PROPERTIES WHICH DO NOT PASS TO CLASSICAL RINGS OF QUOTIENTS

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Abstract. We construct examples of Ore rings satisfying some standard ring-theoretic properties for which the classical rings of quotients do not satisfy those properties. Examples of properties which do not pass to rings of quotients include: Abelian, Dedekind-finite, semicommutative, 2-primal, and NI. In the process of constructing these examples we correct the literature. We also introduce two important construction techniques.

1. Motivating Question

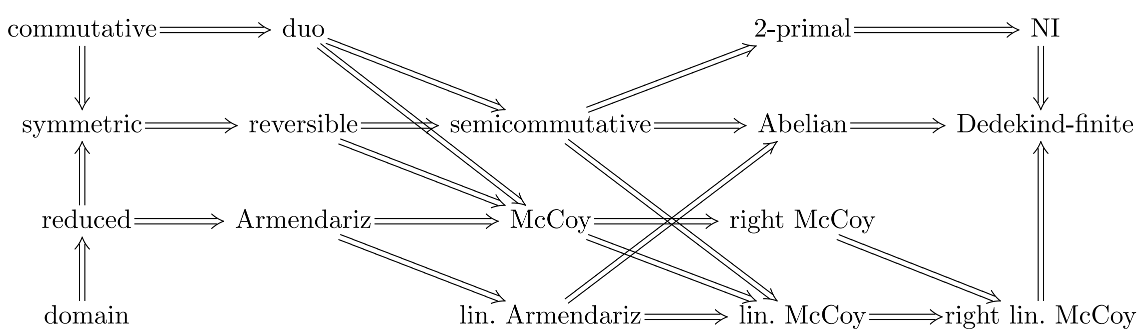
A fundamental construction in ring theory is the formation of the field of quotients of a commutative domain. The utility of this construction lies in the fact that one moves from a decent ring structure (integral domain) to a nearly perfect structure (field). One of Ore's celebrated theorems states that something similar is true for noncommutative rings. In modern terminology this theorem says that the right Ore domains are exactly the right orders in division rings.

More generally, if we are given a right Ore ring R with a property $\mathcal{P}$ we can ask whether its (total) right classical ring of quotients Q also has property $\mathcal{P}$. It turns out that in general the answer is no, and one can lose nice properties by passing to quotients. In this paper we focus on constructing examples demonstrating that many well-known zero-divisor properties do not pass to rings of quotients.

All of the standard definitions of ring properties and facts about localizations can be found in [12] and [13]. By C_R we mean the set of regular elements in R. If R is a right Ore ring, we will let $Q^r_{cl}(R)$ denote its right classical ring of quotients. We also let $E_{i,j}$ denote the usual matrix units (with 1 in the (i, j) position, and zeros elsewhere).

2. What does pass to rings of quotients?

Consider the following diagram of classic ring-theoretic properties, taken from [4] and slightly expanded:



There are no additional implications from one property to another, except those forced by transitivity. The fact that NI rings are Dedekind-finite is [8, Proposition 2.7], and an example of an Armendariz

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ring which is not NI is given by [4, Example 9.3]. Many of the other implications are well-known, and we refer the interested reader to [4] for more information.

Which of these properties pass to the classical right ring of quotients? Which pass down from the classical ring, to a right order? The following proposition includes all previous answers to these two questions, and a few new pieces as well.

Proposition 1. *Let R be a right Ore ring, and let $Q = Q^r_{cl}(R)$.*

1. *The ring R has $\mathcal{P}$ if and only if Q has $\mathcal{P}$, when $\mathcal{P}$ is one of the following: commutative, symmetric, reduced, domain, reversible, (linearly) Armendariz, or (linearly) right McCoy.*
2. *If the ring Q has $\mathcal{P}$ then R has $\mathcal{P}$, when $\mathcal{P}$ is one of the following: semicommutative, Abelian, 2-primal, NI, or Dedekind-finite.*
3. *If the ring R is (linearly) left McCoy then Q is (linearly) left McCoy.*

Proof. (1) Going from Q down to R follows from the fact that $\mathcal{P}$ passes to (nonzero) subrings, except for the (linearly) right McCoy case which we will deal with shortly. So we focus on going up from R to Q. The statement for commutative rings is classical. For domains this is Ore's theorem [12, Proposition 10.21]. The case when R is reduced is [9, Theorem 16]. Similar computations for reversible rings are found in [10, Theorem 2.6], and for symmetric rings in [6, Theorem 4.1].

The Armendariz case is dealt with in [7, Theorem 12]. The proof was generalized to work for McCoy rings independently in [11], [16], and [17], also covering the case going down from Q to R. The proofs of all of these results work when restricted to polynomials of specific degrees, and this covers the linearly Armendariz and linearly right McCoy cases.

(2) All the relevant conditions pass to subrings. The 2-primal case is covered by [2, Proposition 2.2], and all others are easy.

(3) This implication is new. Assume $R \neq 0$ is left McCoy. Further assume that $F(x), G(x) \in Q[x]$ satisfy $F(x)G(x) = 0$. After finding common denominators, we can write $F(x) = f(x)u^{-1}$ and $G(x) = g(x)v^{-1}$ for some $f(x), g(x) \in R[x]$ and regular elements $u, v \in R$. We can also write $u^{-1}g(x) = g'(x)w^{-1}$ for some polynomial $g'(x) \in R[x]$ and some regular element $w \in R$, using the right Ore condition on each of the coefficients of $g(x)$ and then finding a common denominator.

We now have $0 = F(x)G(x) = f(x)g'(x)w^{-1}v^{-1}$. Hence $f(x)g'(x) = 0$. From the left McCoy condition, there exists $r \in R$, $r \neq 0$ so that $rg'(x) = 0$. But then $0 = rg'(x)w^{-1}v^{-1} = (ru^{-1})G(x)$ and $ru^{-1} \in Q$ is nonzero. This proves that Q is left McCoy. If we restrict to linear polynomials (or more generally, polynomials of any given fixed degrees) the same result holds. $\square$

The remainder of the paper is devoted to constructing examples and developing the theoretical framework to prove that in many of the remaining cases the relevant properties do not pass to rings of quotients.

3. Abelian and Dedekind-finite rings

Recall that a ring is *Abelian* when its idempotents are central, and that a ring R is *Dedekind-finite* if for $a, b \in R$, $ab = 1$ implies $ba = 1$. Abelian rings are Dedekind-finite. Using upper-triangular matrix rings we can show that both properties do not pass to rings of quotients, and more.

Example 2. *An Abelian Ore ring R with $Q = Q^r_{cl}(R)$ not Abelian.*

Construction and proof. Let t be an indeterminate over a field F. Let $Q = \mathbb{T}_2(F(t))$ denote the ring of 2×2 upper-triangular matrices over the field $F(t)$. Let I be the identity matrix and $R = FI + t\mathbb{T}_2(F[t])$, which is a subring of Q.

It is straightforward to check that all regular elements in R are units in Q. Further, Q is obtained from R by inverting the central, regular elements $f(t)I$ (where $f(t) \in F[t]$ is a nonzero polynomial).

To see this, let $\alpha, \beta, \gamma \in F(t)$. There exist polynomials $f(t), g(t), h(t), d(t) \in F[t]$ with $d(t) \neq 0$ so that $\alpha = f(t)/d(t)$, $\beta = g(t)/d(t)$, and $\gamma = h(t)/d(t)$. We compute

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} tf(t) & tg(t) \\ 0 & th(t) \end{pmatrix} (td(t)I)^{-1}.$$

Thus Q is the two-sided classical quotient ring of R. Next, notice that Q is not Abelian since the idempotent $E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not central. Further, the diagonal entries of each idempotent of Q belong to $\{0,1\}$, and if the diagonal entries agree the idempotent must be trivial. Thus the only idempotents of R are trivial, so R is Abelian. □

While the example above is quite straightforward, it is even easier to construct Abelian rings which are not necessarily Ore, but for which there exist localizations which are not Abelian. In fact, such localizations can be accomplished by simply localizing at one element.

Example 3. *A large class of Abelian rings R, with a central, regular element $a \in R$, so that if $S = \{a, a^2, a^3, \ldots\}$ then the localization $T = RS^{-1}$ is not Abelian.*

Construction and proof. Let A be a commutative ring which has an ideal I with the following two properties: I contains an element a which is regular and not a unit, and I contains no nonzero idempotents. Let B be the ring obtained from A by inverting the central multiplicative set $\{a, a^2, a^3, \ldots\}$. Let M be any nonzero B-B-bimodule. Define the ring R as follows:

$$R = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \ : \ r - s \in I, m \in M \right\}.$$

We claim that R is Abelian. Suppose that

$$E = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix}$$

is an idempotent in R. Clearly e and f must be idempotents in A. Additionally, since $e - f \in I$ we have $(e-f)^2 \in I$. But $(e-f)^2$ is idempotent so $(e-f)^2 = 0$. Expanding, we have $e + f - 2ef = 0$. Multiplying by e, we see that $e = ef$. Similarly $f = ef$. Thus $e = f$. Then since $E^2 = E$, $2xe = x$. Since $2e - 1$ is a unit, $x = 0$. Thus our idempotent is

$$\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$$

which is central.

Let T be the ring obtained from R by inverting the central, regular element aI. We then have

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}^{-1} \in T$$

is a noncentral idempotent (that behaves like $E_{1,1}$). Thus T is not Abelian. □

The next question we want to ask is whether the Dedekind-finite property passes to rings of quotients. According to [5, Example 2.2], there is a right Ore ring which is Dedekind-finite but its right classical quotient ring is not Dedekind-finite. Unfortunately there is an error in the proof which we describe now.

Given a ring A let $\mathrm{CFM}(A)$ denote the ring of $\omega \times \omega$ matrices with only finitely many entries in any column nonzero. This ring is called the ring of column-finite matrices, and is isomorphic to the endomorphism ring of the countable rank free right A-module $A^{(\mathbb{N})}$. It is well-known that $\mathrm{CFM}(A)$ is not Dedekind-finite when $A \neq 0$, for if x is the matrix with 1's down the super-diagonal and 0's elsewhere, while y is the matrix with 1's down the sub-diagonal and 0's elsewhere, we have $xy = 1 \neq yx$. Let

$T = \mathrm{CFM}(\mathbb{Q})$. It is also well-known that regular elements in T are units, which is seen by thinking of T as the set of linear transformations on the $\mathbb{Q}$-vector space $V = \mathbb{Q}^{(\mathbb{N})}$. Hence, T is a classical ring.

Let R be the subring of T consisting of column-finite matrices with off-diagonal entries even. Given $t \in T$ there is a common denominator n_i for the entries in the ith column. If we set

$$d = \begin{pmatrix} 2n_1 & & & \\ & 2n_2 & & \\ & & 2n_3 & \\ & & & \ddots \end{pmatrix} \in R$$

then $td \in R$. Let S be the set of diagonal matrices in R with nonzero diagonal entries. We see that $T = RS^{-1}$. Clearly S consists of regular elements since each element is a bijective linear transformation on V. At this point we may be tempted to conclude that T is the classical right quotient ring of R. The problem is that regular elements in R do not necessarily remain regular in T. For example, the matrix

$$s = \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ & -2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \in R$$

is regular in R but is not regular in T as it is annihilated on the left by

$$\begin{pmatrix} 1 & 1/2 & 1/4 & 1/8 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Fortunately, we can not only fix this problem but also improve the example to obtain:

Example 4. *An Ore ring R which is Abelian but $Q = Q^r_{cl}(R)$ is not Dedekind-finite.*

Construction and proof. Let Q be the ring of column finite matrices over the field $\mathbb{Q}$, where the first column is zero except possibly in the $(1,1)$ entry. We view Q as a subring of $T = \mathrm{CFM}(\mathbb{Q})$. Denote the identity in Q by I, and let

$$R = \mathbb{Z}I + \begin{pmatrix} 2\mathbb{Z} & \mathbb{Q} \quad \mathbb{Q} \quad \cdots \\ 0 & \mathrm{CFM}(2\mathbb{Z}) \end{pmatrix}.$$

In other words, these are the column finite matrices, where the entries of the first column off of the main diagonal are all zero, the entries of the first row off of the main diagonal are rational, all other entries off of the main diagonal are even integers, and the entries on the main diagonal are integers which agree modulo 2. The ring R is a subring of Q, and we will see that R is an Ore ring by proving it is an order in the ring Q and that Q is classical. Just as before, given $q \in Q$ there is a common denominator n_i for the entries in the ith column. If we set

$$d = \begin{pmatrix} 2n_1 & & & \\ & 2n_2 & & \\ & & 2n_3 & \\ & & & \ddots \end{pmatrix} \in R$$

then $qd \in R$. Let S be the set of diagonal matrices in R with nonzero diagonal entries. We see that $Q = RS^{-1}$.

We next want to show that regular elements in R are regular in Q, and in fact we will show they are regular in T. Fix $r \in R$ and suppose there is some $t \in T$, $t \neq 0$, with $rt = 0$. Let t' be a nonzero column from t. After scaling if necessary, we may assume the entries of t' belong to $2\mathbb{Z}$. If we let $s \in R$ denote the matrix with t' as the second column and zeros elsewhere then $rs = 0$. Similarly suppose there is some $t \in T$, $t \neq 0$ with $tr = 0$. Let t' be a nonzero row of r. After scaling if necessary, we may assume that the first entry in t' belongs to $2\mathbb{Z}$ (but not necessarily the rest of the entries). If we let $s \in R$ denote the matrix with t' as the first row and zeros elsewhere then $sr = 0$.

Thus $Q = Q^r_{cl}(R)$ will follow once we know that Q is a classical ring. If $q \in Q$ is regular then, as we saw above, q is a unit in T. But an easy computation shows that $q^{-1} \in T$ must be of the same form as q, and thus $q^{-1} \in Q$. Note that Q is not Dedekind-finite for essentially the same reason that T is not Dedekind-finite.

Finally, we will show that R is Abelian by proving that the only idempotents in R are the trivial ones. After replacing e by $1 - e$ if necessary, we may assume that the entries (off the first row) are all even. If the entries of e off the first row are nonzero, then the entries of e^2 below the first row are divisible by more powers of 2 than those in e, contradicting the fact that $e^2 = e$. Thus $e = E_{1,1}e$. In particular, $e_{1,1} = E_{1,1}eE_{1,1}$ is an idempotent and is even, hence is zero. But then $e = e_{1,1}e = 0$ as desired. □

This example can be generalized if we replace $\mathbb{Z}$ by a commutative domain D with an element that behaves as 2 does above, and $\mathbb{Q}$ by the fraction field of D.

4. Semicommutative rings

Let R be a ring and let $a, b \in R$. Recall that R is *semicommutative* if $ab = 0$ implies $aRb = 0$, and reversible if $ab = 0$ implies $ba = 0$. Reversible rings are semicommutative, and the two conditions are quite similar.

We will show that given an Ore ring R which is semicommutative, the right classical quotient ring Q need not be semicommutative. As we want Q to properly contain R, we want a regular element x which is not invertible in R. We also want to simplify matters and make $\{x, x^2, \ldots\}$ be the set of non-unit regular elements in R. Thus, we want "polynomials" like $1 - x$ to already be invertible in R; and this naturally leads to looking at power series rings. Also, we need to be able to move x around in the correct manner in order to satisfy the right Ore condition. Working out what is needed, we arrive at the following theorem (cf. [12, Exercise 10.10]):

Theorem 5. *Let $A = A_0 + A_1$ be a ring where A_0 is a field and A_1 is a nilpotent ideal. Suppose there is an automorphism σ on A which sends $A_i \to A_i$ for each i. The ring $R = A[\![x;\sigma]\!]$ of skew power series is right Ore, with right classical ring of quotients $Q = A[\![x, x^{-1};\sigma]\!]$ the skew Laurent power series ring.*

Proof. Fix $m \geq 1$ so that $A_1^m = 0$. Also, we extend σ to an automorphism on R by taking $\sigma(x) = x$. Given $f \in R$, $f \neq 0$, write $f = f_0 + f_1$ where the coefficients of f_i belong to A_i. If $f_0 = 0$ then $f^m = 0$ and so f is not regular. On the other hand, if $f_0 \neq 0$ we will show below that there is both a left multiple of f and a right multiple of f which are powers of x. As x is easily seen to not be a zero-divisor, this shows that f is regular when $f_0 \neq 0$.

We now show that if $f_0 \neq 0$ then there exists $g \in R$ such that fg is a power of x (depending only on f). First notice that $f_0 = x^k v$ where v is a power series in x whose constant term is a unit. In particular, v is a unit in R. So after multiplying on the right by v^{-1}, we may assume without loss of generality that $f = x^k - f_1$ (changing the sign on f_1 to simplify the formula below). If we set

$$
\begin{aligned}
g &= \sum_{i=0}^{m-1} \left(\sigma^{-k}(f_1)\sigma^{-2k}(f_1)\cdots\sigma^{-ik}(f_1)\right) x^{(m-1-i)k} \\
&= x^{(m-1)k} + \sigma^{-k}(f_1)x^{(m-2)k} + \sigma^{-k}(f_1)\sigma^{-2k}(f_1)x^{(m-3)k} + \sigma^{-k}(f_1)\sigma^{-2k}(f_1)\sigma^{-3k}(f_1)x^{(m-4)k} + \cdots
\end{aligned}
$$

then we easily compute $fg = x^{mk}$, as the product turns into a telescoping sum with the last term belonging to $A_1^m = 0$. A symmetric computation shows that there is a left multiple of f which is a power of x.

Now, to show that R is right Ore let $f \in R$ be a regular element and let $r \in R$ be an arbitrary element. We want to show the existence of two elements $f', r' \in R$, with f' regular, such that $fr' = rf'$. Given g and k as in the computation above, we may choose $f' = x^{mk}$ and $r' = g\sigma^{-mk}(r)$. Let $Q = Q^r_{cl}(R)$, and note that if $f \in R$ is regular then $f^{-1} = gx^{-mk} \in Q$. Thus, we have $Q = A[\![x, x^{-1}; \sigma]\!] = RS^{-1}$ where $S = \{x, x^2, \ldots\}$. □

It is interesting to note in the theorem above that R is also left Ore, and Q is the two-sided classical quotient ring. We are now ready for our example.

Example 6. *A ring R which is (right) Ore and semicommutative but $Q = Q^r_{cl}(R)$ is not semicommutative.*

Construction and proof. Let F be a field. Let $A = F\langle a_i, b_i \ : \ i \in \mathbb{Z}\rangle$ with the noncommuting variables subject to the following conditions:

- all monomials of grade 3 or larger are zero,
- for any $i, j \in \mathbb{Z}$ we have $b_i b_j = 0$, $a_i a_j = 0$, $b_i a_j = 0$,
- for any $i, k \in \mathbb{Z}$ with $k \geq 0$ we have $a_i b_{i+k} = 0$.

In other words, A is the set of finite formal sums with coefficients in F over the set of monomials $B = \{1, a_i, b_i, a_i b_{i+k} \ : \ i \in \mathbb{Z}, k < 0\}$, with formal multiplication subject to the zeroing constraints above.

Let σ be the automorphism defined on A which fixes F and sends $a_i \mapsto a_{i+1}$ and $b_i \mapsto b_{i+1}$. If we let $R = A[\![x; \sigma]\!]$ then by the previous theorem we have that R is Ore and $Q = A[\![x, x^{-1}; \sigma]\!]$. Further, we compute that $a_0 x^{-1} b_0 = a_0 b_{-1} x^{-1} \neq 0 = a_0 b_0$. Thus Q is not semicommutative.

It remains to show that R is semicommutative. Suppose $cd = 0$ with $c, d \in R$. Write $c = c_0 + c_1 + c_2$ where c_i is the part of c with coefficients in grade i, and do similarly for d. Then $c_0 = d_0 = 0$, and hence $c_1 d_1 = 0$. Let $r \in R$ be arbitrary. We want to prove $crd = 0$. This quickly reduces to checking that $c_1 r_0 d_1 = 0$. This in turn reduces to checking that $c_1 x^k d_1 = c_1 \sigma^k(d_1) x^k = 0$ for every $k \geq 1$. We leave it to the reader to verify that this follows from the equation $c_1 d_1 = 0$ and the relations on the variables a_i and b_j. □

Recall that by [10, Theorem 2.6] if R is a right Ore ring then R is reversible if and only if $Q^r_{cl}(R)$ is reversible. In fact, the proof shows more generally that if S is a regular and right permutable subset of R, then R is a reversible ring if and only if $T = RS^{-1}$ is reversible. Thus, since reversibility and semicommutativity are closely related, the previous example is especially strong. Additionally, the next result puts limits on how much one might improve the example.

Theorem 7. *Let R be a semicommutative ring and let $S \subseteq R$ be a regular, right denominator set. The ring $T = RS^{-1}$ is Abelian, hence also Dedekind-finite.*

Proof. Suppose $au^{-1} \in T$ is an idempotent, where $a \in R$ and $u \in S$. For later use, since S is a right denominator set there exist $b \in R$ and $v \in S \subseteq C_R$ so that $av = ub$. We compute that $au^{-1}au^{-1} = au^{-1}$ implies $abv^{-1} = a$ hence $ab = av = ub$. Since $a(b-v) = 0$ and R is semicommutative, we have $0 = au(b - v) = a(a - u)v$. As v is regular, $a^2 = au$. Similarly, $(u - a)b = 0$ implies $0 = (u - a)ub = (u - a)av$. Thus $ua = a^2 = au$. In particular, a, u, and u^{-1} all commute with each other.

Given $r \in R$, from $a(a-u) = 0 = (a-u)a$ we have $ar(a-u) = 0 = (a-u)ra$. Hence $aru = ara = ura$, and multiplying on both sides by u^{-1} we have $rau^{-1} = u^{-1}ar = au^{-1}r$. Thus au^{-1} commutes with all elements of R, and by implication also the inverses of elements in S. This means that $au^{-1} \in T$ is central. □

In Example 6, the ring constructed was both left and right Ore. It turns out that when this happens one can conclude that part of the semicommutative property does pass to the full ring of quotients.

Proposition 8. *Let R be a semicommutative left and right Ore ring, and let Q be the classical quotient ring of R. If $\alpha, \beta \in Q$ satisfy $\alpha\beta = 0$ then $\alpha R\beta = 0$.*

Proof. Fix $\alpha, \beta \in Q$ with $\alpha\beta = 0$. As Q is a two-sided quotient ring, we can write $\alpha = u^{-1}a$, $\beta = bv^{-1}$ for $a, b \in R$, $u, v \in C_R$. Thus $u^{-1}abv^{-1} = 0$, and hence $ab = 0$. Since R is semicommutative we have $aRb = 0$ and hence $\alpha R\beta = 0$. □

5. NI and 2-primal rings

Recall that a ring R is said to be NI if the set of nilpotent elements, $N(R)$, is an ideal. A ring R is *2-primal* if the prime radical equals the set of nilpotent elements, $P(R) = N(R)$. It turns out neither of these properties passes to rings of quotients in general. Recall also, from our implication diagram, semicommutative rings are 2-primal and Abelian, 2-primal rings are NI, and NI rings are Dedekind-finite.

Example 9. *A ring R with an automorphism σ so that the skew polynomial ring $R[x;\sigma]$ is 2-primal, but $R[x, x^{-1};\sigma]$ is not NI.*

Construction and proof. Let F be a field and let $P = \prod_{i\in\mathbb{Z}} F$. Let $D \subseteq P$ be the set of constant sequences. In other words, if $e = (\ldots, 1, 1, 1, \ldots) \in P$ then $D = Fe$. Let $I = \bigoplus_{i\in\mathbb{Z}} F \subseteq P$. Set $R = D + I$, which is the ring constructed in [14, Example 4.5]. Let e_i be the sequence with 1 in the ith spot and 0's elsewhere, and define an automorphism σ on R by fixing D and $\sigma(e_i) = e_{i+1}$ for each $i \in \mathbb{Z}$. In other words, σ is the "right shift" action.

We first see that R is 2-primal because R is a commutative ring, and in fact $P(R) = N(R) = 0$. Next, let J be an ideal of R which is (i) proper, (ii) $\sigma(J) = J$, and (iii) for all ideals A and B of R with $\sigma(B) \subseteq B$ if $AB \subseteq J$ then $A \subseteq J$ or $B \subseteq J$. In the literature, such an ideal is called *strongly σ-prime.* If $J \neq 0$ fix $\alpha \in J$, $\alpha \neq 0$. After scaling, α has a 1, say, in the ith coordinate. But then $e_i = \alpha e_i \in J$. As $J = \sigma(J)$ we see that $J \supseteq I$. But I is a maximal ideal and J is proper, so $J = I$. Conversely, we see that I is strongly σ-prime, for it is (i) proper, (ii) $\sigma(I) = I$, and (iii) it is completely prime since $R/I \cong F$ is a field. Finally, note that (0) is not strongly σ-prime, for if we set A to be the ideal generated by e_1 and B to be the ideal generated by $\{e_2, e_3, \ldots\}$ we have $\sigma(B) \subseteq B$ and $AB = (0)$. Thus I is the only strongly σ-prime ideal in R, and it is completely prime.

By [15, Proposition 3.2], the ring $R[x;\sigma]$ is 2-primal. The ring $R[x, x^{-1};\sigma]$ is not NI, since e_0x is nilpotent but $e_0 = e_0xx^{-1}$ is not. □

The ring R in the previous example is a classical ring, but the ring $R[x;\sigma]$ is not right Ore. To see this note that $1 - x \in R[x;\sigma]$ is regular, and so it suffices to show $(1 - x)R[x;\sigma] \cap e_0C_{R[x;\sigma]} = \emptyset$. Let $f(x) \in R[x;\sigma]$ be nonzero and write $f(x) = a_mx^m + a_{m+1}x^{m+1} + \cdots + a_nx^n$ where $a_m, a_n \neq 0$ (and it may be the case that $n = m$). Suppose $(1 - x)f(x) = e_0r(x)$ with $r(x)$ regular. We compute

$$(1 - x)f(x) = a_mx^m + (a_{m+1} - \sigma(a_m))x^{m+1} + \cdots + (a_n - \sigma(a_{n-1}))x^n + (-\sigma(a_n))x^{n+1}.$$

For this to equal $e_0r(x)$ each of the coefficients must live in e_0R. This forces $a_m = e_0\beta_m$ for some $\beta_m \in F$. We then have $a_{m+1} = e_0\beta_{m+1} + e_1\beta_m$ for some $\beta_{m+1} \in F$. Repeating this process we obtain

$$a_n = e_0\beta_n + e_1\beta_{n-1} + \cdots + e_{n-m}\beta_m.$$

But then the last coefficient of $(1 - x)f(x)$, which is $-\sigma(a_n)$, cannot live in e_0R.

This computation suggests that if we want to construct an Ore ring which is NI, but for which the classical quotient ring is not NI, we may not want to use skew power series constructions (as $1 - x$ would be a unit). However, there is another alternative which we map out in the next two sections.

6. Freely adding zero-divisor conditions

The general question we are investigating is: If we are given a right Ore ring R with a property $\mathcal{P}$ then does its right classical ring of quotients Q also have property $\mathcal{P}$? In particular, if possible we wish to construct examples where $\mathcal{P}$ does not pass to Q. We also want to assume Q is constructed in as simple a manner as possible.

One of the easiest situation is when the set of regular elements of R are central. However, this case is not suited to constructing very many counterexamples as property $\mathcal{P}$ often does pass to Q in that case. Further, if the set of regular elements equals the units, then $R = Q$ which is also a problem. So we might as well assume the existence of a regular element x which will not be central in R nor a unit in R. Perhaps we could hope that all elements of Q are of the form ax^{-n} with $a \in R$. In other words, after we invert x, we have all of Q. Therefore, we might wish that $S = \{x, x^2, x^3, \ldots\}$ is both a left and a right denominator set.

In practice we have seen that this is often possible. Notice that these conditions imply that in R elements are either zero-divisors, or units up to a power of x. In particular, the element $1-x$ (and many other polynomials in x) must either be a unit or a zero-divisor. We can make it a unit by passing to a type of skew power series ring in the variable x; and this was done in a previous example. Alternatively, in this and the next section we will consider what happens if we try to force such polynomials to be a zero-divisors.

Lemma 10. *Let F be a commutative domain. Let $f(x) \in F[x]$ be a nonconstant, monic polynomial not divisible by x. Let $R = F[u,v]/(v^2 = 0, vf(u) = 0)$. Then v is not zero in R, $F[u]$ sits as a subring of R, and u is regular in R.*

Proof. We first show that v is not zero. For ease, we identify each variable with its image in the quotient ring, and assume they satisfy the given relations. Since $f(x)$ is nonconstant let $n > 0$ be the degree and write $f(x) = \sum_{i=0}^{n} \alpha_i x^i$, where $\alpha_n = 1$ since $f(x)$ is monic. A reduction system for monomials in R is given by

$$uv = vu,\ v^2 = 0,\ vu^n = -\sum_{i=0}^{n-1} \alpha_i v u^i.$$

(In the language of Bergman's Diamond Lemma [1], there are no "inclusions" and the three "overlaps" all easily resolve. The ordering on monomials is by total degree, with reverse lexicographical ordering.) Thus $v \neq 0$.

Next, if we specialize v to be zero then we see that all the relations in R are the trivial ones, and so $F[u]$ sits inside R. Finally, suppose that $ru = 0$ where $r \in R$. We can write r uniquely as a finite linear combination of monomials over F in reduced form, and so

$$r = \sum_{i=0}^{m} \beta_i u^i + \sum_{i=0}^{n-1} \gamma_i v u^i,$$

where $\beta_i, \gamma_i \in F$. When writing ru in reduced form the only reduction to make is to replace $\gamma_{n-1} v u^n$ by $-\sum_{i=0}^{n-1} \gamma_{n-1} \alpha_i v u^i$. But as $ru = 0$ this implies that $\beta_i = 0$ for all i, and

$$\gamma_{n-1}\alpha_i = \gamma_{i-1}$$

for each i in the range $0 \leq i \leq n-1$ (where $\gamma_{-1} = 0$). As $f(x)$ is not divisible by x, $\alpha_0 \neq 0$. As F is a domain, this implies that $\gamma_{n-1} = 0$. We then can conclude that $\gamma_i = 0$ for all i. Thus $r = 0$ as desired. As R is commutative this proves that u is regular. □

We have thus shown that any given monic, nonconstant polynomial in u, not divisible by u, can be turned into a (two-sided) zero-divisor without destroying the regularity of u. Without much more work, we can do the same for all such polynomials, simultaneously.

Lemma 11. *Let F be a field. Let I be the set of monic, nonconstant polynomials in $F[x]$, not divisible by x. Let*
$$R = F[u, v_f \ : \ f \in I]/(v_f v_g = 0, v_f f(u) = 0 \ : \ f, g \in I).$$
An element $r \in R$ is regular if and only if there is an element $s \in R$ such that $rs = sr = u^n$ for some $n \geq 0$.

Proof. First, note that R is commutative. Second, if we fix some $f \in I$ and specialize v_g to be zero for each $g \in I - \{f\}$ then we see that the ring from the previous lemma sits inside R. In particular, $v_f \neq 0$ for each $f \in I$. Next, a simple reduction system exists just as before, and one shows just as before that u is not a zero-divisor.

Finally, fix $r \in R$. If there is some $s \in R$ with $rs = u^n$ then clearly r is not a zero-divisor. Conversely, suppose that r is not a zero-divisor. Writing r as a linear combination of monomials in reduced form, we have
$$r = \sum_{i=0}^{m} \beta_i u^i + \sum_{f \in J} \sum_{i=0}^{\deg(f)-1} \gamma_{i,f} v_f u^i$$
where $J \subset I$ is a finite set. It is clear that if it were the case that $\beta_i = 0$ for all i, then r would be a zero-divisor (try multiplying by v_g for any $g \in I$). So, we may assume $\beta_m \neq 0$. As F is a field, replacing r by $\beta_m^{-1} r$ if necessary, we lose no generality in assuming $\beta_m = 1$. (Making this change just forces us to multiply s by β_m.) If $\beta_i \neq 0$ for some $i < m$ then $\sum_{i=0}^{m} \beta_i u^i = g(u) u^\ell$ for some $g \in I$ and some $\ell \geq 0$, as is seen by dividing out by the highest power of u possible. But then $v_g r = 0$. Thus $\beta_i = 0$ for all $i < m$.

We now have $r = u^m + \sum_{f \in F} \sum_{i=0}^{\deg(f)-1} \gamma_{i,f} v_f u^i$. Letting $s = u^m - \sum_{f \in F} \sum_{i=0}^{\deg(f)-1} \gamma_{i,f} v_f u^i$ we have $rs = u^{2m}$ as desired. □

These results allow us to concentrate on inverting u without having to worry about inverting other polynomials in u, and we also do not have to deal with power series considerations. In the next section we equate x and u.

7. Adding more variables

Now we return to the fact that x will not be a central element. Let a_0 be an element that doesn't commute with x. In the ring of quotients we are building we will have $x^{-1} a_0 = a_{-1} x^{-n}$ for some element $a_{-1} \in R$ and $n \geq 0$, since we are only inverting powers of x. To keep things symmetrical we should take $n = 1$ and so $x a_{-1} = a_0 x$. It is useful to think of moving x past a_0 as an automorphism, and so we introduce variables $\{a_i \ : \ i \in \mathbb{Z}\}$ subject to the relations $x a_i = a_{i+1} x$. In other words, we are naturally led to skew polynomial rings.

Except this isn't quite the skew polynomial situation. At some point we will bring back the variables v_f, with their usual relations, but also subject to the conditions $v_f a_i = a_i v_f = 0$. This will force the zero-divisor conditions we had in the previous section on polynomials in x, without interfering with conditions on the a_i.

If we want a_0 to generate a nil ideal but also guarantee $a_0 x^{-1}$ is not nilpotent, an easy way to do so is assume $a_i a_j = 0$ whenever $j \neq i - 1$. In fact, with these conditions in place the ideal generated by all of the a_i is nil.

One might ask what happens if we now take the ring R generated by x, the a_i, and the v_f, subject to the relations we discussed. It turns out there are still problems that need to be dealt with. An arbitrary element $r \in R$ is of the form $r = g(x) x^\ell + s$ where s is in the ideal generated by the a_i and v_f and either $g \in I$ or $g = 1$. If $g \in I$ then r is a zero-divisor (multiply by v_g). So now consider when $r = x^\ell + s$. Occasionally this element will already be a unit in R, and so there is nothing to do. For example, $1 + a_0$ is a unit, with inverse $1 - a_0$. The element $x + a_0x$ is not a unit, but will be once we invert x.

Unfortunately, the element $x-a_0$ may not be a unit in R, even after we invert x. Intuitively, its inverse (if it has an inverse) will look something like the power series $v = x^{-1}+x^{-1}a_0x^{-1}+x^{-1}a_0x^{-1}a_0x^{-1}+\cdots$. This suggests that the element has no "common denominator" which is another reason we do not work with power series rings in this section. More disastrously, an easy computation shows that if $x - a_0$ does have an inverse v in R, and a_0v is nilpotent, then $x^{-1} \in R$ which we don't want.

So, we are forced to either adjoin more denominators than just x^{-1} to form Q, or we need to make elements like $x - a_0$ into zero-divisors. In the following proposition we do the latter. For simplicity, we leave out the variables v_f for now. (One may think of the proposition to follow as what happens if we specialize all of the v_f to be zero.) We begin with an easy lemma.

Lemma 12. *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a $\mathbb{N}$-graded ring. If $r = 1+r'$ where $r' \in R_1 \oplus R_2 \oplus \cdots$ has a right inverse $s \in R$ then s is a two-sided inverse and s is of the same form as r (meaning, $s = 1 + s'$ for some $s' \in R_1 \oplus R_2 \oplus \cdots$).*

Proof. Write $s = s_0 + s'$ where $s_0 \in R_0$ and $s' \in R_1 \oplus R_2 \oplus \cdots$. Since $rs = 1$, looking at grade 0 pieces we find $s_0 = 1$. Thus s has the same form as r.

Let S denote the ring of formal series of the form $\sum_{i=0}^{\infty} x_i$ where $x_i \in R_i$. One checks that r is a unit in S by formally solving for the inverse. But as inverses are unique, and $s \in S$ is a right inverse, we see that s is a two-sided inverse for r. □

Note: The previous lemma is just a generalization of the proof that if R is Dedekind-finite then so is $R[x]$, which is seen by first showing that $R[\![x]\!]$ is Dedekind-finite and then passing to the subring $R[x]$.

Proposition 13. *Let F be a field. Let*

$$A = F\langle a_i, b_i \ : \ i \in \mathbb{Z}\rangle/(a_ia_j = 0 \text{ when } j \neq i-1, b_ib_j = 0, a_ib_j = 0).$$

and let

$$T = A[x, x^{-1}; \sigma]$$

where σ is the unique automorphism on A which fixes F and sends $a_i \mapsto a_{i+1}, b_i \mapsto b_{i+1}$.

Let $r \in 1 + \sum_{i\in\mathbb{Z}} a_iT$ and let $X = T/(b_0r)$. We have $\overline{b_0} = 0$ in X if and only if r is a unit in T.

Proof. It is straightforward to check that A is a nonzero ring and that σ is a well-defined automorphism. Letting $R = A[x;\sigma]$, the element x is regular, and the set $S = \{x, x^2, \ldots\}$ is a left and right denominator set. So we can adjoin an inverse element x^{-1} without any problems, and in fact $T = RS^{-1}$. Thus, the ring T is not the zero ring. Further, the given set of relations for A forms a reduction system for T, after we add the (forced, and obvious) relations $xx^{-1} = x^{-1}x = 1$, $xa_i = a_{i+1}x$, $xb_i = b_{i+1}x$, $x^{-1}a_i = a_{i-1}x^{-1}$, and $x^{-1}b_i = b_{i-1}x^{-1}$.

If r is a unit then in the quotient ring X the equality $\overline{b_0} = 0$ follows by multiplying the relation $\overline{b_0r} = 0$ on the right by $\overline{r^{-1}}$. For the forward implication assume that $\overline{b_0} = 0$. This means that in R we have an equality

$$b_0 = \sum_j m_jb_0rn_j \tag{1}$$

where m_j and n_j are monomials in the letters x, x^{-1}, a_i, b_i. Writing $m_jb_0rn_j$ in reduced form, we see that if m_j involves any a_i or b_i then the summand is zero. So we may assume m_j is a power of x, say $m_j = x^\ell$ for some $\ell \in \mathbb{Z}$. But then, any nonzero monomial appearing in the summand $m_jb_0rn_j$, when written in reduced form, begins with $b_{-\ell}$. As the left hand side of Equation (1) does not have any summands involving b_i except when $i = 0$, we see that all such terms on the right hand side must cancel. So we may as well assume $m_j = 1$ for all j. Thus

$$b_0 = \sum_j b_0 r n_j = b_0 r t$$

where $t = \sum_j n_j$. Hence $b_0(1 - rt) = 0$. An easy computation using reduced forms shows that the right annihilator of b_0 in T is the right ideal $\sum_{i\in\mathbb{Z}} b_i T$, which is nilpotent of index 2. Hence $1 - rt = x$ is nilpotent, and so $rt = 1 - x$ is a unit. This proves that r has a right inverse in T. Fix $s \in T$ to be such a right inverse.

Since $1 = rs$ and $r = 1 + r'$ with $r' \in \sum_{i\in\mathbb{Z}} a_i T$, a quick computation shows that both r and s live in the subring of T generated by x, x^{-1} and the a_i's. In other words if $A' = F\langle a_i \,:\, i \in \mathbb{Z}\rangle/(a_i a_j = 0 \text{ when } j \neq i-1)$ and $T' = A'[x, x^{-1}; \sigma] \subseteq T$ then $r, s \in T'$. The ring T' is a graded ring, where x and x^{-1} live in grade 0, and each a_i lives in grade 1. By the previous lemma s is a two-sided inverse to r and is of the same form as r. □

The dedicated reader might ask why we did not give a reduction system for the ring X. Since r can be complicated and nonhomogeneous, this makes it difficult (but in specific cases, not impossible) to turn the single relation $b_0 r = 0$ into a reduction system. Further, we really don't need a reduction system for X, but rather a reduction system for the quotient of R by the ideal $K = Tb_0rT \cap R$ which is even more complicated. It is for these reasons that we worked in T and passed to the quotient ring X only when necessary.

While it is not strictly necessary, it is conceptually handy to have a generating set for K. Given an element $t \in T$ let $n_t \in \mathbb{Z}$ denote the integer for which $tx^{n_t} \in R$ but $tx^{n_t - 1} \notin R$. (If $t = 0$ we take $n_t = 0$.) In other words, n_t is the exact number of copies of x in the denominator of t. Notice that this quantity is well defined since no nonzero element of R is infinitely divisible by x. A generating set for K is given by:

$$K = \left(b_i \sigma^i(r x^{n_r}), b_i \sigma^i(r a_j x^{n_{r a_j}}) \;:\; i, j \in \mathbb{Z}\right). \tag{2}$$

While it is useful to have this generating set for K, all we really need is that K is the ideal in R making the previous proposition work.

At this point, we can reintroduce the variables v_f and we have the main theorem of this section. As the theorem is complicated to state we introduce the construction in stages.

Theorem 14. *Let F be a field. Let I be the set of monic, nonconstant polynomials in $F[x]$ not divisible by x (which we think of as needing to be made into zero-divisors). Let $A = F\langle x, a_i \,:\, i \in \mathbb{Z}\rangle$ subject to the relations*

1. *$xa_i = a_{i+1}x$ (the skewing condition for x, which shifts subscripts up one), and*
2. *$a_i a_j = 0$ when $j \neq i - 1$ (the condition which forces the quotient ring to not be NI).*

In other words $A \cong (F\langle a_i \,:\, i \in \mathbb{Z}\rangle/(a_i a_j \text{ when } j \neq i-1))\,[x; \sigma]$, where $\sigma : a_i \mapsto a_{i+1}$.

Let $B = F\langle x, a_i, v_f \,:\, i \in \mathbb{Z}, f \in I\rangle \supseteq A$ subject to the additional relations

3. *$xv_f = v_f x$ (a simplifying condition, allowing v_f to lie in the center)*
4. *$v_f v_g = v_f a_i = a_i v_f = 0$ (more simplifying conditions for v_f, allowing it to generate a nilpotent ideal of index 2), and*
5. *$v_f f(x) = 0$ (the condition forcing $f(x)$ to be a zero-divisor).*

Let J be the set of elements $r \in A$ where

- (i) *$r = x^\ell + r'$ with $r' \in \sum_j a_j A$, and*
- (ii) *there is no $s \in A$ (of the same form as in (i)) with rs a power of x.*

Let $R = F\langle x, v_f, a_i, b_{i,r} \,:\, i \in \mathbb{Z}, f \in I, r \in J\rangle \supseteq B$ with the variables subject to the additional relations

6. $xb_{i,r} = b_{i+1,r}x$ *(more skewing conditions for x, which shifts subscripts up one),*
7. $v_f b_{i,r} = b_{i,r}v_f = 0$ *(more simplifying conditions for v_f, allowing it to generate a nilpotent ideal of index 2),*
8. $a_i b_{j,r} = b_{i,r}b_{j,r'} = 0$ *(simplifying conditions mirroring what happens in the previous proposition), and*
9. $b_{i,r}\sigma^i(rx^{n_r}) = 0 = b_{i,r}\sigma^i(ra_j x^{n_{ra_j}})$ *(the conditions forcing r to be a zero-divisor, but allowing x to remain regular, as in Equation (2) above).*

Then R is a (right) Ore ring and its right classical quotient ring Q is formed by adjoining x^{-1}. Furthermore, R is 2-primal but Q is not NI.

Proof. It is useful to note that specializing all of the v_f's to zero shows that indeed A is a subring of B, and specializing all of the $b_{i,r}$'s to zero shows that B is a subring of R. After specializing all other variables to 0, we see that $x \neq 0$ in R and is not a unit. It is also useful to note that we can extend the map σ to all of R by taking $a_i \mapsto a_{i+1}$, $b_{i,r} \mapsto b_{i+1,r}$, $v_f \mapsto v_f$, and $x \mapsto x$. For convenience, we also denote this extended map by σ.

Next, we prove that $S = \{x, x^2, \ldots\}$ is a right (and left) denominator set in R. This boils down to two main issues. First, one verifies that σ is an automorphism on R, and that $xr = \sigma(r)x$ for every $r \in R$; which we leave to the reader. This demonstrates the right permutability of S. Second, we need to check that x is regular in R. We do more than that by classifying all regular elements of R.

Given $r \in R$ we can write it in the form $r_0+r_1+r_2+r_3$ where $r_0 \in F[x]$, $r_1 \in \sum_{i\in\mathbb{Z}} a_i R = \sum_{i\in\mathbb{Z}} a_i A$, $r_2 \in \sum_{f\in I} v_f R = \bigoplus_{f\in I} v_f F[x]$, and $r_3 \in \bigoplus_{r\in J, i\in\mathbb{Z}} b_{i,r}R$. If $r_0 = 0$ or if $r_0 = x^m g(x)$ for some $m \geq 0, g(x) \in I$, then $v_g r = 0$ and $v_g \neq 0$ (again by specializing and using previous results). Thus r is a zero-divisor in those cases.

Now suppose $r_0 = x^m$ for some $m \geq 0$. If $r' = r_0 + r_1 \in J$ then $b_{0,r'}r = 0$, and $b_{0,r'}$ is not zero. So we may assume $r_0 + r_1 \notin J$. This implies the existence of an element $s \in A$ with $r's = x^n$ (for some $n \geq m$). The previous proposition also yields $sr' = x^n$. We compute that $rs(x^n - \sigma^{-n}(r_2 s + r_3 s)) = x^{2n}$ and we also compute that $(x^n - \sigma^n(sr_2 + sr_3))sr = x^{2n}$. So we will see r is regular once we prove x is regular, and furthermore after we invert x we will have that r is a unit. So suppose that $xy = 0$ for some $y \in R$. Write $y = y_0 + y_1 + y_2 + y_3$ as we did above for r. From $xy = 0$ and noting that multiplying by x preserves this decomposition, we have $xy_0 = 0$, $xy_1 = 0$, $xy_2 = 0$ and $xy_3 = 0$. This implies $y_0 = 0$ as x is not a zero-divisor in $F[x]$. We have $y_1 = 0$ from the reduction system given for A. We also have that $y_2 = 0$ by Lemma 11.

Finally, we wish to prove $y_3 = 0$, and this is the heart of this theorem. Suppose $y_3 \neq 0$. From the relations given by conditions (6)-(9) (especially condition (8)), there exists some $r \in J$ so that if we specialize $b_{i,r'} = 0$ for all $r' \in J - \{r\}$ then y_3 is still nonzero in this specialization. Further, by conditions (3)-(5), we may specialize *all* of the v_f to zero, and y_3 is still nonzero. Notice that our specialized ring is now exactly the ring we work with in Proposition 13. As $xy_3 \in K = R \cap T(b_{0,r}r)T$ (where T here is the ring from that proposition) and x is invertible in T, $y_3 \in K$. Thus, as condition (9) simply asserts the relations in K, $y_3 = 0$ after all. This finishes the proof that x is regular in the ring R, and that S is a right denominator set.

If we take $Q = RS^{-1}$, then all regular elements of R are units in Q and Q is the classical right quotient ring for R. The element $a_0 x^{-1}$ is not nilpotent even though a_0 is. So Q is not NI. On the other hand, the ideal $L \leq R$ generated by all of the variables besides x is a nil ideal and $R/L \cong F[x]$ is a domain. Furthermore, L is a sum of nilpotent right ideals. Thus $L = P(R) = N(R)$. □

Once again, there are limits to how "bad" the example above can be.

Theorem 15. *If R is a semicommutative right Ore ring, $a \in R$ is nilpotent, and $x \in C_R \setminus U(R)$ then $x - a \in C_R \setminus U(R)$.*

Proof. Fix $a \in R$, $x \in C_R$ with a nilpotent (say $a^n = 0$ for some $n \geq 1$) and assume $a \neq 0$. There are two cases to consider.

Case 1: $x - a$ is a zero divisor. Assume $b(x-a) = 0$ for some $b \in R$ and $b \neq 0$ (the case $(x-a)b = 0$ is similar and is left to the reader). As R is semicommutative, we have $0 = ba^{n-1}(x-a) = ba^{n-1}x$. As x is regular, $ba^{n-1} = 0$. Again using the semicommutative property we have $0 = ba^{n-2}(x-a) = ba^{n-2}x$. Again, as x is regular, $ba^{n-2} = 0$. Repeating this argument we reach $b = 0$, a contradiction.

Case 2: $x - a$ is a unit of R, say $v = (x-a)^{-1} \in R$. We compute $1 = (x-a)v = xv - av$. As R is semicommutative it is also NI and so av is nilpotent. Thus $1 - xv$ is nilpotent, say $(1 - xv)^m = 0$. Expanding, we have $0 = 1 - xy$ for some $y \in R$ and so x is right invertible. Repeating the argument with $1 = v(x-a)$ we have that $x \in U(R)$. □

8. Open questions

We end with a few interesting questions we were unable to answer.

Question 1. If R is duo (and hence Ore) is $Q = Q^r_{cl}(R)$ duo? Is Q semicommutative, 2-primal, or NI?

Question 2. If Q is a left McCoy classical ring and R is a right order in Q, is R left McCoy?

Question 3. If R is (right) Ore, Abelian, and 2-primal then is $Q^r_{cl}(R)$ Dedekind-finite?

We note that the answer to the converse of the first part of Question 1 has been answered. Brungs [3] constructed a classical ring Q which is a domain and duo, with a right order R which is not left or right duo, as follows: Let $F = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where ω is a primitive cube-root of unit, let σ be an automorphism of F which sends $\sqrt[3]{2} \mapsto \sqrt[3]{2}\omega$ and fixes $\mathbb{Q}$, let $Q = F[\![x, x^{-1}; \sigma]\!]$, and let $R = (F \cap Q) + xF[\![x; \sigma]\!] \subseteq Q$. One checks that Q is a division ring hence duo, that R is not duo, but that Q is obtained by inverting the regular denominator set $\{x, x^2, \ldots\}$ in R.

The following proposition tells us how to make partial progress on the last part of the first question.

Lemma 16. *Assume R is a semicommutative ring, $S \subseteq C_R$ is a right denominator set, and let $T = RS^{-1}$. If $a \in R$, $t \in T$ with $at = 0$ then $aRt = 0$.*

Proof. Write $t = rs^{-1}$ with $r \in R$ and $s \in S$. Since $at = 0$ we have $ar = 0$. As R is semicommutative, $aRr = 0$ and thus $aRt = 0$. □

Proposition 17. *Assume R is a semicommutative ring, $S \subseteq C_R$ is a right denominator set, and let $T = RS^{-1}$. If $au^{-1} \in T$ is nilpotent then $a \in R$ is nilpotent.*

Proof. Assume $(au^{-1})^n = 0$, so $au^{-1}au^{-1} \cdots au^{-1} = 0$. Using the previous lemma, we can insert u to obtain $0 = auu^{-1}au^{-1} \cdots au^{-1} = aau^{-1} \cdots au^{-1}$. Repeatedly insert u before the leftmost copy of u^{-1} to obtain $a^n = 0$. □

We leave it as an easy exercise for the reader to show that the answer to the second question is yes if R is semicommutative.

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