THE BEHAVIOR OF ASCENDING CHAIN CONDITIONS ON SUBMODULES OF BOUNDED FINITE GENERATION IN DIRECT SUMS

PACE P. NIELSEN

Abstract. We construct a ring R which has the ascending chain condition on n-generated right ideals for each $n \geq 1$ (also called the right pan-acc property), such that no full matrix ring over R has the ascending chain condition on cyclic right ideals. Thus, the right pan-acc property is not a Morita invariant. Moreover, a direct sum of (free) modules with pan-acc does not necessarily even have 1-acc.

1. Introduction and history

Given any ring R, any right R-module M_R, and any positive integer $n \geq 1$, one says that M_R has *n-acc* if every chain of n-generated R-submodules $L_1 \subseteq L_2 \subseteq \ldots \subseteq M$ stabilizes; meaning that $L_k = L_{k+1} = \cdots$ for some $k \geq 1$. Moreover, if M has n-acc for every $n \geq 1$, then M is said to have *pan-acc*. Clearly, pan-acc is a weakening of the usual Noetherian condition. A ring has right n-acc, respectively right pan-acc, if the right regular module R_R has the corresponding property.

Those who have read Cohn's influential book "Free Rings and Their Relations" [6] may have found the first two problems asked in the book exceptionally unyielding. Exercise 0.1.1 enjoins the reader to show that if R has right n-acc, then so does every finitely generated free right R-module. Exercise 0.1.2 challenges the reader to show that if R has right rs-acc (for two integers $r, s \geq 1$), then $\mathbb{M}_r(R)$ has right s-acc. Bonang [3] has pointed out that the first exercise poses an *open problem*, suggested earlier (in the context of modules over commutative rings) by Nicolas [11] (for *all* free modules) and Heinzer and Lantz [9].

In Cohn's subsequent book "Free Ideal Rings and Localization in General Rings" [7], these problems appear in slightly different forms. For instance exercise 0.1.1 is replaced by 0.2.4, a weaker problem which directs one to prove that if R_R has pan-acc, then so does any finitely generated free right R-module. This new problem is also open according to [3].

Further connections between these exercises can be found in [13]. For instance, by [13, Corollary 1.6] we find that R_R^n has n-acc if and only if $\mathbb{M}_n(R)$ has right 1-acc. So, the truth of exercise 0.1.1 would imply the truth of 0.1.2. For a recent overview on this and related topics, see [4].

The purpose of this short note is to settle the aforementioned thirty year-old problem once and for all, in the negative, by proving the following main result:

Theorem 1.1. *There exists a ring R which has right pan-acc, such that R_R^2 does not have 1-acc.*

From the results of [13], we immediately obtain using the same example:

Corollary 1.2. *There exists a ring R with right pan-acc, for which every matrix ring $\mathbb{M}_n(R)$ with $n > 1$ does not have right 1-acc. In particular, the property "right pan-acc" is not a Morita invariant.*

All rings in this paper are associative with 1, but not necessarily commutative. All modules are unital. Notations introduced in one section carry over to subsequent parts of the paper.

2010 *Mathematics Subject Classification.* Primary 16S36, Secondary 16N60, 16N80, 16S20.

Key words and phrases. Morita invariant, n-acc, pan-acc.

2. Description of the ring, and basic properties

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of non-negative integers. Also let $\mathcal{P}$ be the set of all finite, non-empty subsets of $\mathbb{N}$. Given any element $p \in \mathcal{P}$ we write $\max(p)$ for the largest element of p. For simplicity, we will work over the field $\mathbb{F}_2$ of two elements, although this construction works over any field, *mutatis mutandis*. We define

$$\begin{aligned} R \;:=\; \mathbb{F}_2\Big\langle\, & a_i,\ r,\ v_p\ (i \in \mathbb{N}, p \in \mathcal{P}) \;:\; a_{i+1}r = a_i, \\ & \left(\sum_{k\in p} a_k\right) v_p = a_{\max(p)+1},\ a_j a_i = r a_i = v_p a_i = 0 \Big\rangle. \end{aligned}$$

A few words about the relations defining R are in order. The first system of relations, $a_{i+1}r = a_i$, is given in order to create an increasing chain of cyclic submodules of R^2_R; namely,

$$(a_0, a_1)R \subseteq (a_1, a_2)R \subseteq (a_2, a_3)R \subseteq \ldots \subseteq R^2_R. \tag{2.1}$$

We will prove later that this chain is strictly increasing.

The next system of relations (i.e. those involving the letters v_p), in conjunction with the first set of relations, allow us to move from *any* non-trivial $\mathbb{F}_2$-linear combination of the a-letters, to any other non-trivial combination. To see this, let $p \in \mathcal{P}$ be arbitrary. We compute

$$\left(\sum_{k\in p} a_k\right) v_p r^{\max(p)+1} = a_0,$$

so any linear combination can be brought to a_0. Conversely, we find

$$a_0 \sum_{k\in p} v_{\{0\}} v_{\{1\}} \cdots v_{\{k-1\}} = \sum_{k\in p} a_k.$$

(When $k = 0$, the summand is an empty product, and thus should be interpreted as 1.) These relations serve a double purpose, both to simplify some computations and to remove trivial obstructions arising from (2.1) which may prevent R_R from having 1-acc.

The remaining relations say that a_i cannot occur to the right of any other letter, and this simplifies computations. In fact, we see that $R = S \oplus a_0 S$, where $S = \mathbb{F}_2\langle r, v_p\ (p \in \mathcal{P})\rangle$ is a free $\mathbb{F}_2$-algebra. However, we note that the a_i *can* occur to the left of most of the v-letters without being influenced by any relations. (See the reduction rules (2.2) below for more information.)

Fix a well-ordering $\prec$ of $\mathcal{P}$, with order type ω (since this set is countable). Thus, we can write the elements of $\mathcal{P}$ in a chain $p_0 \prec p_1 \prec p_2 \prec \cdots$. We then well-order the letters defining R, using the same symbol $\prec$ (which should lead to no confusion), by putting

$$a_0 \prec a_1 \prec a_2 \prec \cdots \prec r \prec v_{p_0} \prec v_{p_1} \prec v_{p_2} \prec \cdots.$$

Next, we order the set $\mathcal{M}$ of monomials in these letters, first by degree and then lexicographically (still denoting this new ordering by $\prec$). Finally, if f, g are two finite formal sums of monomials in these letters, we say $f \prec g$ if among the monomials which appear in the support of f or g but not both, the largest one occurs in the support of g. The reader can check directly that this describes a well-ordering on the set of such finite formal sums. (Indeed, since we are working over $\mathbb{F}_2$ one can identify such formal sums with finite subsets of the well-ordered set $\mathcal{M}$. The ordering we have described is called the "last difference ordering." It can be used as the first step in defining ordinal exponentiation, as is done, for instance, in Sierpiński's classic text [12] starting on page 309. For a more high-brow approach to the last difference ordering, see the survey article [10].)

We put elements of R in *reduced form* by using the following reduction rules for all $i, j \in \mathbb{N}$ and for all $p \in \mathcal{P}$:

$$a_{i+1}r \mapsto a_i,\ a_j a_i \mapsto 0,\ r a_i \mapsto 0,\ v_p a_i \mapsto 0, \quad \text{and} \quad a_{\max(p)} v_p \mapsto \sum_{k \in p \setminus \{\max(p)\}} a_k v_p + a_{\max(p)+1}. \tag{2.2}$$

It is easy to check that these relations form a reduction system with all ambiguities resolvable, in the sense of [1] (cf. [2]), since there are no inclusions and every overlap resolves to 0.

It is important to notice that every reduction replaces a monomial with a sum of smaller monomials, so we know that this process must eventually stop (since $\prec$ is a well-ordering of finite formal sums of monomials). As all ambiguities of (2.2) are resolvable, Theorem 1.2 of [1] tells us that every element can be reduced using those reductions to a *unique* irreducible normal form.

Hereafter, when writing an element of R we will use its reduced form unless context requires otherwise. This also allows us to well-order the elements of R using $\prec$. By the *leading term* of an element $f \in R$, we mean the $\prec$-maximal monomial m in the support of the reduced form of f.

3. Information about finitely generated right ideals

Let $I_R \subseteq R_R$ be a finitely generated right ideal. In this section we describe a specific choice of a generating set for I, and some relevant properties of this set. If $I = (0)$, the generating set is just the empty set, so hereafter we may assume $I \neq (0)$.

First, consider the case where

$$I \text{ contains no non-zero sum of the letters } \{a_i\ (i \in \mathbb{N})\}. \tag{3.1}$$

In particular, note that no element $f \in I$ has 1 in its support, else $fa_0 = a_0 \in I$. Thus the right R-module structure on I reduces to a right S-module structure (since right multiplication by $a_0 S$ is trivial). Furthermore, if m is the leading term of $f \in I$, and m' is the leading term of any element $g \in S$, then (using (3.1) and the reduction rules) the leading term of fg is mm'.

Let $n \geq 1$ be the minimal number of generators for I, and let $G := G(I)$ be the set of all n-tuples $(f_1, f_2, \ldots, f_n)$ of elements which generate I. Well-order G, and more generally R^n, by saying that for two elements $(f_1, f_2, \ldots, f_n), (g_1, g_2, \ldots, g_n) \in R^n$, we have $(f_1, f_2, \ldots, f_n) < (g_1, g_2, \ldots, g_n)$ exactly when for some $k \geq 1$ it happens that $f_i = g_i$ for all $1 \leq i < k$, and $f_k \prec g_k$. By the *minimized generating set* for I, we mean the smallest element of $(G, <)$.

Let $\mathbf{f} := (f_1, f_2, \ldots, f_n)$ be the minimized generating set for I. Clearly $0 \prec f_1 \prec f_2 \prec \cdots \prec f_n$. An important fact we will use repeatedly is that for all $i \neq j$, there is no monomial in the support of (the reduced form of) f_i which is a right multiple of the leading term of f_j. To see this, suppose that $m = m'm''$ where m is a monomial in the support of f_i, m' is the leading term of f_j, and $m'' \in S$ is some monomial. Then $f_i' := f_i - f_j m'' \prec f_i$, and $(f_1, \ldots, f_{i-1}, f_i', f_{i+1}, \ldots, f_n) \in G$, contradicting the minimality of $\mathbf{f}$. Thus we have proven:

Lemma 3.2. *Let I be a finitely generated right ideal of R satisfying* (3.1)*, and let $(f_1, f_2, \ldots, f_n)$ be the minimized generated set. No right multiple of the leading term of f_j appears in the support of any f_i, for $i \neq j$.*

As pointed out by Bergman (personal communication), this is similar to the work of Cohn in studying free modules over rings with weak algorithm (such as free algebras). Related to this idea, we also have:

Lemma 3.3. *Let I be a finitely generated right ideal of R satisfying* (3.1)*, and let $(f_1, f_2, \ldots, f_n)$ be the minimized generating set. If m is the leading term of a non-zero element $f \in I$, then $m = m'm''$ with m' the leading term of f_{i_0}, for some unique i_0. Moreover, if we write $f = \sum_{i=1}^{n} f_i g_i$ with $g_i \in S$, then m'' is the leading term of g_{i_0} and* $\deg(f) = \max(\deg(f_i) + \deg(g_i)) = \deg(f_{i_0}) + \deg(g_{i_0})$.

Proof. Let $f = f_1g_1 + f_2g_2 + \cdots + f_ng_n$ for some $g_1, g_2, \ldots, g_n \in S$. The leading term of f_ig_i is m_im_i', where m_i is the leading term of f_i and m_i' is the leading term of g_i. (If $g_i = 0$, then there is technically no leading term, and so in the remainder of the proof we implicitly consider only those subscripts $1 \leq i \leq n$ with $g_i \neq 0$.) We cannot have $m_im_i' = m_jm_j'$ for some $i < j$, else (after removing the letters in m_j' from both sides) we get that m_j is a right multiple of m_i, contradicting Lemma 3.2. Thus, the leading term of f is the $\prec$-largest element among the *distinct* list $m_1m_1', \ldots, m_nm_n'$. The last sentence of the lemma is now clear. □

Next, let's consider the case where I contains some non-zero sum of the letters $\{a_i\ (i \in \mathbb{N})\}$. Then, using the relations involving the letters v_p, we see that in fact

$$I \text{ contains } \textit{all} \text{ sums of the letters } \{a_i\ (i \in \mathbb{N})\}. \tag{3.4}$$

Fix any finite generating set $f_1, \ldots, f_n$ for I, and let f_i' be defined by deleting from f_i any monomials in a_0S. We define I' to be the right ideal of S generated by $f_1', \ldots, f_n'$. Clearly, I'_S is finitely generated (with at most the same number of generators as I) and $I = I' \oplus a_0S$. Moreover, from the direct sum decomposition we see that $I' = I \cap S$, so the definition of I' is independent of the generating set we chose for I. The utility of working with I' is that we can "peel off" all the monomials involving the a-letters, and work in the simpler ring S. While we could now define a minimal generating set for I', we will see shortly that this is unnecessary.

4. The ring R has right pan-acc

In this section we prove that R_R has pan-acc. To this end, let $I_1 \subseteq I_2 \subseteq \ldots \subseteq R_R$ be a chain of right R-submodules, with each I_k generated by at most n elements, for some $n \geq 1$. We may as well assume that each I_k is not $(n-1)$-generated, by passing to a subsequence and decreasing n if necessary.

Case 1: Assume that (3.1) holds for each I_k, $k \geq 1$.

Let $\mathbf{f}_k := (f_{k,1}, f_{k,2}, \ldots, f_{k,n})$ be the minimized generating set for I_k. As mentioned in the previous section, we have a well-ordering $(R^n, <)$. It suffices to show that $\mathbf{f}_{k+1} \leq \mathbf{f}_k$, for every $k \geq 1$; for since there are no infinite descending chains in $(R^n, <)$, this will prove that the ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$ must stabilize.

Suppose, to the contrary, that $\mathbf{f}_k < \mathbf{f}_{k+1}$. Fix $1 \leq \ell \leq n$ such that $f_{k,i} = f_{k+1,i}$ for $1 \leq i < \ell$, but $f_{k,\ell} \prec f_{k+1,\ell}$. The leading term of $f_{k,\ell}$ is not right a multiple of any of the leading terms of $\{f_{k,1}, f_{k,2}, \ldots, f_{k,\ell-1}\}$, by Lemma 3.2 and the fact that $\mathbf{f}_k$ is a minimized generating set. By Lemma 3.3, since $f_{k,\ell} \in I_{k+1}$ and since

$$f_{k,\ell} \prec f_{k+1,\ell} \prec f_{k+1,\ell+1} \prec \cdots \prec f_{k+1,n},$$

we know that the leading term of $f_{k,\ell}$ is the same as that of $f_{k+1,\ell}$. Moreover, as we are able to write $f_{k,\ell} = \sum_{i=1}^{n} f_{k+1,i}g_i$ for some $g_i \in S$, Lemma 3.3 also implies that $g_\ell = 1$. But then, $(f_{k+1,1}, \ldots, f_{k+1,\ell-1}, f_{k,\ell}, f_{k+1,\ell+1}, \ldots, f_{k+1,n})$ is a generating set for I_{k+1}, and is smaller than $\mathbf{f}_{k+1}$, which gives us the needed contradiction.

Case 2: Assume that I_ℓ satisfies (3.4) for all $\ell \geq k$, for some sufficiently large k.

For each $\ell \geq k$, define I_ℓ' as in the previous section. These are n-generated right ideals of S. It suffices to show that the chain $I_k' \subseteq I_{k+1}' \subseteq \ldots$ stabilizes. Viewing this as a chain of right ideals in S, an argument similar to case 1 suffices. Alternatively, since S is a free algebra it is a fir [5, Corollary 2], and thus S_S has pan-acc by [7, Theorem 2.2.3]. (Inversely, one could view the argument in case 1 as a tweaking of this argument.) Therefore, this chain of right ideals in S stabilizes.

5. The free rank 2 modules over R do not have 1-acc

Recall that we have an increasing chain of cyclic submodules of R_R^2 given in (2.1). To finish the proof of Theorem 1.1, it suffices to show that $(a_{k+1}, a_{k+2}) \notin (a_k, a_{k+1})R$ for every $k \geq 0$. From the relations $a_j a_i = 0$, we reduce to proving $(a_{k+1}, a_{k+2}) \notin (a_k, a_{k+1})S$.

Fix $g \in S$. We can uniquely write

$$g = g_k + r g_{k-1} + r^2 g_{k-2} + \cdots + r^k g_0,$$

where $g_i \in S$ has no monomial in its support beginning with r unless $i = 0$. Let ℓ be the largest index with $\deg(g_\ell)$ maximized, and let m be the $\prec$-maximal monomial in the support of g_ℓ.

Observe that $a_k r^{k-\ell} g_\ell$ and $a_{k+1} r^{k-\ell} g_\ell$ reduce to $a_\ell g_\ell$ and $a_{\ell+1} g_\ell$ respectively, which before any further reduction have leading terms $a_\ell m$ and $a_{\ell+1} m$. We will next prove that at least one of these terms is already in reduced form. If m begins with some v_p, then since v_p is involved in a reduction with only one a_i (on the left), the claim follows in this case. Otherwise m begins with r, which forces $\ell = 0$; and since $a_\ell = a_0$ does not interact with r from the left, we see that $a_\ell m$ is reduced.

Fix $i \in \{0, 1\}$ such that $a_{\ell+i} m$ is in reduced form. We will be done once we show that this term appears in the support of

$$a_{k+i} g = a_{k+i} g_k + a_{k+i-1} g_{k-1} + \cdots + a_i g_0,$$

because $a_{\ell+i} m \neq a_{k+i+1}$ (since by comparing lengths, equality could hold only if $m = 1$, but the resulting equality is then excluded because $\ell \leq k$). Let m' be any monomial in the support of g_j, for some j. If $j > \ell$ then $\deg(m') < \deg(m)$, and so $a_{j+i} m' \prec a_{\ell+i} m$ (even before reducing the left-hand side). If $j < \ell$ then $\deg(m') \leq \deg(m)$ while $a_{j+i} \prec a_{\ell+i}$, and so again $a_{j+i} m' \prec a_{\ell+i} m$. Finally, if $j = \ell$ and $m \neq m'$, then the maximality condition on m implies $a_{\ell+i} m' \prec a_{\ell+i} m$. Thus the monomial $a_{\ell+i} m$ cannot be cancelled from $a_{k+i} g$, as claimed.

6. Two-sided construction?

Throughout this paper we have worked exclusively "on the right." This naturally raises the question of whether or not a left-right symmetric construction is possible. Moreover, it would be rewarding to begin answering the questions raised by Heinzer, Lantz, and Nicolas, by addressing the following:

Question 6.1. Does there exist a commutative ring R with (both left and right) pan-acc, such that R_R^2 does not have pan-acc?

It is an easy exercise, apparently first worked out by Frohn [8, Lemma] (which he says "is probably well-known"), that if R is commutative and M, N are right R-modules with 1-acc, then $M \oplus N$ also has 1-acc. In particular, Theorem 1.1 *requires* that the ring R fail to be commutative.

Acknowledgements

Many improvements to this text are the result of conversations with George Bergman, to whom I express my gratitude. I thank the anonymous referee for comments which improved the exposition. This work was partially supported by a grant from the Simons Foundation (#315828 to Pace Nielsen). The project was sponsored by the National Security Agency under Grant Number H98230-16-1-0048.

References

1. George M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), no. 2, 178–218. MR 506890 (81b:16001)
2. L. A. Bokut′, *Imbeddings into simple associative algebras*, Algebra i Logika **15** (1976), no. 2, 117–142, 245. MR 0506423 (58 #22167)
3. F. Bonang, *Noetherian rings whose subidealizer subrings have pan-a.c.c.*, Comm. Algebra **17** (1989), no. 5, 1137–1146. MR 993392 (90d:16015)

4. Esperanza Sánchez Campos and Patrick F. Smith, *Modules satisfying the ascending chain condition on submodules with bounded uniform dimension*, Rings, Modules and Representations, Contemp. Math., vol. 480, Amer. Math. Soc., Providence, RI, 2009, pp. 57–71. MR 2508144 (2010m:16028)
5. P. M. Cohn, *Free ideal rings*, J. Algebra **1** (1964), 47–69. MR 0161891 (28 #5095)
6. ______, *Free Rings and Their Relations*, second ed., London Mathematical Society Monographs, vol. 19, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1985. MR 800091 (87e:16006)
7. ______, *Free Ideal Rings and Localization in General Rings*, New Mathematical Monographs, vol. 3, Cambridge University Press, Cambridge, 2006. MR 2246388 (2007k:16020)
8. Daniel Frohn, *A counterexample concerning ACCP in power series rings*, Comm. Algebra **30** (2002), no. 6, 2961–2966. MR 1908250 (2003h:13029)
9. William Heinzer and David Lantz, *Commutative rings with ACC on n-generated ideals*, J. Algebra **80** (1983), no. 1, 261–278. MR 690717 (84g:13030)
10. Joseph B. Kruskal, *The theory of well-quasi-ordering: A frequently discovered concept*, J. Combinatorial Theory Ser. A **13** (1972), 297–305. MR 0306057 (46 #5184)
11. Anne-Marie Nicolas, *Sur les modules tels que toute suite croissante de sous-modules engendrés par n générateurs soit stationnaire*, J. Algebra **60** (1979), no. 1, 249–260. MR 549110 (80j:13011)
12. Wacław Sierpiński, *Cardinal and Ordinal Numbers*, Polska Akademia Nauk, Monografie Matematyczne. Tom 34, Państwowe Wydawnictwo Naukowe, Warsaw, 1958. MR 0095787 (20 #2288)
13. Maria Elisa Antunes Simões and Patrick F. Smith, *Rings whose free modules satisfy the ascending chain condition on submodules with a bounded number of generators*, J. Pure Appl. Algebra **123** (1998), no. 1-3, 51–66. MR 1492895 (98k:16032)

Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
E-mail address: pace@math.byu.edu