AMITSUR'S PROPERTY FOR SKEW POLYNOMIALS OF DERIVATION TYPE

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Abstract. We investigate when radicals $\mathfrak{F}$ satisfy Amitsur's property on skew polynomials of derivation type, namely $\mathfrak{F}(R[x;\delta]) = (\mathfrak{F}(R[x;\delta]) \cap R)[x;\delta]$. In particular, we give a new argument that the Brown-McCoy radical has this property. We also give a new characterization of the prime radical of $R[x;\delta]$.

1. Introduction

A radical $\mathfrak{F}$ is said to satisfy *Amitsur's property* if for every ring R we have $\mathfrak{F}(R[x]) = (\mathfrak{F}(R[x]) \cap R)[x]$. The terminology is a consequence of Amitsur's work in [1], where he showed that the Jacobson radical, prime radical, Levitzki radical, and the upper nilradical all have this property. Amitsur also gave the following two criteria (as distilled in [10, Lemma 1]), which together are sufficient to guarantee Amitsur's property, and in practice are often easily verified: (1) $\mathfrak{F}$ is hereditary and (2) whenever R has characteristic p we have $\mathfrak{F}(R[x]) \cap R[x^p - x] \subseteq \mathfrak{F}(R[x^p - x])$. These criteria were used for instance in [10, Theorem 3] and [12, Theorem 3.2] to show respectively that the Brown-McCoy radical and Behrens radical have Amitsur's property. For three recent and interesting studies of Amitsur's property in connection with other natural radical properties, see [18, 13, 9]. For information on general radical theory, and Amitsur's property in particular, we recommend [7, Section 4.9] as a good reference.

In the present paper we are concerned with radicals of skew polynomial rings of derivation type (which we will also call "differential polynomial rings"). Let R be a ring, and let δ be a derivation on R, meaning an additive map satisfying the product rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. We can then define the ring $R[x;\delta]$ consisting of left polynomials with standard addition, and multiplication subject to the skewed constraint $xa = ax + \delta(a)$. Radicals of this ring were first studied in the work of Ferrero, Kishimoto, and Motose [6], where it was shown that the Jacobson, prime, and Wedderburn radicals again possess (an analogue of) Amitsur's property. Letting $\mathfrak{J}$ denote the Jacobson radical, they raised the question of whether $\mathfrak{J}(R[x;\delta]) \cap R$ is a nil ideal of R, and this question remains open.

In this paper, we generalize the work in [6] to give a set of general criteria for the analogue of Amitsur's property to hold over $R[x;\delta]$. We say that a radical $\mathfrak{F}$ satisfies the *δ-Amitsur property* if:

$$\text{(1.1)}\quad \begin{array}{c}\text{For all rings } R \text{ and for all derivations } \delta \text{ of } R,\\ (\mathfrak{F}(R[x;\delta]) \cap R)[x;\delta] = \mathfrak{F}(R[x;\delta]).\end{array}$$

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When possible, we also describe explicitly the resulting ideal $\mathfrak{F}(R[x;\delta]) \cap R \trianglelefteq R$.

Throughout the paper, R will be an arbitrary associative ring possibly without 1, and δ will be an arbitrary derivation on R. We reserve fractal letters for radicals, capitalized English letters for rings and sets, and lowercase English letters for ring elements or variables. We write $I \trianglelefteq R$ to mean that I is a two-sided ideal of R. When we use the word "radical" we will mean a radical in the sense of Kurosh and Amitsur [7, Definition 2.1.1]. (Two exceptions to this convention are when we speak of the "Wedderburn radical" and "bounded nilradical" whose names have been established in the literature, but these are not technically radicals.) To be precise, $\mathfrak{F}$ is a radical if it assigns to each ring R an ideal $\mathfrak{F}(R) \trianglelefteq R$ satisfying the next three conditions:

(R1) If $\mathfrak{F}(R) = R$ and $R \to S$ is a surjective ring homomorphism, then $\mathfrak{F}(S) = S$.
(R2) We have $\mathfrak{F}(\mathfrak{F}(R)) = \mathfrak{F}(R)$, and if $I \trianglelefteq R$ with $\mathfrak{F}(I) = I$, then $\mathfrak{F}(I) \subseteq \mathfrak{F}(R)$.
(R3) The equality $\mathfrak{F}(R/\mathfrak{F}(R)) = 0$ always holds.

As usual, we say the radical $\mathfrak{F}$ is *hereditary* if it satisfies the further condition:

(R4) If $I \trianglelefteq R$, then $\mathfrak{F}(I) = \mathfrak{F}(R) \cap I$.

2. Preliminaries on rings with derivations

In this short section we collect a few definitions and results that are important when working with skew polynomials of derivation type. Given a derivation δ on a ring R, following the literature we say that a subset $S \subseteq R$ is a *δ-subset* if $\delta(S) \subseteq S$. When $I \trianglelefteq R$ is an ideal and a δ-subset, we simply say that I is a *δ-ideal*. The next lemma (whose proof we omit) relates the ideals of R and $R[x;\delta]$.

Lemma 2.1. *Let R be a ring and δ a derivation of R.*

(1) If I is a right ideal of R, then $I[x;\delta]$ is a right ideal of $R[x;\delta]$ and $I[x;\delta] \cap R = I$.
(2) If I is a δ-ideal of R, then $I[x;\delta]$ is an ideal of $R[x;\delta]$.
(3) If J is an ideal of $R[x;\delta]$, then $J \cap R$ is an ideal of R.

Given $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x;\delta]$, with $a_i \in R$ for each $0 \leq i \leq n$, we abuse notation by writing $\delta^j(f(x)) = \sum_{i=0}^{n} \delta^j(a_i)x^i$. (Note that δ is not a derivation on $R[x;\delta]$.) We will find many occasions to make use of the following extremely useful result, which capitalizes on the startlingly pretty formula $xf(x) - f(x)x = \delta(f(x))$ which recursively leads to

$$\delta^j(f(x)) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} x^i f(x) x^{j-i}. \tag{2.1}$$

Lemma 2.2. *If J is an ideal of $R[x;\delta]$ and $f(x) \in J$, then*

$$R\delta^j(f(x))R \subseteq J$$

for every $j \geq 0$. Moreover, if J is closed under multiplication by x, then $\delta^j(f(x)) \in J$ for all $j \geq 0$ and $J \cap R$ is a δ-ideal of R.

Proof. Given $r, s \in R$, we obtain $r\delta^j(f(x))s \in J$ by multiplying (2.1) on the left by r and the right by s. To prove the last sentence, note that $J \cap R$ is clearly an ideal in R, and when J is closed under multiplication by x then (2.1) proves the δ-invariance claims. □

3. The prime radical

The prime radical $\mathfrak{P}(R)$ of a ring R has many equivalent definitions:

- The lower radical described by the class of nilpotent rings, and thus the limit of the (higher) Wedderburn radicals.
- The intersection of all the prime ideals.
- The set of strongly nilpotent elements.

- The limit of the (higher) left (or right) T-nilpotent radideals, see [8].
- The limit of the (higher) bounded nilradicals.

If we write $\mathfrak{P}_\delta(R) := \mathfrak{P}(R[x;\delta]) \cap R$, the ideal $\mathfrak{P}_\delta(R) \trianglelefteq R$ can similarly be described as a limit of δ-Wedderburn ideals [6, Theorem 2.1 and Corollary 2.2], as the intersection of δ-prime δ-ideals [5, Theorem 1.1], and as the set of strongly δ-nilpotent elements [11, Proposition 1.11]. In this section we will pursue an analogue of the fourth bullet point.

A subset $S \subseteq R$ is *left T-nilpotent* if for every sequence of elements $s_1, s_2, \ldots \in S$ there is some index $n \geq 1$ such that $s_1 s_2 \cdots s_n = 0$. Such subsets are quite well-behaved, as evidenced by the next two lemmas.

Lemma 3.1. *Let R be a ring, $I \subseteq R$, and $J \trianglelefteq R$. If J is left T-nilpotent and $\overline{I}$ is left T-nilpotent in R/J, then $I + J$ is left T-nilpotent.*

Proof. This is a slight strengthening of [8, Lemma 4.2], with the same proof, *mutatis mutandis.* □

Lemma 3.2. *Let R be a ring, $I \subseteq R$, and J a one-sided ideal of R.*

(1) If J is left T-nilpotent, then the two-sided ideal generated by J is left T-nilpotent.
(2) If I and J are left T-nilpotent, then $I + J$ is left T-nilpotent.

Proof. This is [8, Proposition 4.3]. □

The *left T-nilpotent radideal* is defined by $\mathfrak{T}_\ell(R) := \{a \in R \,:\, aR \text{ is left } T\text{-nilpotent}\}$ or equivalently as the set

$$\{a \in R \,:\, \text{the ideal generated by } a \text{ is left } T\text{-nilpotent}\}.$$

For more information and basic facts, see [8, Section 4]. This radical-like ideal satisfies the δ-Amitsur property, and moreover we can explicitly describe the derived ideal as follows.

Theorem 3.3. *Given a ring R with a derivation δ, then $\mathfrak{T}_\ell(R[x;\delta]) = \mathfrak{T}_{\ell,\delta}(R)[x;\delta]$, where*

$$\begin{aligned} \mathfrak{T}_{\ell,\delta}(R) &:= \left\{ a \in R \,:\, \sum_{j=0}^{\infty} \delta^j(a)R \text{ is left } T\text{-nilpotent} \right\} \\ &= \mathfrak{T}_\ell(R[x;\delta]) \cap R. \end{aligned} \tag{3.1}$$

Proof. Define $\mathfrak{T}_{\ell,\delta}(R)$ as above, and note that this is an ideal of R by Lemma 3.2(1), and hence a δ-ideal. We first show that $\mathfrak{T}_\ell(R[x;\delta]) \subseteq \mathfrak{T}_{\ell,\delta}(R)[x;\delta]$. Fix $f(x) \in \mathfrak{T}_\ell(R[x;\delta])$, and write $f(x) = \sum_{i=0}^n a_i x^i$ with $a_i \in R$ for each $0 \leq i \leq n$. Set $J_i = \sum_{j=0}^\infty \delta^j(a_i)R$ for $0 \leq i \leq n$. We will show that J_n is left T-nilpotent.

Fix a sequence of elements $r_1, r_2, \ldots \in R$ and a sequence of non-negative integers $i_1, i_2, \ldots$ and set $t_k := \delta^{i_1}(a_n) r_1 \delta^{i_2}(a_n) r_2 \cdots \delta^{i_k}(a_n) r_k$ for each $k \geq 1$. Each of the elements

$$\delta^{i_1}(f(x)) r_1 \delta^{i_2}(f(x)) r_2, \delta^{i_3}(f(x)) r_3 \delta^{i_4}(f(x)) r_4, \ldots$$

belongs to $\mathfrak{T}_\ell(R[x;\delta])$ by Lemma 2.2, and so there exists some index k such that

$$\delta^{i_1}(f(x)) r_1 \delta^{i_2}(f(x)) r_2 \cdots \delta^{i_k}(f(x)) r_k = 0.$$

The degree nk coefficient in this product is exactly t_k, and thus $t_k = 0$.

We now show that J_m is left T-nilpotent for any $0 \leq m \leq n$. By a recursive argument, we may assume that $J_{m+1}, J_{m+2}, \ldots, J_n$ are left T-nilpotent, and thus the two-sided ideal J generated by $J_{m+1} + J_{m+2} + \cdots + J_n$ is a left T-nilpotent ideal by Lemma 3.2. Lemma 3.1 tells us that in order to prove J_m is left T-nilpotent we can pass to the quotient ring R/J, and thus we may assume m is the leading index of $\overline{f(x)}$. But then the methods of the previous paragraph apply, and so J_m is left T-nilpotent as desired, which proves the inclusion $\mathfrak{T}_\ell(R[x;\delta]) \subseteq \mathfrak{T}_{\ell,\delta}(R)[x;\delta]$.

We now show the opposite inclusion $\mathfrak{T}_\ell(R[x;\delta]) \supseteq \mathfrak{T}_{\ell,\delta}(R)[x;\delta]$. Fix $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathfrak{T}_{\ell,\delta}(R)[x;\delta]$, and also fix a sequence of polynomials $g_1(x), g_2(x), \ldots \in R[x;\delta]$. Every coefficient in the product $f(x)g_1(x)f(x)g_2(x)\cdots f(x)g_k(x)$ is a $\mathbb{Z}$-linear combination of terms of the form

$$\delta^{j_1}(a_{i_1})\delta^{j_1'}(r_1)\delta^{j_2}(a_{i_2})\delta^{j_2'}(r_2)\cdots\delta^{j_k}(a_{i_k})\delta^{j_k'}(r_k) \tag{3.2}$$

where $j_1, j_1', j_2, j_2', \ldots, j_k, j_k'$ are non-negative integers, for each $m \geq 1$ we have that r_m is a coefficient of $g_m(x)$, that $a_{i_m} \in \{a_0, a_1, \ldots, a_n\}$, and that

$$j_m, j_m' < \sum_{p=1}^{m} \deg(g_p(x)) + m\deg(f(x)) + 1.$$

In particular, note that there are only *finitely* many choices for each j_m, j_m', i_m, and r_m (given that we have already chosen the sequence $g_1(x), g_2(x), \ldots$).

Suppose by way of contradiction that for each $k \geq 1$ there is a term as in (3.2) which is nonzero. By an application of König's tree lemma (see, for instance, how it is applied in [8]), we can *fix* sequences $i_1, i_2, \ldots, j_1, j_2, \ldots, j_1', j_2', \ldots \in \mathbb{N}$ and $r_1, r_2, \ldots \in R$ such that

$$s_k := \delta^{j_1}(a_{i_1})\delta^{j_1'}(r_1)\delta^{j_2}(a_{i_2})\delta^{j_2'}(r_2)\cdots\delta^{j_k}(a_{i_k})\delta^{j_k'}(r_k) \neq 0$$

for *every* $k \geq 1$. On the other hand, some a_i must occur infinitely often in the sequence $a_{i_1}, a_{i_2}, \ldots$, and since $a_i \in \mathfrak{T}_{\ell,\delta}(R)$ we must have $s_k = 0$ for k large enough, giving us the needed contradiction. □

Example 3.4. The ideal $\mathfrak{T}_{\ell,\delta}(R)$ is not always the maximal δ-ideal contained in $\mathfrak{T}_\ell(R)$. Indeed, let $R = \mathbb{F}_2[x_0, x_1, \ldots : x_i^2 = 0]$ with derivation $\delta(x_i) = x_{i+1}$. We see that $\mathfrak{T}_\ell(R) = \mathfrak{P}(R) = (x_0, x_1, \ldots)$ is already a δ-ideal, but $x_0 \notin \mathfrak{T}_{\ell,\delta}(R)$ since

$$x_0\delta(x_0)\delta^2(x_0)\cdots\delta^k(x_0) = x_0x_1x_2\cdots x_k \neq 0$$

for each $k \geq 1$.

Following [8, Section 5], put $\mathfrak{T}_\ell^{(0)} = (0)$ and recursively define the higher left T-nilpotent radideals

$$\mathfrak{T}_\ell^{(\alpha)}(R) = \{a \in R \;:\; a + \mathfrak{T}_\ell^{(\beta)}(R) \in \mathfrak{T}_\ell(R/\mathfrak{T}_\ell^{(\beta)}(R))\}$$

if α is the successor of β, and if α is a limit ordinal

$$\mathfrak{T}_\ell^{(\alpha)}(R) = \bigcup_{\beta<\alpha} \mathfrak{T}_\ell^{(\beta)}(R).$$

We can now extend Theorem 3.3 to the higher left T-nilpotent radideals, by a simple transfinite induction.

Corollary 3.5. *The higher left T-nilpotent radideals $\mathfrak{T}_\ell^{(\alpha)}$ satisfy the δ-Amitsur property* (1.1). *Thus, we can write*

$$\mathfrak{T}_\ell^{(\alpha)}(R[x;\delta]) = \mathfrak{T}_{\ell,\delta}^{(\alpha)}(R)[x;\delta]$$

for a unique δ-ideal $\mathfrak{T}_{\ell,\delta}^{(\alpha)}(R) \trianglelefteq R$.

Proof. Assume the statement is true for all ordinals $\beta < \alpha$. Note that we have a natural surjection $R[x;\delta] \to R[x;\delta]/\mathfrak{T}_{\ell,\delta}^{(\beta)}(R)[x;\delta]$, and a natural isomorphism $R[x;\delta]/\mathfrak{T}_{\ell,\delta}^{(\beta)}(R)[x;\delta] \cong (R/\mathfrak{T}_{\ell,\delta}^{(\beta)}(R))[x;\overline{\delta}]$, where $\overline{\delta}$ is the derivation induced on the factor ring by δ, since $\mathfrak{T}_{\ell,\delta}^{(\beta)}(R)$ is a δ-ideal.

First consider the case when α is the successor of some ordinal β. By Theorem 3.3, we know $\mathfrak{T}_\ell$ satisfies the δ-Amitsur property. Thus, the elements of $\mathfrak{T}_\ell((R/\mathfrak{T}_{\ell,\delta}^{(\beta)}(R))[x;\overline{\delta}])$ are determined by the constant polynomials in this ideal. Lifting these constant polynomials through the natural isomorphism and surjection above, we obtain the desired conclusion.

Finally, if α is a limit ordinal, we have

$$\begin{aligned}\mathfrak{T}_\ell^{(\alpha)}(R[x;\delta]) &= \bigcup_{\beta<\alpha}\mathfrak{T}_\ell^{(\beta)}(R[x;\delta]) = \bigcup_{\beta<\alpha}\mathfrak{T}_{\ell,\delta}^{(\beta)}(R)[x;\delta]\\ &= \left(\bigcup_{\beta<\alpha}\mathfrak{T}_{\ell,\delta}^{(\beta)}(R)\right)[x;\delta].\end{aligned}$$

Thus, we can take $\mathfrak{T}_{\ell,\delta}^{(\alpha)}(R) = \bigcup_{\beta<\alpha}\mathfrak{T}_{\ell,\delta}^{(\beta)}(R)$. □

As noted at the beginning of this section, these higher left T-nilpotent radideals stabilize to the prime radical. Thus, we obtain a new characterization of the prime radical of $R[x;\delta]$, and we also recover the result of Ferroro, Kishimoto, and Motose that the prime radical satisfies the δ-Amitsur property [6].

Proposition 3.6. *Given a ring R with a derivation δ, then we have $\mathfrak{P}(R[x;\delta]) = \mathfrak{P}_\delta(R)[x;\delta]$, where*

$$\mathfrak{P}_\delta(R) = \mathfrak{P}(R[x;\delta])\cap R$$

is the limit of the δ-ideals $\mathfrak{T}_{\ell,\delta}^{(\alpha)}(R)$. In particular, the prime radical satisfies the δ-Amitsur property.

4. An alternate characterization of the δ-Amitsur property

Perhaps the most well-known and utilized condition for checking the Amitsur property is that of Krempa in [10, Theorem 1], which states that a radical $\mathfrak{F}$ has Amitsur's property if and only if for every ring R,

$$\mathfrak{F}(R[x])\cap R = 0 \;\text{ implies }\; \mathfrak{F}(R[x]) = 0.$$

This equivalence holds true for differential polynomial rings, with only minimal changes to the proofs, as we will now show. To begin, we first need the basic fact that radicals are closed under multiplications in unital extensions.

Lemma 4.1. *Let $\mathfrak{F}$ be a radical and let R be a ring. If $I \trianglelefteq R$, then $\mathfrak{F}(I) \trianglelefteq R$ and hence $\mathfrak{F}(I)\subseteq\mathfrak{F}(R)$.*

Proof. This is [2, Theorem 1]. □

To apply this to unital extensions, we make the following definition. If R is unital we set $R^1 = R$, otherwise we let $R^1 = R\oplus\mathbb{Z}$ be the Dorroh extension of R by $\mathbb{Z}$, where addition is component-wise and multiplication is given by the rule $(r,m)(s,n) = (rs+ms+nr, mn)$ for all $r,s\in R$ and $m,n\in\mathbb{Z}$. Note that $R\trianglelefteq R^1$.

Corollary 4.2. *Let $\mathfrak{F}$ be a radical, let R be a ring, and let δ be a derivation on R. It happens that $x\mathfrak{F}(R[x;\delta]) + \mathfrak{F}(R[x;\delta])x \subseteq \mathfrak{F}(R[x;\delta])$. Consequently, $\mathfrak{F}(R[x;\delta])\cap R$ is a δ-ideal of R.*

Proof. If R contains 1, then this result is trivial. If R does not contain 1, let R^1 be the Dorroh extension as above. We extend δ to R^1 in the only possible way which preserves the fact that δ is a derivation, by having δ act trivially on $\mathbb{Z}$. It is easy to check that $R[x;\delta]\trianglelefteq R^1[x;\delta]$. Thus, by Lemma 4.1, $\mathfrak{F}(R[x;\delta])$ is an ideal in $R^1[x;\delta]$, and in particular is closed under multiplication by x. Finally, apply Lemma 2.2. □

Note that this corollary holds true for more general extensions (such as Ore extensions, see [11] for the definition), not just those of derivation type. This fact may be useful in studying radicals of such rings, but we won't make use of such generality here.

With this corollary in place, we are now in a position to prove that Krempa's characterization holds for differential polynomial rings.

Proposition 4.3. *Let $\mathfrak{F}$ be a radical. This radical has the δ-Amitsur property* (1.1) *if and only if for every ring R and every derivation δ on R it happens that*

$$\mathfrak{F}(R[x;\delta]) \cap R = 0 \text{ implies } \mathfrak{F}(R[x;\delta]) = 0. \tag{4.1}$$

Proof. This follows by modifying the proof of [10, Theorem 1], but we include the proof for completeness. The forward direction is clear, so we only do the reverse.

Set $A = \mathfrak{F}(R[x;\delta]) \cap R$, which is a δ-ideal of R and so $A[x;\delta]$ is a well-defined ring. By Corollary 4.2, $A[x;\delta] \subseteq \mathfrak{F}(R[x;\delta])$. Thus, $\mathfrak{F}(R[x;\delta]/A[x;\delta]) = \mathfrak{F}(R[x;\delta])/A[x;\delta]$ by standard radical arguments. On the other hand $R[x;\delta]/A[x;\delta] \cong \overline{R}[x;\overline{\delta}]$ where $\overline{R} = R/A$ and $\overline{\delta}$ is the induced derivation on the factor ring (which exists since A is a δ-ideal). We then have a string of isomorphisms

$$\begin{aligned}
\mathfrak{F}(\overline{R}[x;\overline{\delta}]) \cap \overline{R} &\cong \mathfrak{F}(R[x;\delta]/A[x;\delta]) \cap R/A \\
&\cong (\mathfrak{F}(R[x;\delta])/A[x;\delta]) \cap ((R + A[x;\delta])/A[x;\delta]) \\
&= (\mathfrak{F}(R[x;\delta]) \cap (R + A[x;\delta]))\,/A[x;\delta] \\
&= (\mathfrak{F}(R[x;\delta]) \cap R + A[x;\delta])/A[x;\delta] \\
&= (A + A[x;\delta])/A[x;\delta] = 0.
\end{aligned}$$

By assuming the implication in the statement of the proposition, we must have

$$0 = \mathfrak{F}(\overline{R}[x;\overline{\delta}]) \cong \mathfrak{F}(R[x;\delta])/A[x;\delta]$$

and hence

$$\mathfrak{F}(R[x;\delta]) = A[x;\delta] = (\mathfrak{F}(R[x;\delta]) \cap R)[x;\delta].$$

In other words, $\mathfrak{F}$ has the δ-Amitsur property. □

5. Two different attacks on radicals

There are two distinct ways of showing that the Jacobson radical $\mathfrak{J}$ satisfies Amitsur's property. The first comes from Amitsur's original characterization found in [1] of the Jacobson radical as $\mathfrak{J}(R[x]) = N[x]$ for some nil ideal $N \trianglelefteq R$, and involves a clever isomorphism trick. The second method has its roots in unpublished work of Bergman [3], and utilizes the fact that the Jacobson radical behaves well with respect to what are called in the literature *finite centralizing extensions* (see [15, Section 10.1]). We will now describe how to transfer these arguments to the situation of skew polynomial extensions of derivation type, which allows us to show that many radicals have the δ-Amitsur property.

Both arguments begin in the same way. Let $\mathfrak{F}$ be a radical, and assume that $\mathfrak{F}$ does not have the δ-Amitsur property. Thus, we may fix a ring R and a derivation δ on R such that there exists a polynomial

$$f(x) \in \mathfrak{F}(R[x;\delta]) \setminus (\mathfrak{F}(R[x;\delta]) \cap R)[x;\delta]$$

with $n := \deg(f(x)) \geq 1$ minimal among all choices of R and δ. After passing to a factor ring if necessary, we may as well assume $\mathfrak{F}(R[x;\delta]) \cap R = 0$, and hence this minimal degree n occurs over a ring where the differential version of Krempa's criterion (4.1) fails. Write $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathfrak{F}(R[x;\delta])$ with each $a_i \in R$. Now, one quickly verifies that the map sending $h(x) \to h(x+1)$ is an automorphism of $R[x;\delta]$ (even in the case when R is non-unital), and radicals are invariant under automorphisms, so $f(x+1) \in \mathfrak{F}(R[x;\delta])$. But since $\deg(f(x+1) - f(x)) < \deg(f(x))$, minimality of n implies $f(x+1) - f(x) = 0$.

Expanding yields $f(x+1) - f(x) = na_n x^{n-1} +$ lower order terms, and so $na_n = 0$. If either $n = 1$ or R is a $\mathbb{Q}$-algebra we have a contradiction (to the fact that $a_n \neq 0$), so we may assume $n \geq 2$. Letting $m > 1$ be the smallest integer with $ma_n = 0$ and fixing a prime $p|m$, we can replace $f(x)$ by $(m/p)f(x)$, and thus $pa_n = 0$. But $pf(x) \in \mathfrak{F}(R[x;\delta])$, so by minimality of degree we in fact have that $pf(x) = 0$.

It is at this juncture that the arguments of Amitsur and Bergman diverge, so we first describe Amitsur's argument. We want to reduce to the case $pR = 0$. To facilitate such a reduction, we make the additional assumption that $\mathfrak{F}$ is hereditary. Then letting

$$R_p = \{r \in R \ : \ pr = 0\} \trianglelefteq R$$

the hereditary assumption gives us

$$\mathfrak{F}(R_p) = \mathfrak{F}(R) \cap R_p. \tag{5.1}$$

From the harder of the two inclusions in (5.1) we get $f(x) \in \mathfrak{F}(R) \cap R_p \subseteq \mathfrak{F}(R_p)$, and from the (easier) other inclusion we see that there are no nonzero polynomials of smaller degree in $\mathfrak{F}(R_p)$. Thus, after replacing R by R_p if necessary, we may reduce to the case $pR = 0$.

In the usual polynomial case one now shows that since $f(x+1) = f(x)$ we have $f(x) = g(x^p - x)$ for some polynomial $g(x) \in R[x]$. Not surprisingly, in the differential polynomial case one can still show that $f(x)$ is a left polynomial in the variable $t := x^p - x$ (see [1] for the quick argument). Note that for any $r \in R$ we have

$$(x^p - x)r = r(x^p - x) + (\delta^p - \delta)r,$$

since $pR = 0$. Moreover, because R is an $\mathbb{F}_p$-algebra it is straightforward to check that δ^p is a derivation on R, and thus $\delta^p - \delta$ is a derivation as well. Hence, $g(x^p - x) = g(t) \in R[t; \delta^p - \delta]$ and clearly $\deg(g) < \deg(f)$. In the standard polynomial case, one argues that $g(t)$ belongs to $\mathfrak{F}(R[t])$ (often, by appealing to additional assumptions on the radical $\mathfrak{F}$) and then since $R[t] \cong R[x]$ we have $g(x) \in \mathfrak{F}(R[x])$, contradicting the minimality of n. Unfortunately, there is in general no degree preserving isomorphism between $R[x; \delta]$ and $R[t; \delta^p - \delta]$. Thus, Amitsur's argument often breaks at this point for differential polynomial rings.

We now turn to the other method. Propitiously, in Bergman's argument there is no roadblock. (Note that we are now not necessarily assuming $\mathfrak{F}$ is hereditary, although if that assumption holds then the reductions in the previous two paragraphs can still be made.) The idea is simply that we need more automorphisms which will give us more control over $f(x)$. In that spirit, let $q > n$ be an integer prime and let $\zeta = \zeta_q$ be a primitive qth root of unity (from $\mathbb{C}$). Given any ring A, define $A' := A[\zeta] = A \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]$. (Here we are abusing notation slightly since A may already contain the complex qth roots of unity; so one must remember that $A[\zeta]$ is shorthand for $A \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]$.) Note that A sits (isomorphically) as a (possibly non-unital) subring of A'. In our case, we focus on the ring $R[x; \delta] \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta] \cong R'[x; \delta']$ where δ' is the derivation on $R' = R[\zeta]$ determined by the rule $\delta'(r \otimes \alpha) = \delta(r) \otimes \alpha$. In this ring we have the automorphisms

$$\sigma_j(h(x)) = h(x + \zeta^j) \tag{5.2}$$

for $j \in \mathbb{N}$. Identifying $f(x)$ with $f(x) \otimes 1 \in R'[x; \delta']$, the constant term of $f(x + \zeta^j) - f(x)$ is exactly

$$\sum_{i=1}^{n} a_i \otimes \zeta^{ij}.$$

Thus

$$\begin{aligned} g(x) &:= \sum_{j=0}^{q-1} \zeta^{-nj} \left(f(x + \zeta^j) - f(x) \right) \\ &= q a_n \otimes 1 + \sum_{j=0}^{q-1} \sum_{i=1}^{n-1} a_i \otimes \zeta^{j(i-n)} + \text{ higher order terms,} \end{aligned}$$

but by switching the order of summation and noting that ζ^{i-n} is also a primitive qth root of unity (for $1 \leq i \leq n-1$ since $n < q$), we have

$$\sum_{j=0}^{q-1}\sum_{i=1}^{n-1} a_i \otimes \zeta^{j(i-n)} = \sum_{i=1}^{n-1}\left(a_i \otimes \sum_{j=0}^{q-1}\left(\zeta^{i-n}\right)^j\right) = \sum_{i=1}^{n-1} a_i \otimes 0 = 0.$$

Thus qa_n is the constant term in $g(x)$.

We now define a property which holds for many radicals. We say that $\mathfrak{F}$ *respects finite cyclotomic extensions* when it happens that

$$\text{for all rings } A, \text{ and all integer primes } q, \ \ \mathfrak{F}(A) = \mathfrak{F}(A[\zeta_q]) \cap A, \tag{5.3}$$

with $\zeta_q \in \mathbb{C}$ a primitive qth root of unity and $A[\zeta_q] := A \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_q]$. Under this assumption, we see that

$$f(x) \in \mathfrak{F}(R[x;\delta]) \subseteq \mathfrak{F}((R[x;\delta])[\zeta]) = \mathfrak{F}(R'[x;\delta'])$$

and hence by Lemma 4.1 (to get closure under multiplication by powers of ζ) we have $g(x) \in \mathfrak{F}(R'[x;\delta'])$. But $\deg(g(x)) < \deg(f(x))$ and so the minimality condition on n tells us that every coefficients of $g(x)$ is an elements of $\mathfrak{F}(R'[x;\delta'])$. In particular $qa_n \in \mathfrak{F}((R[x;\delta])[\zeta]) \cap R[x;\delta] = \mathfrak{F}(R[x;\delta])$. But $pa_n = 0 \in \mathfrak{F}(R[x;\delta])$ and $\gcd(p,q) = 1$, hence $a_n \in \mathfrak{F}(R[x;\delta]) \cap R = 0$, yielding a contradiction. Putting this all together we have the following result.

Theorem 5.1. *If $\mathfrak{F}$ is a radical which respects finite cyclotomic extensions, then $\mathfrak{F}$ has the δ-Amitsur property.*

In the case of hereditary radicals, we can make an even nicer statement by arguing along the lines of [6], as was done in [4] using the stronger notion of normalizing extensions.

Theorem 5.2. *Let $\mathfrak{F}$ be a hereditary radical. Also assume that $\mathfrak{F}$ respects finite field extensions, meaning:*

$$\begin{gathered}\text{For every } \mathbb{F}_p\text{-algebra } A, \text{ and every integer } m \geq 1, \\ \mathfrak{F}(A) = \mathfrak{F}(A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}) \cap A.\end{gathered} \tag{5.4}$$

Then $\mathfrak{F}$ has the δ-Amitsur property.

Proof. We start by working contrapositively. Assume $\mathfrak{F}$ is a hereditary radical and does not have the δ-Amitsur property. As in the argument above, we can reduce to the case where (i) R is an $\mathbb{F}_p$-algebra, (ii) $\mathfrak{F}(R[x;\delta]) \cap R = 0$, and (iii) $f(x) = \sum_{i=0}^{n} a_i x^i$ in the set $\mathfrak{F}(R[x;\delta]) \setminus (\mathfrak{F}(R[x;\delta]) \cap R)[x;\delta]$ has minimal degree $n \geq 2$.

Now, assume by way of contradiction that $\mathfrak{F}$ respects finite field extensions. Setting $R' = R \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}$ with $m \geq n$, we obtain

$$f(x) \in \mathfrak{F}(R[x;\delta] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}) \cong \mathfrak{F}(R'[x;\delta'])$$

where δ' is the natural extended derivation. On the ring $R'[x;\delta']$, the map $h(x) \mapsto h(x+t)$ is an automorphism for every $t \in \mathbb{F}_{p^m}$. Thus $f(x+t) - f(x) \in \mathfrak{F}(R'[x;\delta'])$ for every $t \in \mathbb{F}_{p^m}$. Since $\deg(f(x+t) - f(x)) < \deg(f(x))$, the minimality condition on n implies that the constant term of $f(x+t) - f(x)$, which is just $\sum_{i=1}^{n} a_i t^i$, belongs to $\mathfrak{F}(R'[x;\delta'])$. A Vandermonde matrix argument (using the fact that $p^m > n$) tells us that each $a_i \in \mathfrak{F}(R'[x;\delta'])$ for $1 \leq i \leq n$. Hence

$$f(x) \in (\mathfrak{F}(R[x;\delta] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}) \cap R)[x;\delta] = (\mathfrak{F}(R[x;\delta]) \cap R)[x;\delta],$$

giving us the needed contradiction. $\square$

Remark 5.3. (1) In the argument of Theorem 5.2 we really only need $\mathfrak{F}$ to respect finite field extensions $\mathbb{F}_{p^m}/\mathbb{F}_p$ for a sequence of strictly increasing positive integers $m = m_1 < m_2 < \ldots$. We will make use of this small improvement shortly.

(2) Suppose A is an $\mathbb{F}_p$-algebra and $\mathfrak{F}$ is a radical. Let $q \neq p$ be prime, and let $\Phi_q(x) = (x^q - 1)/(x-1)$ be the qth cyclotomic polynomial. Since $\Phi_q(x)$ is separable modulo p, we can write $\Phi_q(x) \equiv \prod_{i=1}^t f_i(x)$ (mod p) where the f_i are relatively prime, monic polynomials which are irreducible modulo p. We then have

$$A \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_q] = A \otimes_{\mathbb{F}_p} \mathbb{F}_p[y]/(\Phi_q(y)) = \prod_{i=1}^{t} \left(A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^{n_i}}\right)$$

where $n_i = \deg(f_i)$ for each i. An easy computation now shows that if $\mathfrak{F}$ respects finite field extensions, then it respect finite cyclotomic extensions (at least when $q \neq p$).

(3) Finite cyclotomic extensions and finite field extensions are just special cases of finite centralizing extensions. See Section 10 of [15] for some further nice examples.

We now give an easy example of where these methods apply. (Note that each of the radicals in the corollary below was already covered by [4].)

Corollary 5.4. *The Jacobson radical, Levitzki radical, prime radical, and Brown-McCoy radical each have the δ-Amitsur property.*

Proof. Let $\mathfrak{F}$ be any of the above radicals. First suppose R does not have 1. We then have $\mathfrak{F}(R) \subseteq \mathfrak{F}(R^1)$ since $\mathfrak{F}$ is hereditary. But $R^1/R \cong \mathbb{Z}$ is $\mathfrak{F}$-semisimple; so we must have equality $\mathfrak{F}(R) = \mathfrak{F}(R^1)$. A similar argument holds if we replace R by $R[x;\delta]$ (since, in this case, $\mathbb{Z}[x]$ is $\mathfrak{F}$-semisimple). So without loss of generality, we can reduce to the case that R is unital. It is well known that each of these radicals respects finite cyclotomic extensions in case the ring is unital, see for instance [16, p. 454]. Now just apply Theorem 5.1. $\square$

As the proof that the Brown-McCoy radical respects finite cyclotomic extensions is by no means trivial, we give an alternate proof in that case, which may be of independent interest.

Proposition 5.5. *The Brown-McCoy radical $\mathfrak{G}$ has the δ-Amitsur property.*

Proof. Since $\mathfrak{G}$ is a hereditary radical, by Remark 5.3(1) and Theorem 5.2 it suffices to show that whenever F is a finite field, R is an F-algebra, and K/F is a field extension with $[K : F] = 2$, then $\mathfrak{G}(R) = \mathfrak{G}(S) \cap R$ where $S = R \otimes_F K$. Further, if R is non-unital and R^* is the Dorroh extension of R by F, then $\mathfrak{G}(R) \subseteq \mathfrak{G}(R^*)$ since R is an ideal in R^*. On the other hand $R^*/R \cong F$ is Brown-McCoy semisimple. Thus $\mathfrak{G}(R) = \mathfrak{G}(R^*)$, and hence we only need to prove the equality holds in the case when R is a unital F-algebra and S is a unital overring. Fix an F-basis $B = \{1, b\}$ for the extension K/F. We now prove the equality $\mathfrak{G}(R) = \mathfrak{G}(S) \cap R$.

($\subseteq$): Assume by way of contradiction that $r \in \mathfrak{G}(R) \setminus \mathfrak{G}(S)$. Then $r \notin M$ for some maximal ideal of S. Since S/M is a simple ring, $1 - \sum_{i=1}^m s_i r s_i' \in M$ for some $s_i, s_i \in S$. Expanding this sum in terms of the basis B, we have $\sum_{i=1}^m s_i r s_i' = r_1 + r_2 b$ for some $r_1, r_2 \in RrR \subseteq \mathfrak{G}(R)$, and also $1 - r_1 - r_2 b \in M$.

Consider the two-sided ideal $R(1 - r_1)R \trianglelefteq R$. If this were a proper ideal, it would be contained in a maximal ideal M' of R. But since $r_1 \in \mathfrak{G}(R)$ we would also have $r_1 \in M'$ and hence $1 = (1-r_1)+r_1 \in M'$ which is impossible. Thus $R(1 - r_1)R = R$, so we can write $1 = \sum_{j=1}^n t_j(1 - r_1)t_j'$ for some elements $t_j, t_j' \in R$. Hence

$$\sum_{j=1}^{n} t_j(1 - r_1 - r_2 b)t_j' = 1 - \left(\sum_{j=1}^{n} t_j r_2 t_j'\right) b = 1 - ub \in M$$

for some element $u \in Rr_2R \subseteq \mathfrak{G}(R)$.

Since K is a finite field, we can fix an integer $k \geq 2$ such that $b^k = 1$. Thus

$$1 - u^k = (1 - ub)(1 + ub + u^2b^2 + \cdots + u^{k-1}b^{k-1}) \in M.$$

But $u^k \in \mathfrak{G}(R)$, and so by the same argument as at the beginning of the previous paragraph we have $R(1 - u^k)R = R$. Thus $1 \in M$, giving us the needed contradiction.

($\supseteq$): Assume by way of contradiction $r \in (\mathfrak{G}(S) \cap R) \setminus \mathfrak{G}(R)$. Fix a maximal ideal I of R, with $r \notin I$. Notice that $I \otimes_F K = I \oplus Ib$ is a proper ideal of S, and so is contained in a maximal ideal M of S. Since $r \in \mathfrak{G}(S)$ we have $r \in M$. But $I \subseteq M$ as well, so $R = I + RrR \subseteq M$. This yields $1 \in M$, a contradiction. □

Remark 5.6. These same techniques fail for the Behrens radical β. By [12, Proposition 3.1] we have $\beta(S) \cap R \subseteq \beta(R)$ for *any* unital finite centralizing extension, and so we only care about the reverse inclusion. However, from the Example following Proposition 3.1 in [12] there is a Behrens radical ring R which is an $\mathbb{R}$-algebra and $\beta(R) \not\subseteq \beta(R \otimes_{\mathbb{R}} \mathbb{C})$. This shows that β does not even respect quadratic field extensions. One might still hold out hope in the finite field case, but in fact the example can be modified to disprove that case as follows.

Let K be the field of rational functions in the commuting variables x_i (for $i \in \mathbb{Z}$), with coefficients in $\mathbb{F}_3(y)$. Let σ by the $\mathbb{F}_3(y)$-automorphism of K which sends $x_i \mapsto x_{i+1}$. One verifies that the skew Laurent polynomial ring (of automorphism type) $K[x, x^{-1}; \sigma]$ is a simple domain with $\mathbb{F}_3(y)$ as the center.

Define $\mathbb{H} = \left(\frac{2,y}{F}\right)$ as the quaternion algebra over $F = \mathbb{F}_3(y)$ with generators i, j and relations given by $i^2 = 2$, $j^2 = y$, and $ij = -ji$. This is a non-split algebra, hence is a 4-dimensional division algebra with center $\mathbb{F}_3(y)$. The tensor product $T = \mathbb{H} \otimes_{\mathbb{F}_3(y)} K[x, x^{-1}; \delta]$ is also a central simple $\mathbb{F}_3(y)$-algebra. The remainder of the construction is essentially unchanged from [12] after replacing $\mathbb{R}$ by $\mathbb{F}_3(y)$ and $\mathbb{C}$ by $\mathbb{F}_9(y)$ everywhere, so we leave those details to the interested reader. Finally, note that $\mathbb{H} \otimes_{\mathbb{F}_3(y)} \mathbb{F}_9(y) \cong \mathbb{M}_2(\mathbb{F}_9(y))$ since $\mathbb{F}_9$ contains a square-root of 2, and also note that $S = R \otimes_{\mathbb{F}_3(y)} \mathbb{F}_9(y) \cong R \otimes_{\mathbb{F}_3} \mathbb{F}_9$.

6. Open Questions

One radical which is conspicuously missing from Corollary 5.4 is the upper nilradical. Amitsur in 1956 proved that the upper nilradical has Amitsur's property [1], but only in 2014 was a proof finally found by Smoktunowicz for the fact that the upper nilradical is homogeneous in $\mathbb{Z}$-graded rings [17]. This work was subsequently extended to gradings over semigroups in [14]. Unfortunately, these methods are somewhat orthogonal to those employed in this paper, and so we ask:

Question 6.1. Does the δ-Amitsur property hold for the upper nilradical?

Perhaps even more difficult is the case of the Behrens radical, for we know that this radical behaves poorly with respect to field extensions.

Another set of questions involves element-wise characterizations of the ideal $\mathfrak{F}(R[x;\delta]) \cap R$, when R has a derivation δ. While much work has been done in the case of the prime radical, there are other basic radicals which might yield to similar analyses. In particular, we ask:

Question 6.2. If $\mathfrak{L}$ is the Levitzki radical, is there a simple description of the ideal $\mathfrak{L}(R[x;\delta]) \cap R$, for any ring R with a derivation δ?

Finally, in Section 3 we gave a list of five characterizations of the prime radical, four of which are now known to generalize extremely well to the differential polynomial case. Thus we ask:

Question 6.3. How does the bounded nilradical behave in differential polynomial rings? Is there a simple element-wise description of this ideal in terms of the coefficient ring?

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