality F_U. Therefore, F_U must be empty, and $e_i \in Re^2 + U$ for all $i \in I$ as claimed. Finally, by Lemma 3.2 part (3), and the fact that $Re^2 + U$ is closed, we see $e \in Re^2 + U$. Since U is arbitrary we are done. $\square$

Lemma 6.2. *Assume the hypotheses of Lemma 6.1. If $e^n r = 0$, for some $r \in R$ and some $n \in \mathbb{Z}_+$, then we have $e_i r = 0$ for all $i \in I$. In particular, $er = 0$.*

Proof. This follows from the previous lemma. In fact, one may weaken the hypothesis $e^n r = 0$ to $\lim_{n\to\infty} e^n r = 0$. The point is that if $\mathfrak{U}$ is a basis of left ideal neighborhoods, then for each $U \in \mathfrak{U}$ one must have $e^n r \in U$ for some $n \in \mathbb{Z}_+$. Then since each e_i lies in the closure of Re^2 an easy induction yields the fact that each e_i lies in the closure of Re^n for each $n \geqslant 2$. By continuity of right multiplication we have each $e_i r$ lies in the closure of $Re^n r$. But U is a closed left ideal containing $e^n r$, and hence must contain each $e_i r$. $\square$

Theorem 6.3. *Let R be an exchange ring with a linear Hausdorff Σ-complete topology. If R satisfies conditions $(*_1)$ and $(*_2)$ (defined and boxed below) then R is a full exchange ring.*

Proof. Let $\{x_i\}_{i\in I}$ be a summable collection of elements of R, summing to 1, with I an indexing set of arbitrary cardinality. Without loss of generality, we may assume that I is a well-ordered set with first element 1. Further, we may assume I has a last element κ, which we will use near the end of the proof. We call those elements of I which are not successors, limit elements. Put $y_j = \sum_{i>j} x_i$ and $y'_j = x_j + y_j = \sum_{i\geqslant j} x_i$.

For each $j \in I$ we will inductively construct elements $e_{i,j} \in Rx_i$ (for $i \leqslant j$), $f_j \in Ry_j$, and $v_j \in U(R)$ such that the following two conditions hold:

(A) For each $j \in I$, the family $\{e_{i,j}\}_{i\leqslant j} \cup \{f_j\}$ consists of summable, orthogonal idempotents, summing to 1.

(B) For each $i \leqslant j$, $v_j e_{i,i} = e_{i,j}$ and $v_j f_j = f_j$.

Put $v_1 = 1$. Since R is an exchange ring the equation $x_1 + y_1 = 1$ implies that there are orthogonal idempotents $e_{1,1} \in Rx_1$ and $f_1 \in Ry_1$ with $e_{1,1} + f_1 = 1$. This provides the base step of our inductive definition. Now suppose (by transfinite induction) that for all $j < \alpha$ we have constructed elements $e_{i,j}$ (for all $i \leqslant j$), f_j, and v_j satisfying the conditions above. We have two cases.

Case 1. α is a successor element.

In this case we proceed exactly as in the proof of Theorem 5.1. Let $\alpha - 1$ be the predecessor of α. Writing $f_{\alpha-1} = r_{\alpha-1} y_{\alpha-1}$ for some $r_{\alpha-1} \in R$, we have

$$1 = \sum_{i<\alpha} e_{i,\alpha-1} + f_{\alpha-1} = \sum_{i<\alpha} e_{i,\alpha-1} + r_{\alpha-1} x_\alpha + r_{\alpha-1} y_\alpha.$$

Lemma 4.4 allows us to pick orthogonal idempotents

$$h_1 \in R\left(\sum_{i<\alpha} e_{i,\alpha-1}\right), \qquad h_2 \in Rr_{\alpha-1} x_\alpha, \qquad h_3 \in Rr_{\alpha-1} y_\alpha$$

with $h_1 + h_2 + h_3 = 1$ and $h_1 \sim_\ell \sum_{i<\alpha} e_{i,\alpha-1}$. By Lemma 4.2, property (5), there exists $u_\alpha \in U(R)$ such that $h_1 = u_\alpha \left(\sum_{i<\alpha} e_{i,\alpha-1}\right)$ and $u_\alpha f_{\alpha-1} = f_{\alpha-1}$. If we put $e_{i,\alpha} = u_\alpha e_{i,\alpha-1} \in Rx_i$ (for $i < \alpha$), $e_{\alpha,\alpha} = h_2 \in Rx_\alpha$, and $f_\alpha = h_3 \in Ry_\alpha$, then Lemma 4.5 implies that these are orthogonal idempotents. Also clearly

$$\sum_{i \leqslant \alpha} e_{i,\alpha} + f_\alpha = 1.$$

Therefore, condition (A) holds when $j = \alpha$. Checking that condition (B) holds for $v_\alpha = u_\alpha v_{\alpha-1}$ is done exactly as before. This completes the inductive definition of the elements we need, when α is a successor element.

Case 2. α is a limit element.

This case is much harder. Set $\varphi = \sum_{i<\alpha} e_{i,i}$. We calculate that if $i < j < \alpha$, then $e_{i,i} e_{j,j} = v_j^{-1} v_j e_{i,i} e_{j,j} = v_j^{-1} e_{i,j} e_{j,j} = 0$. So φ is a sum of "almost" orthogonal idempotents. We assume

$$\boxed{(*_1) \quad \text{There exists an idempotent } p \text{ such that } \varphi' = \varphi + p \text{ is a unit and } \varphi p = 0.}$$

For notational ease put $v'_\alpha = (\varphi')^{-1}$. From our work above, and Lemma 6.2, we see that $e_{i,i}p = 0$ for all $i < \alpha$. This means that the decomposition $\varphi' = \sum_{i<\alpha} e_{i,i} + p$ satisfies the hypotheses of Proposition 4.8. This yields $\sum_{i<\alpha} v'_\alpha e_{i,i} + v'_\alpha p = 1$, where the summands are orthogonal idempotents. Put $e'_{i,\alpha} = v'_\alpha e_{i,i}$ for all $i < \alpha$, and $f'_\alpha = v'_\alpha p$. The following argument shows that $f'_\alpha = p$, and in particular $v'_\alpha f'_\alpha = f'_\alpha$, which we will need later. First note that by orthogonality $v'_\alpha p e'_{i,\alpha} = f'_\alpha e'_{i,\alpha} = 0$ and so $pe'_{i,\alpha} = 0$. We already saw $e_{i,i}p = 0$ and so $e'_{i,\alpha}p = 0$ for all $i < \alpha$. Hence $r = \sum_{i<\alpha} e'_{i,\alpha} + p$ is a sum of orthogonal idempotents, and hence is an idempotent. Further, if $rs = 0$ for some $s \in R$, then orthogonality implies $e'_{i,\alpha}s = 0$ and $ps = 0$. Hence $(\varphi + p)s = 0$ and so $s = 0$. But this means r is a left non-zero-divisor and an idempotent, so $r = 1$. Thus, $p = 1 - \sum_{i<\alpha} e'_{i,\alpha} = f'_\alpha$.

We also claim $f_j f'_\alpha = f'_\alpha$ for all $j < \alpha$. To see this we compute

$$e_{i,j} f'_\alpha = v_j e_{i,i} f'_\alpha = v_j v'^{-1}_\alpha v'_\alpha e_{i,i} f'_\alpha = v_j v'^{-1}_\alpha e'_{i,\alpha} f'_\alpha = 0$$

and so

$$f_j f'_\alpha = \left(1 - \sum_{i \leqslant j} e_{i,j}\right) f'_\alpha = f'_\alpha. \tag{6.1}$$

Notice that we put prime marks on the idempotents we constructed. This is because they are not quite the ones we set out to construct. We need a few more modifications. The first problem with the idempotents we constructed above is that f'_α is not a left multiple of $y'_\alpha = \sum_{i \geqslant \alpha} x_i$. We can fix this problem by finding a new idempotent $f''_\alpha \in Ry'_\alpha$, which is right associate to f'_α. The construction is as follows:

For use shortly, we note

$$\lim_{i\to\alpha} y_i = y'_\alpha, \tag{6.2}$$

where by $\lim_{i\to\alpha} y_i$ we mean the limit in the ring topology on R. (More formally, if $\mathfrak{U}$ is the given basis of neighborhoods of zero, then for each $U \in \mathfrak{U}$ there is some index $j \in I$, $j < \alpha$, so that for each $i \in (j, \alpha)$, $y_i - y'_\alpha \in U$.) Also by construction, for $i < \alpha$ we have $f_i \in Ry_i$, and so we can fix elements $r_i \in R$ with $f_i = r_i y_i$.

We claim that left multiplication by y'_α gives an isomorphism $f'_\alpha R \to y'_\alpha f'_\alpha R$. It suffices to show that if $y'_\alpha f'_\alpha r = 0$ then $f'_\alpha r = 0$. Using equations (6.1) and (6.2) above, we see

$$f'_\alpha r = \lim_{i\to\alpha} f_i f'_\alpha r = \lim_{i\to\alpha} r_i y_i f'_\alpha r = \lim_{i\to\alpha} r_i y'_\alpha f'_\alpha r = 0$$

as claimed. (Note that $\lim_{i\to\alpha} r_i y_i = \lim_{i\to\alpha} r_i y'_\alpha$ follows from equation (6.2) and the linearity of the topology, so then the third equality in the above equation follows from continuity of multiplication.) Let $r'_\alpha : y'_\alpha f'_\alpha R \to f'_\alpha R$ be the isomorphism which is the inverse to $y'_\alpha|_{f'_\alpha R}$. In our calculation above, we saw that $r'_\alpha = \lim_{i\to\alpha} r_i|_{y'_\alpha f'_\alpha R}$. Notice that, *a priori*, the map r'_α does not extend to an element in $R = \mathrm{End}(R_R)$, since this limit might not converge on all of R. However, if r'_α did extend to an element in R, that would be equivalent to:

$$\boxed{(*_2)\quad \text{There exists } r'_\alpha \in R \text{ such that } r'_\alpha y'_\alpha f'_\alpha = f'_\alpha.}$$

We assume $(*_2)$ holds.

Set $f''_\alpha = f'_\alpha r'_\alpha y'_\alpha$. We do the calculations to check that f''_α is right associate to f'_α and is an idempotent. First,

$$f''_\alpha f''_\alpha = f'_\alpha r'_\alpha y'_\alpha f'_\alpha r'_\alpha y'_\alpha = f'_\alpha (r'_\alpha y'_\alpha f'_\alpha) r'_\alpha y'_\alpha = f'_\alpha f'_\alpha r'_\alpha y'_\alpha = f''_\alpha.$$

Second, one easily sees $f'_\alpha f''_\alpha = f''_\alpha$. Finally,

$$f''_\alpha f'_\alpha = f'_\alpha r'_\alpha y'_\alpha f'_\alpha = f'_\alpha (r'_\alpha y'_\alpha f'_\alpha) = (f'_\alpha)^2 = f'_\alpha.$$

We have shown $f'_\alpha \sim_r f''_\alpha$. Therefore the equivalence of properties (1) and (6) in Lemma 4.2 implies $(1-f'_\alpha) \sim_\ell (1-f''_\alpha)$. So, again by Lemma 4.2, property (5), we pick some unit v''_α such that $v''_\alpha(1-f'_\alpha) = 1-f''_\alpha$ and $v''_\alpha f'_\alpha = f'_\alpha$. Set $e''_{i,\alpha} = v''_\alpha e'_{i,\alpha}$, for $i<\alpha$. We have $\sum_{i<\alpha} e''_{i,\alpha} + f''_\alpha = 1$, and $\{e''_{i,\alpha}\}_{i<\alpha} \cup \{f''_\alpha\}$ is a summable family of orthogonal idempotents by Lemma 4.5.

With all the machinery we have built up, it is now an easy matter to construct $e_{i,\alpha}$ (for each $i \leqslant \alpha$), f_α, and v_α. To do so, notice we have the equation

$$1 = \sum_{i<\alpha} e''_{i,\alpha} + f''_\alpha = \sum_{i<\alpha} e''_{i,\alpha} + r'_\alpha x_\alpha + r'_\alpha y_\alpha.$$

Now we use exactly the same ideas as in Case 1 to construct the elements we need. Lemma 4.4 allows us to pick orthogonal idempotents

$$h_1 \in R\left(\sum_{i<\alpha} e''_{i,\alpha}\right), \qquad h_2 \in Rr'_\alpha x_\alpha, \qquad h_3 \in Rr'_\alpha y_\alpha$$

with $h_1 + h_2 + h_3 = 1$ and $h_1 \sim_\ell \sum_{i<\alpha} e''_{i,\alpha}$. By Lemma 4.2, property (5), there exists $u_\alpha \in U(R)$ such that $h_1 = u_\alpha\left(\sum_{i<\alpha} e''_{i,\alpha}\right)$ and $u_\alpha f''_\alpha = f''_\alpha$. Putting $e_{i,\alpha} = u_\alpha e''_{i,\alpha} \in Rx_i$ (for $i < \alpha$), $e_{\alpha,\alpha} = h_2 \in Rx_\alpha$, and $f_\alpha = h_3 \in Ry_\alpha$, then Lemma 4.5 implies that these are orthogonal idempotents. Also clearly

$$\sum_{i \leqslant \alpha} e_{i,\alpha} + f_\alpha = 1.$$

Therefore, condition (A) holds when $j = \alpha$.

We put $v_\alpha = u_\alpha v''_\alpha v'_\alpha$. It is clear that $v_\alpha e_{i,i} = e_{i,\alpha}$ for $i < \alpha$, so we just need to see that left multiplication by v_α acts as the identity on $e_{\alpha,\alpha}$ and f_α. First, remember $f'_\alpha = v'_\alpha f'_\alpha$. Second, we chose v''_α so that $v''_\alpha f'_\alpha = f'_\alpha$ holds. Third, u_α was chosen so that $u_\alpha f''_\alpha = f''_\alpha$. Finally, $e_{\alpha,\alpha}$ and f_α are both fixed by left multiplication by f''_α and f'_α since $(e_{\alpha,\alpha} + f_\alpha) \sim_r f''_\alpha \sim_r f'_\alpha$. Therefore,

$$\begin{aligned} v_\alpha f_\alpha &= (u_\alpha v''_\alpha v'_\alpha)(f'_\alpha f_\alpha) = u_\alpha(v''_\alpha v'_\alpha f'_\alpha) f_\alpha \\ &= u_\alpha f'_\alpha f_\alpha = u_\alpha f_\alpha = u_\alpha(f''_\alpha f_\alpha) = f''_\alpha f_\alpha = f_\alpha \end{aligned}$$

and similarly, $v_\alpha e_{\alpha,\alpha} = e_{\alpha,\alpha}$. This finishes Case 2.

By transfinite induction, we have constructed the elements we wanted for all $j \in I$. Recall that we well-ordered I so that it had a last element κ. Let $e_i = e_{i,\kappa}$ for each $i \leqslant \kappa$. Then $\{e_i\}_{i \in I}$ is a summable family of orthogonal idempotents, summing to $1 - f_\kappa = 1$ (since $f_\kappa \in Ry_\kappa = (0)$), with $e_i \in Rx_i$ for each $i \in I$. This completes the proof. □

Notice, once again, that the conditions $(*_1)$ and $(*_2)$ have quantifications that were not made explicit in the body of the proof. We do so now. We say that $(*_1)$ holds in the ring R in the case that for each possible choice of φ (and each limit element $\alpha \in I$) arising as in the construction, there exists an idempotent p such that $\varphi + p$ is a unit and $\varphi p = 0$. Similarly, $(*_2)$ holds when there exists $r'_\alpha \in R$ such that $r'_\alpha y'_\alpha f'_\alpha = f'_\alpha$ for each possible choice of y'_α and f'_α satisfying the construction restraints in the theorem.

7 Lifting through the Jacobson Radical

Mohamed and Müller have shown in [10] that if M is a module such that $E/J(E)$ is regular and abelian (that is, all idempotents are central), with idempotents lifting modulo $J(E)$, then M has full exchange. In particular, they use this to establish that continuous modules (see the next section for the definition) have full exchange. Similarly, one way of further generalizing Theorem 6.3 is to try and lift the argument through the Jacobson radical.

If R is a ring with a linear Hausdorff topology, then the quotient topology on $R/J(R)$ (which by definition is linear) is again Hausdorff by Lemma 3.2 part (2). If R is Σ-complete it isn't clear that $R/J(R)$ needs to be Σ-complete. However, if $\{x_i\}_{i\in I}$ is a left multiple summable family in R, then $\{\overline{x}_i\}_{i\in I}$ is left multiple summable in $R/J(R)$. So we can push the important part of our topology through the radical. Can we lift back up? The following lemma tells us how to do so.

Lemma 7.1. *Let R be an exchange ring, and put $\overline{R} = R/I$ for some ideal $I \subseteq J(R)$. If $\varepsilon \in \overline{R}\overline{x}$ is an idempotent, then there is an idempotent $e \in xRx$ with $\overline{e} = \varepsilon$.*

Proof. Follows easily from [12, Lemma 6]. □

Let R be an exchange ring with a linear Hausdorff Σ-complete topology, and set $\overline{R} = R/J(R)$. Using the same constructions and terminology as in Theorem 6.3, consider the following two conditions:

$$\boxed{(*_1') \quad \text{There exists an idempotent } \pi \in \overline{R} \text{ such that } \overline{\varphi} + \pi \text{ is a unit and } \overline{\varphi}\pi = 0.}$$

and

$$\boxed{(*_2') \quad \text{There exists } \overline{r}'_\alpha \in \overline{R} \text{ such that } \overline{r}'_\alpha \overline{y}'_\alpha \overline{f}'_\alpha = \overline{f}'_\alpha.}$$

Here, again, we are implicitly assuming that these conditions hold for each possible construction of φ, y'_α, and f'_α as in Theorem 6.3.

Theorem 7.2. *Let R be an exchange ring with a linear Hausdorff Σ-complete topology. Then $(*_1') \iff (*_1)$. Also, if $(*_1')$ and $(*_2')$ both hold then R is a full exchange ring.*

Proof. We show $(*_1') \implies (*_1)$, noting that the converse is trivial. Suppose $\pi \in \overline{R}$ is chosen so that $(*_1')$ holds. Since R is an exchange ring, idempotents lift modulo $J(R)$. Hence, there is some idempotent $\widetilde{p} \in R$ such that $\overline{\widetilde{p}} = \pi$. Set $u = \varphi + \widetilde{p}$. Notice that u is a unit, since it is a unit modulo $J(R)$ by hypothesis. Lemma 4.5 tells us that $\sum_{i<\alpha} u^{-1}e_{i,i} + u^{-1}\widetilde{p} = 1$, where the summands are orthogonal idempotents. Setting $p = u^{-1}\widetilde{p}$, we have $u^{-1}e_{i,i}p = 0$ for all $i < \alpha$, and hence $e_{i,i}p = 0$ for all $i < \alpha$. In particular, $\varphi p = 0$.

The same argument we used in Theorem 6.3 to show that $p = f'_\alpha$ (under the old use of p) shows that $\pi = \overline{u}^{-1}\pi$. Therefore, $\varphi + p$ modulo $J(R)$ equals $\overline{\varphi} + \pi$, and thus must be a unit.

We now show $(*_1') + (*_2') \implies R$ is a full exchange ring. Let $\{x_i\}_{i\in I}$ be a summable family in R, summing to 1. Then, $\{\overline{x}_i\}_{i\in I}$ is a summable family summing to $\overline{1}$, and is left multiple summable since $\{x_i\}_{i\in I}$ is. Therefore, the same argument as used in the proof of Theorem 6.3, now applied to $\overline{R}$, shows that we can find orthogonal idempotents $\varepsilon_i \in \overline{R}\overline{x}_i$ summing to $\overline{1}$. (In Theorem 6.3 we didn't need R to have a left multiple summable topology, only that $\{x_i\}_{i\in I}$ is a left multiple summable family.)

By Lemma 7.1, we can lift each ε_i to an idempotent $e'_i \in Rx_i$. These are still summable idempotents (because the x_i are summable), summing to a unit (since modulo $J(R)$ they sum to $\overline{1}$). Letting $u = \sum_{i\in I} e'_i$, Proposition 4.8 says that $\{u^{-1}e'_i\}_{i\in I}$ is a summable family of orthogonal idempotents summing to 1. Clearly, $e_i = u^{-1}e'_i \in Rx_i$, so we are done. $\square$

8 Examples of Full Exchange Rings

We need to find rings that satisfy $(*_1)$ and $(*_2)$ (or their counterparts). We first make the following definitions:

Definition 8.1. Let M be a module. Recall the definition of "essential submodule" which was introduced in Section 1. Consider the following properties:

(C_1) If $N \subseteq_e M$ then $N \subseteq^{\oplus} M$.

(C_2) If $N \subseteq M$, $N' \subseteq^{\oplus} M$, and $N \cong N'$, then $N \subseteq^{\oplus} M$.

(C_3) If $A, B \subseteq^{\oplus} M$ and $A \cap B = (0)$ then $A \oplus B \subseteq^{\oplus} M$.

It is well known that $(C_2) \Rightarrow (C_3)$. A module is said to be *continuous* (respectively *quasi-continuous*) if it has properties (C_1) and (C_2) (respectively (C_1) and (C_3)). A good introductory text discussing these module properties is [11]. Every continuous module has full exchange [10], but not so for quasi-continuous modules (for example, $\mathbb{Z}_{\mathbb{Z}}$).

If we assume R_R has (C_2), then since we have $y'_\alpha f'_\alpha R \cong f'_\alpha R \subseteq^{\oplus} R_R$ that means $y'_\alpha f'_\alpha R$ is a direct summand. Hence $(*_2)$ holds, since isomorphisms between summands lift to endomorphisms of the module. Similarly, if these same relations hold with bars everywhere, we obtain $(*'_2)$. Not surprisingly, Dedekind-finite, semi-π-regular rings (see Definition 5.6) satisfy $(*'_1)$. So we have:

Theorem 8.2. *Let R be a ring with a linear Hausdorff Σ-complete topology. If R is Dedekind-finite, semi-π-regular, and either R_R or $\overline{R}_{\overline{R}}$ has (C_2), then R is a full exchange ring.*

Proof. As noted in Definition 5.6, R is an exchange ring. Since either $(*_2)$ or $(*'_2)$ holds, it suffices by Theorems 6.3 and 7.2 to show $(*'_1)$ holds.

For φ (and $\alpha \in I$) arising as in Theorem 6.3, we have $\overline{\varphi}$ is π-regular, and so $\varphi^n\psi\varphi^n - \varphi^n \in J(R)$ for some $\psi \in R$ and some $n \geqslant 1$. In particular, $\pi = \overline{1} - \overline{\psi}\overline{\varphi}^n$

is an idempotent. Now, $\overline{\varphi} = \sum_{i<\alpha} \overline{e}_{i,i}$, where $e_{i,i}e_{j,j} = 0$ for $i < j$, and $e_{i,i}$ is an idempotent. Since $\overline{\varphi}^n\pi = 0$, Lemma 6.2 tells us $\overline{\varphi}\pi = 0$, and $\overline{e}_{i,i}\pi = 0$ for all $i < \alpha$.

All we need is that $u = \overline{\varphi} + \pi$ is a unit. From previous arguments, such as in Corollary 5.5, it suffices to show that u is a left non-zero-divisor. Suppose $us = 0$ for some $s \in \overline{R}$. Then

$$0 = \overline{\varphi}^n us = \overline{\varphi}^n \left(\overline{\varphi} + (\overline{1} - \overline{\psi}\overline{\varphi}^n)\right) s = \overline{\varphi}^{n+1} s$$

and so by Lemma 6.2 we have $\overline{\varphi}s = 0$. But then

$$0 = us = \overline{\varphi}s + (\overline{1} - \overline{\psi}\overline{\varphi}^n)s = s.$$

Hence, u is a left non-zero-divisor as claimed. $\square$

Corollary 8.3. *Let R be a ring with a linear Hausdorff Σ-complete topology. If R is Dedekind-finite and semi-regular, then R is a full exchange ring.*

Proof. In this case $\overline{R}_{\overline{R}}$ has (C_2) since $\overline{R}$ is regular. Thus Theorem 8.2 applies. $\square$

Corollary 8.4. *Let R be a ring with a linear Hausdorff Σ-complete topology. If R is a unit regular ring, then R is a full exchange ring.*

Proof. Unit regular rings are always Dedekind-finite and regular. $\square$

Theorem 8.5. *Let R be a ring with a linear Hausdorff Σ-complete topology. If $\overline{R}$ is strongly clean, idempotents lift modulo $J(R)$, and either R_R or $\overline{R}_{\overline{R}}$ has (C_2), then R is a full exchange ring.*

Proof. Just as in Theorem 8.2, it suffices to prove $(*_1')$. Write $\overline{\varphi} = u + e$ where $u \in U(\overline{R})$, $e^2 = e \in \overline{R}$, and $ue = eu$. We just take $\pi = e$. By the same argument as in the second paragraph of the proof of Corollary 5.9, we obtain $\overline{\varphi}e = 0$. (We can't say $e = 0$ since φ might not be a left non-zero-divisor in this case.) Thus $ue = -e$, and so

$$\overline{\varphi} + \pi = u + 2e = u(\overline{1} - e) + e$$

which is a unit with inverse $u^{-1}(\overline{1} - e) + e$, since u and e commute. This yields $(*_1')$. $\square$

9 Exchange Modules

What do the previous results say concerning modules? We have the following unsettling asymmetry, motivated by [9, Proposition 8.11].

Lemma 9.1. *Let M_k be a module and $E = \mathrm{End}(M_k)$ as usual. If E_E is cohopfian, respectively has (C_2), then the same is true for M. The converses do not hold.*

Proof. First, suppose that E_E is cohopfian. Let $x \in E$ be an injective endomorphism on M. If $xr = 0$ for some $r \in E$ then $xr(m) = 0$ for all $m \in M$. But x being injective implies $r(m) = 0$ for all $m \in M$. Therefore $r = 0$. Since r was arbitrary, x is a left non-zero-divisor. The cohopfian condition on E_E then implies x is a unit. This shows that M is cohopfian.

Now suppose instead that E_E has (C_2). Consider the situation where $N' \subseteq M$ and $N' \cong N \subseteq^{\oplus} M$. Let $e \in E$ be an idempotent with $e(M) = N$, and let $\varphi : N \to N'$ be an isomorphism. Without loss of generality, we may assume $\varphi \in E$ by setting φ equal to 0 on $(1-e)(M)$.

Consider the map, $eE \to \varphi eE$, given by left multiplication by φ. Clearly this is surjective. To show injectivity, suppose that $\varphi er = 0$ for some $r \in E$. Then $\varphi er(m) = 0$ for all $m \in M$. In particular, $\varphi(er(M)) = 0$. But $er(M) \subseteq e(M)$ and φ is injective on $e(M) = N$, therefore $er(M) = 0$. But then $er = 0$. This shows injectivity.

Thus φeE is isomorphic to eE, a direct summand of E_E. Therefore φeE is generated by an idempotent, say f. Clearly $f\varphi e = \varphi e$, and $f = \varphi ey$ for some $y \in E$. So $f(M) = \varphi ey(M) \subseteq \varphi e(M) = N'$, and $f(M) \supseteq f(\varphi e(M)) = \varphi e(M) = N'$. Hence $N' = f(M)$ is a direct summand.

A single example will show that both converses do not hold. Let $k = \mathbb{Z}$ and let M be the Prüfer p-group $\mathbb{Z}_{p^\infty}$, for some prime p. Then E is isomorphic to the ring of p-adic integers. The module M_k is cohopfian while E_E is not, by [9, Proposition 8.11]. Notice that the only idempotents in E are 0 and 1. Thus, the only direct summands in either M_k or E_E are the trivial ones. One easily sees that multiplication by p yields $E_E \cong pE$, but pE is not a summand. Therefore E_E does not have the (C_2) property. On the other hand, no proper subgroup of M has infinite length. Thus, the only submodule of M isomorphic to M is M itself, and trivially the only submodule of M isomorphic to (0) is (0). Hence M has the (C_2) property. □

Due to this lemma, it would appear that one could not work with the weaker notion of a cohopfian module and hope to prove a result analogous to Corollary 5.4. However, in an endomorphism ring a limit of units is very special.

Theorem 9.2. *Let M be a cohopfian module with finite exchange. Then M has countable exchange.*

Proof. In the endomorphism ring, E, with the finite topology (as defined in Definition 3.3) a convergent limit of units must be an injective endomorphism (since nothing in the limit process has a kernel). But then the cohopfian condition forces this endomorphism to be an isomorphism, or in other words a unit in E. Thus convergent limits of units are units. Note that $(*)$ holds (since, as explained in the paragraph after Example 5.2, φ is always limit of units) so M has countable exchange by Theorem 5.1. □

While the cohopfian condition was sufficient to show that φ is a unit, it isn't necessary. In the general case, we have an embedding $\varphi : M \to M$. But, since $\varphi = \lim_{n\to\infty} v_n^{-1}$, this embedding is locally split by the units v_n. (By a locally split monomorphism we mean that if we are given a monomorphism $f : A \to B$, then for each finitely generated submodule $B' \subseteq B$ there exists a map $g_{B'} : B \to A$ such that $f \circ g_{B'}|_{B'} = 1|_{B'}$.) In particular, in Theorem 9.2 we could replace the cohopfian condition with the assumption that all locally split monomorphisms (split by units) from M to M are globally split by a unit.

We now ask if one can also tweak Theorems 8.2 and 8.5 so that we are working with the weaker hypothesis that M has the (C_2) property. The answer is yes.

Theorem 9.3. *Let M be a finite exchange module with the (C_2) property, and also be such that $(*_1)$ holds for E in the finite topology. Then M has full exchange.*

Proof. Following Theorem 6.3, with $R = E$, the only thing we need to show is that $(*_2)$ holds.

Consider the map $f'_\alpha(M) \to y'_\alpha f'_\alpha(M)$, given by left multiplication by y'_α. It is clearly surjective. For any $m \in M$ we have

$$f'_\alpha(m) = \lim_{i\to\alpha} f_i f'_\alpha(m) = \lim_{i\to\alpha} r_i y_i f'_\alpha(m) = \lim_{i\to\alpha} r_i y'_\alpha f'_\alpha(m),$$

and so the map above must also be injective. The (C_2) hypothesis now implies $y'_\alpha f'_\alpha(M) = g_\alpha(M)$ for some idempotent g_α.

Define r'_α by the rule $r'_\alpha|_{(1-g_\alpha)(M)} = 0$ and $r'_\alpha|_{g_\alpha(M)} = \lim_{i\to\alpha} r_i|_{y'_\alpha f'_\alpha(M)}$, and extend linearly to M. While it is true that $\lim_{i\to\alpha} r_i$ does not necessarily converge in general, it does converge on $y'_\alpha f'_\alpha(M) = g_\alpha(M)$ since we saw above that $r'_\alpha y'_\alpha f'_\alpha(m) = \lim_{i\to\alpha} r_i y'_\alpha f'_\alpha(m) = f'_\alpha(m)$. (One should also check that r'_α is a well-defined homomorphism, which we leave to the reader.) Since m is arbitrary, $r'_\alpha y'_\alpha f'_\alpha = f'_\alpha$. This gives $(*_2)$. $\square$

Theorems 8.2 and 8.5, and a few corollaries, immediately translate over to the endomorphism ring case. In particular, we have:

Corollary 9.4. *If M has a Dedekind-finite, semi-π-regular endomorphism ring then M has countable exchange. Similarly, if M has finite exchange and $E/J(E)$ is strongly clean, then M has countable exchange. In either of these cases, if we add the hypothesis that M has (C_2) then M has full exchange.*

Corollary 9.5. *If M has a Dedekind-finite, semi-regular endomorphism ring, then M has full exchange.*

Recall we have the isomorphism $y'_\alpha : f'_\alpha M \to y'_\alpha f'_\alpha M$. This induces a monomorphism $f'_\alpha M \to y'_\alpha f'_\alpha M \subseteq M$ which is locally split by the maps $f'_\alpha r_i$. But $(*_2)$ is equivalent to this monomorphism being globally split. So we could replace the (C_2)

hypothesis in Theorem 9.3 and Corollary 9.4 with the weaker assumption that all locally split monomorphisms from a summand of M into M are globally split.

10 Abelian Rings and Commutative Rings

Definition 10.1. Following the literature, as in Lam [8], a ring is said to be *abelian* if all its idempotents are central. A module is *abelian* if its endomorphism ring is abelian.

We can prove abelian modules with finite exchange have full exchange, but the proof doesn't easily translate into the language of linear Hausdorff topologies (since we don't obtain $(*_2)$ in general). To demonstrate this difficulty, suppose R is an exchange ring with a linear Hausdorff Σ-complete topology. As in the proof of Theorem 6.3, we have the equations

$$1 = \sum_{i \leqslant \beta} e_{i,\beta} + f_\beta = \sum_{i \leqslant \beta} e_{i,\beta} + r_\beta \left(\sum_{i \in (\beta,\alpha)} x_i \right) + r_\beta y'_\alpha$$

for each $\beta < \alpha$. Hence, by the exchange property, we can find orthogonal idempotents $a_\beta \in R\left(\sum_{i \leqslant \beta} e_{i,\beta}\right)$, $b_\beta \in R\left(\sum_{i \in (\beta,\alpha)} x_\beta\right)$, and $c_\beta \in Ry'_\alpha$, with $a_\beta + b_\beta + c_\beta = 1$ and $a_\beta \sim_\ell \sum_{i \leqslant \beta} e_{i,\beta}$.

If we assume that R satisfies $(*_1)$ then we can define f'_α, just as before. Now notice that $a_\beta f'_\alpha = 0$ since $e_{i,\beta} f'_\alpha = v_\beta e_{i,i} f'_\alpha = 0$. Also, $\lim_{\beta \to \alpha} \sum_{i \in (\beta,\alpha)} x_i = 0$ and hence $\lim_{\beta \to \alpha} b_\beta = 0$. Thus

$$\lim_{\beta \to \alpha} c_\beta f'_\alpha = \lim_{\beta \to \alpha} a_\beta f'_\alpha + b_\beta f'_\alpha + c_\beta f'_\alpha = f'_\alpha.$$

Writing $c_\beta = s_\beta y'_\alpha$, this says

$$\lim_{\beta \to \alpha} s_\beta y'_\alpha f'_\alpha = f'_\alpha. \tag{10.1}$$

Now suppose that R is abelian. We will show that we get $(*_1)$ for free. Idempotents commute and so $\varphi = \sum_{i < \alpha} e_{i,i}$ is a sum of orthogonal idempotents, and hence φ is an idempotent. We can put $p = 1 - \varphi$, and then $(*_1)$ holds.

What about $(*_2)$? After replacing s_β by $s_\beta y'_\alpha s_\beta$ if necessary, we have $c_\beta = s_\beta y'_\alpha = y'_\alpha s_\beta$. And hence equation (10.1) yields

$$f'_\alpha = \lim_{\beta \to \alpha} f'_\alpha s_\beta f'_\alpha y'_\alpha f'_\alpha = \lim_{\beta \to \alpha} f'_\alpha y'_\alpha f'_\alpha s_\beta f'_\alpha.$$

At this point we don't obtain $(*_2)$. However, if we further assume that $R = \mathrm{End}(M)$ is an endomorphism ring in the finite topology, then the equation $f'_\alpha = \lim_{\beta\to\alpha} f'_\alpha s_\beta f'_\alpha y'_\alpha f'_\alpha$ implies that $f'_\alpha y'_\alpha f'_\alpha$ is an injective endomorphism on the $f'_\alpha(M)$. Similarly, the equation $f'_\alpha = \lim_{\beta\to\alpha} f'_\alpha y'_\alpha f'_\alpha s_\beta f'_\alpha$ implies $f'_\alpha y'_\alpha f'_\alpha$ is surjective on $f'_\alpha(M)$. Therefore $f'_\alpha y'_\alpha f'_\alpha \in U(f'_\alpha R f'_\alpha)$. We just let r'_α be the inverse (in the corner ring $f'_\alpha R f'_\alpha$) and then $(*_2)$ holds. This yields:

Proposition 10.2 ([15, Theorem 2])**.** *An abelian module with finite exchange has full exchange.*

More generally, let I be a directed set and let R be a ring with a linear Hausdorff topology. Suppose we are given elements $s_i \in R$ for each $i \in I$. We can ask what hypotheses need to be placed on R so that any element $r \in R$ satisfying the equations

$$1 = \lim_{i\in I} r s_i = \lim_{i\in I} s_i r$$

must be a unit. We saw above that it sufficed to assume R is an endomorphism ring in the finite topology (since in that case r is injective and surjective). Unfortunately even if we assume $R = \mathrm{End}(M)/J(\mathrm{End}(M))$, using the finite topology on $\mathrm{End}(M)$ and then the quotient topology on R, it doesn't appear that these equations imply $r \in U(R)$.

There is another avenue we can explore if we assume a little more than R being abelian.

Lemma 10.3. *Let R be a ring with a complete, linear Hausdorff topology. If every open neighborhood of zero contains an open (two-sided) ideal, then convergent limits of units are units.*

Proof. Let $u = \lim_{i\in I} u_i$ be a limit in R, with I a directed set and $u_i \in U(R)$. Let U be an arbitrary, open (left ideal) neighborhood of 0, and let $V \subseteq U$ be an open two-sided ideal. There is some $k \in I$ so that for all $j \geqslant k$ we have $u_j - u_k \in V$. Left multiply by u_j^{-1} and right multiply by u_k^{-1} to obtain

$$u_k^{-1} - u_j^{-1} = u_j^{-1}(u_j - u_k)u_k^{-1} \in u_j^{-1} V u_k^{-1} = V \subseteq U.$$

Therefore, $\{u_i^{-1}\}_{i\in I}$ is a Cauchy system. Since the topology is complete, $v = \lim_{i\in I} u_i^{-1}$ exists. Then one verifies

$$uv = \lim_{i\in I} u_i \lim_{i\in I} u_i^{-1} = \lim_{i\in I} u_i u_i^{-1} = 1$$

and similarly $vu = 1$. Therefore, $u \in U(R)$ as claimed. $\square$

With all this talk of convergent limits, one might expect to use Theorem 5.1 to show countable exchange. However, we can do better.

Proposition 10.4. *Let R be a commutative exchange ring with a complete, linear Hausdorff topology. Then R is a full exchange ring.*

Proof. First notice that for any idempotent $e \in R$ the corner ring eRe is a ring with a complete, linear Hausdorff topology. Second, it is well known that commutative exchange rings are strongly clean. From the argument above we see that $(*_1)$ holds. Therefore, it suffices to show that $(*_2)$ holds.

Using the notation of Theorem 6.3, we have

$$y'_\alpha f'_\alpha = \lim_{i\to\alpha} y_i f'_\alpha$$

where $y_i f'_\alpha \in U(f'_\alpha R f'_\alpha)$ with inverse $f'_\alpha r_i$. (Here we are using the fact that $r_i y_i = f_i \sim_r f'_\alpha$, and that idempotents commute.) Thus $y'_\alpha f'_\alpha$ is a limit of units, and hence is a unit by Lemma 10.3, in the corner ring $U(f'_\alpha R f'_\alpha)$. But we then just set $r'_\alpha = (f'_\alpha y'_\alpha f'_\alpha)^{-1}$ (again in the corner ring) and we have $(*_2)$ holding. $\square$

If a family $\{x_i\}_{i\in I}$ is left multiple summable in R, then the same holds true of the family $\{\overline{x}_i\}_{i\in I}$ in $R/J(R)$. However, in general the completeness of a topology does not pass to quotient rings, and so we can't lift the previous proposition through the radical.

Chapter 2

From a Module-theoretic Point of View

11 A Theorem Derived from a Result of Kaplansky

It is well known that projective modules can be written as direct sums of countably generated submodules, and this was first proven by Kaplansky [6, Theorem 1]. In the proof he shows that a summand of a direct sum of countably generated modules is still a direct sum of countably generated modules. In fact, he generalizes the argument to a proof that a summand of a direct sum of $\aleph$-generated modules (where $\aleph$ is any infinite cardinal) is still a direct sum of $\aleph$-generated modules.

One can generalize Kaplansky's theorem even further by defining the notions of $\aleph$-small and $\aleph$-closed. Following Facchini [5], we say that M is $\aleph$-*small* if, for every family of modules $\{A_i \mid i \in I\}$, and every homomorphism $\varphi : M \to \bigoplus_{i\in I} A_i$, the set $\{j \in I \mid p_j\varphi(M) \neq 0\}$ has cardinality no bigger than $\aleph$ (where $p_j : \bigoplus_{i\in I} A_i \to A_j$ is the canonical projection). In other words, M can hit at most $\aleph$ many summands of any direct sum. For example, finitely generated and countably generated modules are $\aleph_0$-small. If $\aleph$ is a finite cardinal, then the only module that is $\aleph$-small is the zero module.

We say that a non-empty class of modules, $\mathfrak{M}$, is $\aleph$-*closed* if it satisfies the following three conditions:

(i) If $M \in \mathfrak{M}$ then M is $\aleph$-small.

(ii) If $M \in \mathfrak{M}$ and there exists an epimorphism $\varphi : M \twoheadrightarrow N$, then $N \in \mathfrak{M}$.

(iii) If I is an index set with $|I| \leqslant \aleph$, and $M_i \in \mathfrak{M}$ for every $i \in I$, then $\bigoplus_{i\in I} M_i \in \mathfrak{M}$.

Notice that the class of $\aleph$-generated modules is $\aleph$-closed, as long as $\aleph$ is an infinite cardinal. For more examples of $\aleph$-closed classes see [5, § 2.9]. Kaplansky's theorem can now be recast in the form:

Proposition 11.1 ([5, Theorem 2.47]). *Let $\mathfrak{M}$ be an $\aleph$-closed class. If M is a direct sum of modules belonging to $\mathfrak{M}$ then every summand of M is also a direct sum of modules belonging to $\mathfrak{M}$.*

We are now ready to state and prove our main result of this chapter, which is in the spirit of both Kaplansky's theorem and Crawley and Jónsson's original paper on exchange [3]. After proving our result, we learned of a very similar though not quite as general result of Stock, which seems to have been neglected in the literature. We include our proof here, both to generalize and publicize his theorem.

Theorem 11.2 (cf. [17, Proposition 2.1]). *Let $\aleph$ be a cardinal, and let $\mathfrak{M}$ be an $\aleph$-closed class. Suppose $M = \bigoplus_{j\in J} M_j$, with $M_j \in \mathfrak{M}$ for all $j \in J$. Let $\aleph'$ be another cardinal. If each direct summand in M that lies in $\mathfrak{M}$ has $\aleph'$-exchange, then M has $\aleph'$-exchange.*[1]

Proof. If $\aleph$ is finite, then $M = (0)$, and everything is trivial. So, we may assume that $\aleph$ is an infinite cardinal. We may also suppose $\aleph' \geqslant 2$. Write $M = \bigoplus_{j\in J} M_j$ with $M_j \in \mathfrak{M}$. By Proposition 2.7 above, to show that M has $\aleph'$-exchange it suffices to consider the case $A = M \oplus N = \bigoplus_{i\in I} A_i$, with $A_i \cong M$ for each $i \in I$, and with $|I| \leqslant \aleph'$. Since M is a direct sum of members of $\mathfrak{M}$, and $A_i \cong M$, condition (ii) of the definition of an $\aleph$-closed class implies that A_i is a direct sum of members of $\mathfrak{M}$. So, for each $i \in I$ there is an indexing set, Λ_i, such that $A_i = \bigoplus_{\lambda\in\Lambda_i} A_\lambda$, with $A_\lambda \in \mathfrak{M}$ for each $\lambda \in \Lambda_i$. Furthermore, A is now a direct sum of members of $\mathfrak{M}$, and hence so is N by Proposition 11.1. Therefore, we may write $N = \bigoplus_{k\in K} N_k$, with $N_k \in \mathfrak{M}$ for all $k \in K$.

We may suppose that the sets Λ_i are disjoint, and put $\Lambda = \bigcup_{i\in I} \Lambda_i$. Throughout this proof we will repeatedly use two direct sum decompositions of A, namely $A = \bigoplus_{\lambda\in\Lambda} A_\lambda = (\bigoplus_{j\in J} M_j) \oplus (\bigoplus_{k\in K} N_k)$. For notational ease, we will let π_λ

[1] In the theorem, we could have weakened the hypothesis "each direct summand in M that lies in $\mathfrak{M}$ has $\aleph'$-exchange" to "for all $\overline{J} \subseteq J$ with $|\overline{J}| \leqslant \aleph$, the sum $\bigoplus_{j\in\overline{J}} M_j$ has $\aleph'$-exchange." However, by a simple application of [5, Proposition 2.50], it turns out these are equivalent conditions.

(respectively π_j, π_k) denote the projection to A_λ (respectively M_j, N_k) in the corresponding direct sum decomposition of A.

Consider tuples of the form

$$(J', K', \Lambda', (A_i')_{i\in I})$$

where $J' \subseteq J$, $K' \subseteq K$, $\Lambda' \subseteq \Lambda$, and for each $i \in I$, $A_i' \subseteq \bigoplus_{\lambda\in\Lambda_i'} A_\lambda \subseteq A_i$ where $\Lambda_i' = \Lambda' \cap \Lambda_i$. Further, consider the following equalities:

$$\left(\bigoplus_{j\in J'} M_j\right) \oplus \left(\bigoplus_{k\in K'} N_k\right) = \left(\bigoplus_{j\in J'} M_j\right) \oplus \left(\bigoplus_{i\in I} A_i'\right) = \bigoplus_{\lambda\in\Lambda'} A_\lambda. \tag{11.1}$$

Let $\mathfrak{S}$ be the set of all such tuples, where the equalities in equation (11.1) hold. We can partially order $\mathfrak{S}$ by setting

$$(J', K', \Lambda', (A_i')_{i\in I}) \leqslant (J'', K'', \Lambda'', (A_i'')_{i\in I})$$

if and only if $J' \subseteq J''$, $K' \subseteq K''$, $\Lambda' \subseteq \Lambda''$, and $A_i' \subseteq A_i''$ for each $i \in I$.

We now Zornify. First note that $\mathfrak{S} \neq \emptyset$ since the trivial tuple lies in $\mathfrak{S}$. Further, the union of the elements of any chain still lies in $\mathfrak{S}$, since the equalities in equation (11.1) rely only on finitistic information. Thus, all chains have upper-bounds, and so $\mathfrak{S}$ has maximal elements. Let $(J^*, K^*, \Lambda^*, (A_i^*)_{i\in I})$ be such a maximal element. We have

$$\left(\bigoplus_{j\in J^*} M_j\right) \oplus \left(\bigoplus_{k\in K^*} N_k\right) = \left(\bigoplus_{j\in J^*} M_j\right) \oplus \left(\bigoplus_{i\in I} A_i^*\right) = \bigoplus_{\lambda\in\Lambda^*} A_\lambda. \tag{11.2}$$

First suppose that $J^* = J$. In this case, adding $\bigoplus_{\lambda\in\Lambda\setminus\Lambda^*} A_\lambda$ to the middle and right hand side of equation (11.2) shows that M has $\aleph'$-exchange. So, we may assume $J^* \neq J$, and we will derive a contradiction.

Let J_0 be some fixed, non-empty subset of $J\setminus J^*$, with $|J_0| \leqslant \aleph$. Since $\bigoplus_{j\in J_0} M_j \in \mathfrak{M}$ we have $\pi_\lambda(\bigoplus_{j\in J_0} M_j) = 0$ for all but at most $\aleph$ many indices λ. So we have $\bigoplus_{j\in J_0} M_j \subseteq \bigoplus_{\lambda\in\Lambda_0} A_\lambda$ where $\Lambda_0 \subseteq \Lambda$, with $|\Lambda_0| \leqslant \aleph$.

Now, $\pi_j\left(\bigoplus_{\lambda\in\Lambda_0} A_\lambda\right)$ is non-zero for at most $\aleph$ many $j \in J$ (and similarly for π_k). Therefore, we have

$$\bigoplus_{\lambda\in\Lambda_0} A_\lambda \subseteq \left(\bigoplus_{j\in J_1} M_j\right) \oplus \left(\bigoplus_{k\in K_1} N_k\right)$$

for some sets J_1 and K_1 of cardinality $\leqslant \aleph$.

Continuing in this manner, for each $n \in \mathbb{N}$ we have sets J_n, K_n, and Λ_n, each of cardinality $\leqslant \aleph$, satisfying

$$\left(\bigoplus_{j\in J_n} M_j\right) \oplus \left(\bigoplus_{k\in K_n} N_k\right) \subseteq \bigoplus_{\lambda\in\Lambda_n} A_\lambda \subseteq \left(\bigoplus_{j\in J_{n+1}} M_j\right) \oplus \left(\bigoplus_{k\in K_{n+1}} N_k\right). \tag{11.3}$$

(We put $K_0 = \emptyset$.) Set $J_\infty = \bigcup_{n\in\mathbb{N}} J_n$, and define K_∞ and Λ_∞ similarly. Then after taking unions, the containments of formula (11.3) imply

$$\left(\bigoplus_{j\in J_\infty} M_j\right) \oplus \left(\bigoplus_{k\in K_\infty} N_k\right) = \bigoplus_{\lambda\in\Lambda_\infty} A_\lambda. \tag{11.4}$$

Put $J^{**} = J^* \cup J_\infty$, and define K^{**} and Λ^{**} similarly. Then, adding the respective sides of equations (11.2) and (11.4) to each other, we obtain

$$\left(\bigoplus_{j\in J^{**}} M_j\right) \oplus \left(\bigoplus_{k\in K^{**}} N_k\right) = \bigoplus_{\lambda\in\Lambda^{**}} A_\lambda. \tag{11.5}$$

Let $B = \bigoplus_{\lambda\in\Lambda^{**}} A_\lambda$. Equations (11.2) and (11.5) then imply

$$\begin{aligned} B &= \left(\bigoplus_{j\in J^*} M_j\right) \oplus \left(\bigoplus_{i\in I} A_i^*\right) \oplus \left(\bigoplus_{j\in J^{**}\backslash J^*} M_j\right) \oplus \left(\bigoplus_{k\in K^{**}\backslash K^*} N_k\right) \\ &= \left(\bigoplus_{j\in J^*} M_j\right) \oplus \left(\bigoplus_{i\in I} A_i^*\right) \oplus \left(\bigoplus_{\lambda\in\Lambda^{**}\backslash\Lambda^*} A_\lambda\right). \end{aligned} \tag{11.6}$$

We are almost ready to use the exchange hypothesis. We have $|J^{**}\backslash J^*| \leqslant \aleph$ so that $\bigoplus_{j\in J^{**}\backslash J^*} M_j \in \mathfrak{M}$ has $\aleph'$-exchange. For each $i \in I$, put $\overline{\Lambda}_i = (\Lambda^{**}\backslash\Lambda^*) \cap \Lambda_i$, and put $\overline{A}_i = \bigoplus_{\lambda\in\overline{\Lambda}_i} A_\lambda \subseteq A_i$. We can rewrite equation (11.6) in the form:

$$\begin{aligned} B &= \left(\bigoplus_{j\in J^*} M_j\right) \oplus \left(\bigoplus_{i\in I} A_i^*\right) \oplus \left(\bigoplus_{j\in J^{**}\setminus J^*} M_j\right) \oplus \left(\bigoplus_{k\in K^{**}\setminus K^*} N_k\right) \\ &= \left(\bigoplus_{j\in J^*} M_j\right) \oplus \left(\bigoplus_{i\in I} A_i^*\right) \oplus \left(\bigoplus_{i\in I} \overline{A}_i\right). \end{aligned} \tag{11.7}$$

Applying Lemma 2.3 to equation (11.7), with

$$X = \left(\bigoplus_{j\in J^*} M_j\right) \oplus \left(\bigoplus_{i\in I} A_i^*\right),$$

we obtain

$$B = \left(\bigoplus_{j\in J^*} M_j\right) \oplus \left(\bigoplus_{i\in I} A_i^*\right) \oplus \left(\bigoplus_{j\in J^{**}\setminus J^*} M_j\right) \oplus \left(\bigoplus_{i\in I} \overline{A_i'}\right) \tag{11.8}$$

for some $\overline{A_i'} \subseteq \overline{A}_i$ (for each $i \in I$).

Finally, for each $i \in I$ put $A_i^{**} = A_i^* \oplus \overline{A_i'}$. Then equations (11.5) and (11.8) say that

$$\left(\bigoplus_{j\in J^{**}} M_j\right) \oplus \left(\bigoplus_{k\in K^{**}} N_k\right) = \left(\bigoplus_{j\in J^{**}} M_j\right) \oplus \left(\bigoplus_{i\in I} A_i^{**}\right) = \bigoplus_{\lambda\in\Lambda^{**}} A_\lambda.$$

Therefore, the four-tuple $(J^{**}, K^{**}, \Lambda^{**}, (A_i^{**})_{i\in I})$ lies is $\mathfrak{S}$, contradicting the maximality of $(J^*, K^*, \Lambda^*, (A_i^*)_{i\in I})$ (since $J^{**} \supseteq J^* \cup J_0 \supsetneq J^*$). We conclude that $J^* = J$ after all, and M has $\aleph'$-exchange. $\square$

Corollary 11.3. *Let $\aleph$ be an infinite cardinal. Let M be a direct sum of members of $\mathfrak{M}$, an $\aleph$-closed class. If every summand of M that lies in $\mathfrak{M}$ has $\aleph$-exchange then M has full exchange.*

Proof. By Theorem 11.2, it suffices to show that a module in $\mathfrak{M}$ with $\aleph$-exchange has full exchange. So we may assume M is $\aleph$-small and has $\aleph$-exchange.

Suppose $A = M \oplus N = \bigoplus_{i \in I} A_i$. Since M is $\aleph$-small, there is an subset $J \subseteq I$ of cardinality $|J| \leqslant \aleph$, such that $M \subseteq \bigoplus_{i \in J} A_i$. In particular there is a submodule N' such that $M \oplus N' = \bigoplus_{i \in J} A_i$. Let $X = \bigoplus_{i \in I \setminus J} A_i$. Then $A = M \oplus N' \oplus X = \left(\bigoplus_{i \in J} A_i\right) \oplus X$. By Lemma 2.3, it suffices to show that M has the exchange property in the decomposition $M \oplus N' = \bigoplus_{i \in J} A_i$. But this is clear since M has $\aleph$-exchange and $|J| \leqslant \aleph$. $\square$

Corollary 11.4. *If M is a direct sum of countably generated modules, and M has countable exchange, then M has full exchange.*

Proof. If M has countable exchange then so does each of its summands. But then, by the previous corollary, M has full exchange. $\square$

Corollary 11.5 ([17, Korollar 2.2])**.** *Any projective module with countable exchange has full exchange.*

Proof. This follows from Corollary 11.4, since all projective modules are direct sums of countably generated modules. $\square$

Let R be a ring. Following Stock, we say R is a *right P-exchange ring* if every projective right R-module has full exchange. In particular, we see that every right P-exchange ring is an exchange ring (but not conversely).

Proposition 11.6. *A ring R is a right P-exchange ring if and only if $R_R^{(\mathbb{N})}$ has countable exchange.*

Proof. One direction is trivial. For the other, suppose that $R_R^{(\mathbb{N})}$ has countable exchange. Then any countably generated projective R-module has countable exchange (being a direct summand of $R_R^{(\mathbb{N})}$). Since any projective is a direct sum of countably generated projectives, Theorem 11.2 tells us that all projective R-modules have countable exchange. Corollary 11.5 then says that they all have full exchange. $\square$

Let R be a ring. We call R a *right finite-P-exchange ring* if all projective right R-modules have finite exchange. Such rings are also called *weakly-P-exchange rings* in the literature. We can immediately give some equivalent conditions, paralleling Proposition 11.6.

Proposition 11.7. *Let R be a ring. The following are equivalent:*
(1) The ring R is a right finite-P-exchange ring.
(2) The module $R_R^{(\mathbb{N})}$ has finite exchange.
(3) The ring of $\mathbb{N} \times \mathbb{N}$ column finite matrices over R is an exchange ring.

Proof. Property (1) is equivalent to (2) using exactly the same reasoning as in Proposition 11.6. Property (2) is equivalent to (3) since $\mathrm{End}(R_R^{(\mathbb{N})})$ is isomorphic to the ring of column finite matrices over R. $\square$

12 Property $(N)_\aleph$

Following Stock, we say that a module, M, has property $(N)_\aleph$ if for each decomposition $M = \sum_{i\in I} M_i$, with $|I| \leqslant \aleph$, there are modules $N_i \subseteq M_i$ such that $M = \bigoplus_{i\in I} N_i$. If this property holds for all cardinalities, $\aleph$, we say that M has property (N). For projective modules, property $(N)_2$ is equivalent to finite exchange by a result of Nicholson [13, Proposition 2.9]. Stock asked whether or not property (N) is equivalent to full exchange (for projective modules). In general, there are no known implications either way, although Stock claimed without proof that these two properties are equivalent for free modules. We can give a generalization of this result, but first we need a few lemmas.

Lemma 12.1. *Let P be a projective module with $P^{(\mathbb{N})} \cong P$. The following are equivalent:*
(1) The module P has countable exchange.
(2) The module P has property $(N)_{\aleph_0}$.

Proof. This is a specialization of the proof of [18, Proposition 2.6]. □

Lemma 12.2 ([18, Lemma 2.5])**.** *Let A be a projective module, and suppose Q has property $(N)_\aleph$. Let I be an indexing set with $|I| \leqslant \aleph$. If we have $A = P \oplus Q = \sum_{i\in I} A_i$ then there are submodules $A_i' \subseteq A_i$ with $A = P \oplus \bigoplus_{i\in I} A_i'$.*

Proposition 12.3. *For projective modules, property $(N)_{\aleph_0}$ is equivalent to property (N).*

Proof. We mimic the proof of Theorem 11.2. Let P be a projective module with property $(N)_{\aleph_0}$. We want to show that P has property (N), so we consider the situation $P = \sum_{i\in I} A_i$. Refining this sum if necessary, we may suppose each A_i is cyclic. We can write $P = \bigoplus_{j\in J} P_j$ with P_j countably generated.

Now, consider tuples $(J', I', (A_i')_{i\in I'})$ where $J' \subseteq J$, $I' \subseteq I$, $A_i' \subseteq A_i$, and where we have the equalities

$$\bigoplus_{j\in J'} P_j = \bigoplus_{i\in I'} A_i' = \sum_{i\in I'} A_i.$$

Let $\mathfrak{S}$ be the set of all such tuples. We order them by saying

$$(J', I', (A'_i)_{i\in I'}) \leqslant (J'', I'', (A''_i)_{i\in I''})$$

if and only if $J' \subseteq J''$, $I' \subseteq I''$, and $A'_i \subseteq A''_i$ for each $i \in I'$. Note that $\mathfrak{S}$ is non-empty, since it contains the trivial tuple. Further, the union of any chain in $\mathfrak{S}$ again lies in $\mathfrak{S}$. Therefore, there is a maximal tuple $(J^*, I^*, (A^*_i)_{i\in I^*})$, satisfying

$$P^* := \bigoplus_{j\in J^*} P_j = \bigoplus_{i\in I^*} A^*_i = \sum_{i\in I^*} A_i. \tag{12.1}$$

If $J^* = J$, then let $A^*_i = 0$ for $i \in I\backslash I^*$, and we have $P = \bigoplus_{i\in I} A^*_i$, which shows P has property (N). So, we may assume $J^* \neq J$.

Let J_0 be a countable, non-empty subset of $J\backslash J^*$. Then there is a countable subset $I_0 \subseteq I$ with $\bigoplus_{j\in J_0} P_j \subseteq \sum_{i\in I_0} A_i$. But since each A_i is cyclic, there is a countable set $J_1 \subseteq J$ with $\sum_{i\in I_0} A_i \subseteq \bigoplus_{j\in J_1} P_j$.

Repeating this process, for each $n \in \mathbb{N}$ we have countable sets J_n, and I_n, such that the following containments hold

$$\bigoplus_{j\in J_n} P_j \subseteq \sum_{i\in I_n} A_i \subseteq \bigoplus_{j\in J_{n+1}} P_j. \tag{12.2}$$

Let $J_\infty = \bigcup_{n\in\mathbb{N}} J_n$ and define I_∞ similarly. Then, after taking unions, the containments in equation (12.2) imply $\bigoplus_{j\in J_\infty} P_j = \sum_{i\in I_\infty} A_i$. Combining this equality with equation (12.1) yields

$$P^* \oplus \left(\bigoplus_{j\in J_\infty\backslash J^*} P_j \right) = P^* + \sum_{i\in I_\infty\backslash I^*} A_i. \tag{12.3}$$

By Lemma 12.2, there exist submodules $\overline{A_i} \subseteq A_i$ (for each $i \in I_\infty\backslash I^*$) such that

$$P^* \oplus \left(\bigoplus_{j\in J_\infty\backslash J^*} P_j \right) = P^* \oplus \left(\bigoplus_{i\in I_\infty\backslash I^*} \overline{A_i} \right). \tag{12.4}$$

Set $J^{**} = J^* \cup J_\infty$ and $I^{**} = I^* \cup I_\infty$. For each $i \in I^*$ put $A_i^{**} = A_i^*$, and for $i \in I_\infty \backslash I^*$ put $A_i^{**} = \overline{A_i}$. Then, noting $P^* = \bigoplus_{i \in I^*} A_i^*$, equations (12.3) and (12.4) imply

$$\bigoplus_{j \in J^{**}} P_j = \bigoplus_{i \in I^{**}} A_i^{**} = \sum_{i \in I^{**}} A_i.$$

This contradicts the maximality of $(J^*, I^*, (A_i^*)_{i \in I'})$ and so we are done. $\square$

Theorem 12.4. *Let P be a projective module with $P^{(\mathbb{N})} \cong P$. Then P has property (N) if and only if P has full exchange. In particular, the exchange property and property (N) are equivalent for infinite rank free modules.*

Proof. The module P has full exchange if and only if P has countable exchange (by Corollary 11.5), if and only if $P \cong P^{(\mathbb{N})}$ has property $(N)_{\aleph_0}$ (by Lemma 12.1), if and only if P has property (N) (by Proposition 12.3). $\square$

Lemma 12.2 also holds for self-projective modules (which are defined in the next section) by [19, Theorem 3.2]. So, we could generalize Proposition 12.3 further, to show that for self-projective modules which are direct sums of $\aleph$-small modules, then the property $(N)_\aleph$ implies property (N).

13 Removing the Radical

In the previous section we reduced the question of whether or not a projective module, P, has full exchange to the question of whether or not P has countable exchange. In this section we will make a further reduction by showing one may "mod out" by the radical.

In the literature, a module, M, is said to be *self-projective* (sometimes called *quasi-projective*) if given an epimorphism $f : M \twoheadrightarrow L$ and a homomorphism $g : M \to L$, there exists a homomorphism $h : M \to M$ such that $fh = g$ (here L is arbitrary). In terms of diagrams, the following is commutative:

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists h"'] \arrow[d, "g"] \\ M \arrow[r, two heads, "f"'] & L. \end{tikzcd}$$

We have the following nice result concerning the endomorphism ring of a self-projective module. (This proposition uses small submodules, and the radical of a module, as defined in Section 1.)

Proposition 13.1. *Let M_R be a self-projective module. Let $J(E)$ be the Jacobson radical of $E = \operatorname{End}(M_R)$. Also let $I = \{f \in E |\ f(M) \subseteq_s M\}$. Then the following hold:*

(1) $J(E) = I$.

(2) $J(E) \subseteq \operatorname{Hom}_R(M, \operatorname{rad}(M))$.

(3) There exists a canonical ring surjection $\varphi : E \twoheadrightarrow \operatorname{End}((M/\operatorname{rad}(M))_R)$ with $\ker(\varphi) = \operatorname{Hom}_R(M, \operatorname{rad}(M))$.

Further, if $\operatorname{rad}(M) \subseteq_s M$ then

(4) $J(E) = \operatorname{Hom}_R(M, \operatorname{rad}(M))$.

(5) There exists a ring isomorphism $E/J(E) \cong \operatorname{End}(M/\operatorname{rad}(M))$.

Proof. The proof is similar to [21, Proposition 1.1], but for completeness we include it here.

Part (1): Let $f \in J(E)$, and let K be a submodule of M with $f(M) + K = M$. Let π be the natural surjection $M \twoheadrightarrow M/K$. Notice that πf is a surjection, since $f(M)+K = M$. By self-projectivity, we have a filler in the following commutative diagram:

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists\, g"'] \arrow[d, "\pi"] \\ M \arrow[r, two heads, "\pi f"'] & M/K. \end{tikzcd}$$

For each $m \in M$, $\pi(m) = \pi f g(m)$. So $\pi(1 - fg)(m) = 0$, and this means $(1 - fg)(m) \in K$. Thus $(1 - fg)(M) \subseteq K$. But $f \in J(E)$ so $1 - fg \in U(E)$. Therefore $K = M$. Since K was arbitrary, we have $f(M) \subseteq_s M$ and hence $f \in I$. This shows $J(E) \subseteq I$.

Conversely, let $f \in I$. Let $g \in E$ be arbitrary. We will show that $1 - fg \in U(E)$, and hence $f \in J(E)$. First, notice that $f(M) + (1 - fg)(M) = M$. Since $f(M) \subseteq_s M$ we have $(1 - fg)(M) = M$. Thus $1 - fg$ is surjective, and so the self-projectivity hypothesis gives us a filler to the commutative diagram:

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists\, h"'] \arrow[d, "1"] \\ M \arrow[r, two heads, "1-fg"'] & M. \end{tikzcd}$$

In other words, the surjection $1 - fg$ splits via h. Let $L = \ker(1 - fg) \subseteq^{\oplus} M$. Then for each $x \in L$, $(1 - fg)(x) = 0$ so $x = fg(x)$. In particular, $L \subseteq f(M)$. But small modules cannot contain non-zero summands, and hence $L = 0$. Therefore $1 - fg$ is an isomorphism, or in other words $1 - fg \in U(E)$ as claimed.

Part (2): We know $\operatorname{rad}(M)$ is the sum of all small submodules of M. Therefore, $f \in J(E) = I$ implies $f(M) \subseteq_s M$, and so $f \in \operatorname{Hom}_R(M, \operatorname{rad}(M))$.

Part (3): We use the easy fact that for every module homomorphism $\psi : M \to N$

the inclusion $\psi(\mathrm{rad}(M)) \subseteq \mathrm{rad}(N)$ holds. (One only need prove that the image of a small submodule must be small.) Therefore, taking $f \in E$, the map $\overline{f} : M/\mathrm{rad}(M) \to M/\mathrm{rad}(M)$ given by $m + \mathrm{rad}(M) \mapsto f(m) + \mathrm{rad}(M)$ is well-defined. So we have a map $\varphi : E \to \mathrm{End}(M/\mathrm{rad}(M))$ defined by $f \mapsto \overline{f}$. It is straightforward to check that this is a ring homomorphism. Notice that this map does not depend on any properties of M. Letting $\pi : M \twoheadrightarrow M/\mathrm{rad}(M)$ be the natural surjection, also notice that $\overline{f}$ is the unique map making the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ M/\mathrm{rad}(M) & \xrightarrow{\overline{f}} & M/\mathrm{rad}(M). \end{array}$$

However, to show that φ is surjective, we need the hypothesis that M is self-projective. Fix $\theta \in \mathrm{End}(M/\mathrm{rad}(M))$. By self-projectivity, there is a filler:

$$\begin{tikzcd} & M \arrow[d, "\pi"] \arrow[ddl, dashed, "\exists f"'] \\ & M/\mathrm{rad}(M) \arrow[d, "\theta"] \\ M \arrow[r, two heads, "\pi"'] & M/\mathrm{rad}(M). \end{tikzcd}$$

This gives surjectivity of φ, since $\overline{f} = \varphi(f) = \theta$, by the uniqueness property of $\overline{f}$.

Finally, $f \in \ker(\varphi)$ if and only if $\overline{f} = 0$, if and only if $f(M) \subseteq \mathrm{rad}(M)$, if and only if $f \in \mathrm{Hom}_R(M, \mathrm{rad}(M))$.

Part (4): Immediate from (1) and (2), since if $\mathrm{rad}(M) \subseteq_s M$ then $\mathrm{Hom}_R(M, \mathrm{rad}(M)) \subseteq I$.

Part (5): Immediate from (3) and (4). □

Lemma 13.2. *Let M be a self-projective module with small radical. Put $E = \mathrm{End}(M)$ and $E' = \mathrm{End}(M/\mathrm{rad}(M))$, both endowed with the finite topology. Also,* *give $E/J(E)$ the quotient topology. Let $\varphi : E/J(E) \to E'$ be the isomorphism of part (5) of the previous proposition. The map φ is continuous.*

Proof. It is well known that to show φ is continuous it suffices to show that for each open neighborhood U of zero in E', there is an open neighborhood V of zero in $E/J(E)$ such that $\varphi^{-1}(U) \supseteq V$. Since E' is given the finite topology with respect to the structure $E' = \mathrm{End}(M/\mathrm{rad}(M))$ we see that we may take $U = \mathrm{ann}_{E'}(m_1 + \mathrm{rad}(M), \ldots, m_n + \mathrm{rad}(M))$ for some elements $m_1, \ldots, m_n \in M$. But the set $W = \mathrm{ann}_E(m_1, \ldots, m_n)$ is open in E, and so $V = W + J(E)$ is open in $E/J(E)$. One easily checks $\varphi(V) \subseteq U$, or in other words $V \subseteq \varphi^{-1}(U)$. $\square$

We are now ready to prove:

Theorem 13.3. *Let M be a self-projective module with small radical. Suppose M has finite exchange. If $M/\mathrm{rad}(M)$ has $\aleph$-exchange then M has $\aleph$-exchange.*

Proof. Let I be an indexing set with $|I| \leqslant \aleph$. Let $\{x_i\}_{i\in I}$ be a summable family of endomorphisms in $E = \mathrm{End}(M)$, summing to 1. We then know $\{x_i + J(E)\}_{i\in I}$ sums to $1 + J(E)$ in $E/J(E)$, and hence $\{\varphi(x_i)\}_{i\in I}$ sums to 1 in E' by the previous lemma, where φ is the map of Proposition 13.1, part (3). Since $M/\mathrm{rad}(M)$ has $\aleph$-exchange, we can find idempotents $\varepsilon_i \in E'\varphi(x_i)$ for each $i \in I$, with $\sum_{i\in I} \varepsilon_i = 1$. So, $\{\varphi^{-1}(\varepsilon_i)\}_{i\in I}$ is a family of orthogonal idempotents. Notice $\varphi^{-1}(\varepsilon_i) \in (E/J(E))(x_i + J(E))$, and since the family $\{x_i\}_{i\in I}$ is left multiple summable, the family $\{\varphi^{-1}(\varepsilon_i)\}_{i\in I}$ is also summable. Let $x = \sum_{i\in I} \varphi^{-1}(\varepsilon_i)$. Since φ is continuous we know that φ preserves limits and hence sums, so that

$$\varphi(x) = \sum_{i\in I} \varphi(\varphi^{-1}(\varepsilon_i)) = \sum_{i\in I} \varepsilon_i = 1,$$

whence $x = \varphi^{-1}(1) = 1 + J(E) \in E/J(E)$.

We now copy what was done at the end of the proof of Theorem 7.2. By Lemma 7.1, we can lift each $\varphi^{-1}(\varepsilon_i)$ to an idempotent $e_i' \in Ex_i$. These are still summable idempotents (because the x_i are summable), summing to a unit (since modulo $J(E)$ they sum to $1 + J(E)$). Letting $u = \sum_{i\in I} e'_i$, and giving I an arbitrary well-ordering, Proposition 4.8 says that $\{u^{-1}e'_i\}_{i\in I}$ is a summable family of orthogonal idempotents summing to 1. Clearly, $e_i = u^{-1}e'_i \in Ex_i$, so we are done. □

Lemma 13.4. *Let $\mathcal{P}$ be a property of modules that passes to direct summands, and forces a non-zero module to have a maximal submodule. If M has property $\mathcal{P}$ then* $\mathrm{rad}(M)$ *contains no non-zero direct summand of M.*

Proof. Suppose by contradiction we have $0 \neq L \subseteq \mathrm{rad}(M)$ with $L \subseteq^{\oplus} M$. Then L has property $\mathcal{P}$, and $\mathrm{rad}(L) = L \cap \mathrm{rad}(M) = L$. But this implies that L has no maximal submodule, contradicting property $\mathcal{P}$. □

In the previous lemma, we could take for $\mathcal{P}$ the property of being projective, since it is a well-known fact (found in [7], for example) that all non-zero projectives have maximal submodules.

Lemma 13.5. *Let M be a self-projective module. If M has finite exchange then M has property $(N)_2$.*

Proof. As already mentioned, for projective modules this is [13, Proposition 2.9]. For self-projective modules this is [19, Theorem 3.3]. □

Lemma 13.6. *A projective module with finite exchange has small radical.*

Proof. This is [17, Lemma 2.9], but we include the proof as a prelude to a discussion of the self-projective case. Let P be a projective module with finite exchange, and let K be an arbitrary submodule with $\mathrm{rad}(P) + K = P$. By Lemma 13.5 there are submodules $L_1 \subseteq \mathrm{rad}(P)$ and $L_2 \subseteq K$ with $P = L_1 \oplus L_2$. But by Lemma 13.4, $\mathrm{rad}(P)$ cannot contain a non-zero summand, and so $L_1 = 0$. This means $L_2 = P$ and hence $K = P$. But this implies that $\mathrm{rad}(P)$ is small in P. □

Corollary 13.7. *If P is a projective module with finite exchange then so is $P/\mathrm{rad}(P)$. If, furthermore, $P/\mathrm{rad}(P)$ has full exchange then so does P.*

Proof. We know that $\mathrm{End}(P/\mathrm{rad}(P)) \cong \mathrm{End}(P)/J(\mathrm{End}(P))$ which is an exchange ring. So $P/\mathrm{rad}(P)$ has finite exchange. For the second half, combine Theorem 13.3 with Lemma 13.6. □

Corollary 13.7 was already known for free modules by the proof of [18, Proposition 4.1]. The only obstacle to replacing "projective" by "self-projective" in Corollary 13.7 is in showing that self-projective modules with finite exchange must have maximal submodules. Unfortunately, self-projective modules with exchange may not have maximal submodules, due to the following example pointed out to me by G. Bergman.

Example 13.8. Let R be a commutative, complete, discrete valuation ring (such as the p-adic integers). Let K be the field of fractions of R, and let $M = K$, regarded as an R-module. We have $\mathrm{End}(M_R) \cong K$, which is a field. So M_R has full exchange.

If we have a diagram

$$\begin{tikzcd} & M \arrow[d, "g"] \\ M \arrow[r, two heads, "f"'] & L \end{tikzcd}$$

then the only nontrivial case is where f is (up to isomorphism) the R-module quotient map $f : K \to K/R$. But the only R-module map from $K \to K/R$ is the quotient map preceded by multiplication by a constant in K. Thus, the needed filler exists, and so M_R is self-projective. It clearly does not have a maximal submodule. This gives us a counter-example to Lemma 13.6, if "projective" is replaced by "self-projective."

14 Final Remarks

There seems to be a natural difference between modules which are Dedekind-finite and those which are (non-finitely generated) projective, in terms of the exchange property. The methods described in Chapter 1 demonstrate that many classes of modules with finite exchange which are Dedekind-finite automatically have $\aleph_0$-exchange; such as modules with strongly π-regular endomorphism rings. On the other hand when a module is free with infinite rank, and hence contains proper summands isomorphic to itself, going from finite to countable exchange is hard, but going from countable to full exchange follows from Chapter 2.

If one wishes to search for counter-examples to the conjecture "finite exchange implies full exchange" the best open case seems to be that of free modules. In fact, it is still an open problem to classify all right finite-P-exchange rings.

Bibliography

[1] V. I. Arnautov, S. T. Glavatsky, and A. V. Mikhalev. *Introduction to the theory of topological rings and modules*, volume 197 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1996.

[2] Nicolas Bourbaki. *Elements of mathematics. General topology. Part 1.* Hermann, Paris, 1966.

[3] Peter Crawley and Bjarni Jónsson. Refinements for infinite direct decompositions of algebraic systems. *Pacific J. Math.*, 14:797–855, 1964.

[4] Friedrich Dischinger. Sur les anneaux fortement π-réguliers. *C. R. Acad. Sci. Paris Sér. A-B*, 283(8):Aii, A571–A573, 1976.

[5] Alberto Facchini. *Module theory: Endomorphism rings and direct sum decompositions in some classes of modules*, volume 167 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1998.

[6] Irving Kaplansky. Projective modules. *Ann. of Math (2)*, 68:372–377, 1958.

[7] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[8] T. Y. Lam. *A first course in noncommutative rings*, volume 131 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.

[9] T. Y. Lam. A crash course on stable range, cancellation, substitution and exchange. *J. Algebra Appl.*, 3(3):301–343, 2004.

[10] Saad H. Mohamed and Bruno J. Müller. Continuous modules have the exchange property. In *Abelian group theory (Perth, 1987)*, volume 87 of *Contemp. Math.*, pages 285–289. Amer. Math. Soc., Providence, RI, 1989.

[11] Saad H. Mohamed and Bruno J. Müller. *Continuous and discrete modules*, volume 147 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.

[12] Saad H. Mohamed and Bruno J. Müller. $\aleph$-exchange rings. In *Abelian groups, module theory, and topology (Padua, 1997)*, volume 201 of *Lecture Notes in Pure and Appl. Math.*, pages 311–317. Dekker, New York, 1998.

[13] W. K. Nicholson. Lifting idempotents and exchange rings. *Trans. Amer. Math. Soc.*, 229:269–278, 1977.

[14] W. K. Nicholson. Strongly clean rings and Fitting's lemma. *Comm. Algebra*, 27(8):3583–3592, 1999.

[15] Pace P. Nielsen. Abelian exchange modules. *Comm. Algebra*, 33(4):1107–1118, 2005.

[16] Pace P. Nielsen. Countable exchange and full exchange rings. *Comm. Algebra (to appear)*, 2006.

[17] Josef Stock. Über die Austauscheigenschaft von Moduln. *Ph.D. thesis, Universität München*, 1982.

[18] Josef Stock. On rings whose projective modules have the exchange property. *J. Algebra*, 103(2):437–453, 1986.

[19] A. A. Tuganbaev. Rings and modules with exchange properties. *J. Math. Sci. (New York)*, 110(1):2348–2421, 2002. Algebra, 17.

[20] Zhou Wang and Jianlong Chen. On two open problems about strongly clean rings. *Bull. Austral. Math. Soc.*, 70(2):279–282, 2004.

[21] Roger Ware. Endomorphism rings of projective modules. *Trans. Amer. Math. Soc.*, 155:233–256, 1971.

[22] R. B. Warfield, Jr. Exchange rings and decompositions of modules. *Math. Ann.*, 199:31–36, 1972.

[23] Birge Zimmermann-Huisgen and Wolfgang Zimmermann. Classes of modules with the exchange property. *J. Algebra*, 88(2):416–434, 1984.