AN ENSEMBLE OF IDEMPOTENT LIFTING HYPOTHESES

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Abstract. Lifting idempotents modulo ideals is an important tool in studying the structure of rings. This paper lays out the consequences of lifting other properties modulo ideals, including lifting of von Neumann regular elements, lifting isomorphic idempotents, and lifting conjugate idempotents. Applications are given for IC rings, perspective rings, and Dedekind-finite rings, which improve multiple results in the literature. We give a new characterization of the class of exchange rings; they are rings where regular elements lift modulo all left ideals.

We also uncover some hidden connections between these lifting properties. For instance, if regular elements lift modulo an ideal, then so do isomorphic idempotents. The converse is true when units lift. The logical relationships between these and several other important lifting properties are completely characterized. Along the way, multiple examples are developed that illustrate limitations to the theory.

1. Introduction

Given a ring R, an element $a \in R$ is said to be *von Neumann regular* if $a = aba$ for some $b \in R$. In such a case we call b an *inner inverse* of a. It is common to refer to von Neumann regular elements simply as regular elements, and we will follow that convention in this paper. Similarly, a *regular ring* is one where each of its elements is regular. These rings were first introduced in the work of von Neumann on continuous geometries. It is well known that a ring R is regular if and only if every finitely generated left (or right) ideal is generated by an idempotent, if and only if all R-modules are flat (i.e. R has weak dimension 0).

Even when working on the level of elements rather than rings, regularity captures important ring-theoretic and module-theoretic information. To give one example, if R is the endomorphism ring of some module (for instance, by identifying R in the natural way with $\mathrm{End}(R_R)$), then regular elements correspond to those endomorphisms whose kernels and images are direct summands [18, Exercise 4.14A$_1$]. As each direct sum decomposition of a module is determined by an idempotent in the endomorphism ring, we see that regular elements are intricately connected to idempotents. This relationship will be made more precise after we introduce some further terminology in the next section.

Besides the reasons listed above, there is one further justification for interest in regular rings; idempotents lift modulo all left and right (and two-sided) ideals in such rings [23, Proposition 1.6], and consequently we can convert information about the projective modules over a factor ring into information on projective modules over the original ring. The purpose of this paper is to study such lifting more generally, and so we introduce some of the basic definitions here. Letting I be a one-sided ideal of a ring R, we say that $x \in R$ is

$$\left.\begin{array}{l} \textit{regular} \\ \textit{idempotent} \\ \textit{a unit} \end{array}\right\} \text{ modulo } I \text{ if } \left\{\begin{array}{l} \exists\, y \in R:\ x - xyx \in I \\ x - x^2 \in I \\ \exists\, y \in R:\ 1 - xy, 1 - yx \in I. \end{array}\right.$$

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If x is a regular element modulo I, then we say that x *lifts regularly modulo* I if there exists a regular element $a \in R$ such that $x - a \in I$. Lifting of idempotents and lifting of units modulo I are defined similarly.

Many people have studied the lifting of idempotents modulo a one-sided ideal I. For instance, it is classically known that idempotents lift if I is nil, and also lift in the case R is I-adically complete and I is a two-sided ideal. Nicholson [23] has shown that exchange rings are precisely those rings where idempotents lift modulo every one-sided ideal. In [14] some applications of lifting of idempotents in connection with the existence of projective covers are given. The papers [6] and [21] clarify when systems of (half-)orthogonal or commuting idempotents lift to systems that are still, respectively, orthogonal or commuting, and important applications are given.

On the other hand, the lifting of regular elements seems to be a new avenue of research initiated in [15], where it was proven that if idempotents lift modulo every left ideal contained in a two-sided ideal I, then regular elements lift modulo I. We were initially led to the study of lifting of regular elements in our pursuit of finding conditions on a two-sided ideal I such that isomorphic idempotents modulo I lift to isomorphic idempotents.

An outline of the structure of the paper follows. We begin in §2 by recalling some basic facts and definitions concerning idempotents, setting up some of the necessary terminology and machinery for the remainder of the paper. We then commence a systematic study of idempotent lifting hypotheses in §3, paying particular attention to lifting isomorphic or conjugate idempotents. Our main result in this section is the construction of a ring R with a two-sided ideal such that units and idempotents lift, but isomorphic idempotents do not lift. In §4 we continue this theme by connecting the lifting of regular elements with idempotent lifting. Modulo two-sided (but not necessarily one-sided) ideals regular lifting implies that isomorphic idempotents lift. Further, strong lifting (to be defined shortly) implies regular elements lift, even over one-sided ideals. All of this information is brought together in §5, to give a complete picture of the relationships between each of the lifting properties studied in this paper. Important connections between lifting and exchange rings are clarified, including a new characterization of exchange rings as those rings for which regular elements lift modulo all left ideals. We end in §§6-7 by discussing further avenues of study, and pointing out a few limitations of the types of lifting properties considered.

To simplify notation, in this paper we let $\mathrm{idem}(R)$ denote the set of idempotents in a ring R, we let $\mathrm{reg}(R)$ denote the set of regular elements, and we let $U(R)$ denote the set of units. We write $I \trianglelefteq R$ to mean that I is a two-sided ideal of R. We will often use bar notation when writing elements of $\overline{R} = R/I$. All rings in this paper are associative. Most rings in this paper are unital, and the reader should infer that is the case unless it is explicitly assumed otherwise.

2. Basic facts on idempotents

In this section we record some important basic facts about idempotents that we will find useful in later sections. We also recall some of the standard equivalence relations on the set $\mathrm{idem}(R)$.

Definition 2.1. Let R be a ring and let $e, f \in \mathrm{idem}(R)$.

(1) The idempotents e and f are *isomorphic* (in the ring R) if $eR \cong fR$ as right R-modules. In this case we write $e \cong_R f$. (This relation is also commonly called the Murray-von Neumann equivalence, and the idempotents are then said to be algebraically equivalent.)
(2) The idempotents e and f are *conjugate* (in the ring R) if $f = u^{-1}eu$ for some unit $u \in U(R)$. In this case we write $e \sim_R f$.
(3) The idempotents e and f are *left associate* if $Re = Rf$. In this case we write $e \sim_\ell f$. Right associate idempotents are defined by the condition $eR = fR$, and the relation is written $e \sim_r f$. (We will see in the next lemma that these conditions depend only on the multiplicative structure of the subring, or even subsemigroup, generated by e and f.)

It is straightforward to show that each of these relations is an equivalence relation on the set $\mathrm{idem}(R)$. The following lemma collects a number of well-known facts about these relations.

Lemma 2.2. *Let R be a ring and let $e, f \in \mathrm{idem}(R)$.*

(1) *The following are equivalent:*
   (a) *$eR \cong fR$ as right R-modules (that is, $e \cong_R f$).*
   (b) *$Re \cong Rf$ as left R-modules.*
   (c) *There exist elements $a \in eRf$, $b \in fRe$ satisfying $e = ab$ and $f = ba$.*
   (d) *There exist elements $a, b \in R$ satisfying $e = ab$ and $f = ba$.*

(2) *The following are equivalent:*
   (a) *$e \sim_R f$.*
   (b) *$e \cong_R f$ and $1 - e \cong_R 1 - f$.*

(3) *The following are equivalent:*
   (a) *$e \sim_\ell f$.*
   (b) *$f = ue$ for some unit $u \in U(R)$.*
   (c) *$f = e + (1 - e)xe$ for some $x \in R$.*
   (d) *$ef = e$ and $fe = f$.*

Part (1) is precisely [17, Proposition 21.20]. Part (2) is [18, Exercise 21.16(1)], while part (3) is covered by [18, Exercise 21.4]. Also, as an immediate corollary, we obtain the following implications between these properties.

Corollary 2.3. *If R is a ring with $e, f \in \mathrm{idem}(R)$, then:*

- *$e \sim_\ell f \implies e \sim_R f$, and*
- *$e \sim_R f \implies e \cong_R f$.*

Neither of these two implications is reversible in general. However, by Lemma 2.2(2), we see that $e \cong_R f \implies e \sim_R f$ if and only if $e \cong_R f \implies 1-e \cong_R 1-f$. In other words, using the correspondence between idempotents of R and direct sum decompositions, the following two conditions are equivalent:

- Any two isomorphic idempotents of a ring R are conjugate.
- Whenever $R_R = A \oplus B = A' \oplus B'$, then $A \cong A'$ implies $B \cong B'$.

This describes the class of *IC rings*, short for internal cancellation rings; see [15, (1.4)].

Before leaving this section, we mention one further connection between the regular elements and isomorphic idempotents. Given $a \in \mathrm{reg}(R)$ with an inner inverse $b' \in R$, after replacing b' by $b := b'ab'$ we have the pair of equations

$$aba = a, \quad bab = b.$$

When these hold, we will say that (a, b) is a pair of *reflexive* (*inner*) *inverses.*

Proposition 2.4. *Pairs of reflexive inverses in a ring R are in 1–1 correspondence with pairs of direct sum decompositions $R_R = A \oplus B = C \oplus D$ equipped with a fixed isomorphism $A \cong C$.*

Proof sketch. Let (a, b) be a pair of reflexive inverses. Setting $e := ab$ and $f := ba$, we have $a \in eRf$, $b \in fRe$, with $e, f \in \mathrm{idem}(R)$. Moreover, left multiplication by a describes an isomorphism $fR \to eR$, with inverse given by multiplication by b. Thus, we can take $A = eR$, $B = (1 - e)R$, $C = fR$, and $D = (1 - f)R$.

Conversely, fix decompositions $R_R = A \oplus B = C \oplus D$ equipped with an isomorphism $\varphi : A \to C$. Let $e \in \mathrm{idem}(R)$ be the projection with image A and kernel B; and similarly define $f \in \mathrm{idem}(R)$ with image C and kernel D. We can extend φ to an endomorphism of R_R by taking

$$R \xrightarrow{e\cdot} eR \xrightarrow{\varphi} fR \subseteq^{\oplus} R.$$

As $\mathrm{End}(R_R)$ is naturally isomorphic to R, this extended map corresponds to left multiplication by an element $b \in fRe$. Similarly, $\varphi^{-1} : fR \to eR$ can be viewed as left multiplication by an element $a \in eRf$. Moreover $ab = e$ and $ba = f$, since $\varphi^{-1}\varphi = \mathrm{id}_{eR}$ and $\varphi\varphi^{-1} = \mathrm{id}_{fR}$, hence $aba = a$ and $bab = b$.

These constructions are inverses to each other, demonstrating the 1–1 correspondence. □

Proposition 2.4 will be especially useful in Section 4 when discussing the connections between regular lifting and idempotent lifting.

3. Lifting isomorphic idempotents and conjugate idempotents

The previous section saw the introduction of some classical relations on idempotents. As a "pie-in-the-sky" wish, one might hope that when idempotents lift, then isomorphic idempotents lift (to isomorphic idempotents). This hope might further be strengthened by the following positive result; see [21, Lemma 1.4] for the quick proof.

Proposition 3.1. *Let R be a ring, and let $I \trianglelefteq R$. If $x, y \in R$ are left associate idempotents modulo I, and $x+I$ lifts to an idempotent $e \in R$, then $y+I$ lifts to an idempotent left associate to e. In particular, idempotents lift modulo I if and only if left* (*and right*) *associate idempotents lift modulo I.*

However, after a moment's reflection it becomes clear that this hope in the case of isomorphic and conjugate lifting is in vain. The relation of being left associate is independent of the containing ring, unlike the relations of isomorphism and conjugation. The latter two relations depend heavily on the existence of elements apart from the idempotents in question, which satisfy additional equations. Thus, we should expect the possibility of a counter-example.

To be clear, when we say *isomorphic idempotents lift* modulo an ideal $I \trianglelefteq R$, this means that given any $x, y \in R$ such that their images in the factor ring R/I are isomorphic idempotents, then there exist isomorphic idempotents $e, f \in R$ such that $x - e, y - f \in I$. Similarly, *conjugate idempotents lift* modulo I when given $x, y \in R$ such that their images in R/I are conjugate idempotents, then there exist conjugate idempotents $e, f \in R$ such that $x - e, y - f \in I$.

Remark 3.2. When I is merely a one-sided ideal of R, there are significant issues when defining isomorphic idempotents lifting modulo I. Since R/I is not a ring, we cannot talk about isomorphisms of idempotents using Definition 2.1(1). One could instead appeal to some equivalent condition, such as that given in Lemma 2.2(1), part (c) or (d). Care still needs to be taken, since conditions which are equivalent over a ring may not give rise to equivalent conditions modulo a one-sided ideal. Therefore, in the remainder of the paper we will only lift isomorphic and conjugate idempotents through two-sided ideals. We will still endeavor to work with one-sided ideals whenever possible, for full generality.

Example 3.3. We record here a list of basic examples and non-examples, where isomorphic and conjugate idempotents lift.

- (A) If $I = 0$, then isomorphic and conjugate idempotents always lift.
- (B) If R/I has only trivial idempotents, including the case when $I = R$, then conjugate and isomorphic idempotents lift. For instance, if $p \in \mathbb{Z}$ is prime, then conjugate and isomorphic idempotents lift from $\mathbb{Z}/p^n\mathbb{Z}$ to $\mathbb{Z}$.
- (C) If idempotents do not lift modulo I, then neither do isomorphic or conjugate idempotents, since an idempotent is isomorphic and conjugate to itself.

To show that idempotent lifting does not imply isomorphic idempotent lifting, we will use a free construction. For a ring R, we let $\mathcal{S}(R)$ denote the abelian monoid of isomorphism classes of finitely generated projective right R-modules, under the operation $[P] + [Q] = [P \oplus Q]$. (Equivalently, this monoid can be defined on isomorphism classes of matrix idempotents, see Section 3 in the preliminary chapter of [7]. The isomorphism classes of idempotents in R correspond to those classes $[P]$ for which there exists some $[Q]$ with $[P]+[Q]=[R]$.) We need to understand the structure of this monoid in the case when R is a free algebra. The following lemma acts as our main tool in this regard, and works for a quite large class of rings. The idea of the proof is implicit in [3]; in particular, see the last paragraph on page 48 of that paper. We thank George Bergman for his help in formulating this result.

Lemma 3.4. *Let R_0 be a semisimple ring, let Λ be a set, and let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a collection of faithful R_0-rings (i.e. rings with a given injective ring homomorphism $R_0\to R_\lambda$). Let R denote their coproduct over R_0. The monoid $\mathcal{S}(R)$ is the pushout (as abelian monoids amalgamated over $\mathcal{S}(R_0)$, i.e. amalgamated over the free modules) of the monoids $\mathcal{S}(R_\lambda)$.*

Proof. Projective modules are isomorphic to summands of free modules, hence by [4, Theorem 2.2] their isomorphism classes are represented by "standard modules" over the coproduct. In the monoid of isomorphism classes of finitely generated modules, the finitely generated projectives are exactly the summands in finitely generated free modules. Being a summand of a free module is respected in pushouts amalgamated over the free modules. Thus, the lemma follows directly from [4, Corollary 2.8]. □

We demonstrate the usefulness of this lemma with the following example.

Example 3.5. Let F be a field, let Λ be a set, and consider the ring

$$R'=F\langle e_\lambda\ (\lambda\in\Lambda)\,:\,e_\lambda^2=e_\lambda\rangle.$$

In the monoid $\mathcal{S}(R')$ there exist elements p_λ and q_λ satisfying $p_\lambda+q_\lambda=1$, corresponding to the decomposition $R'_{R'}=e_\lambda R'\oplus(1-e_\lambda)R'$. These are the obvious relations, and we claim there are no others.

To see this, note that R' is the coproduct over F of the rings generated by single idempotents $R_\lambda=F\langle e_\lambda\,:\,e_\lambda^2=e_\lambda\rangle\cong F\times F$. Thus we can describe $\mathcal{S}(R_\lambda)$ as the abelian monoid with generators $p_\lambda,q_\lambda,1$ satisfying the single relation $p_\lambda+q_\lambda=1$. The pushout of these monoids, amalgamated over 1, is exactly the monoid we described above. Notice in particular that this monoid is cancellative.

Next let Φ be another set, and consider the following ring (which we will have need of again later)

$$R=F\langle x,\,y,\,u_\varphi,\,v_\varphi,\,e_\lambda\ (\varphi\in\Phi,\ \lambda\in\Lambda)\,:\,u_\varphi v_\varphi=v_\varphi u_\varphi=1,\ e_\lambda^2=e_\lambda\rangle. \tag{3.6}$$

We claim $\mathcal{S}(R')\cong\mathcal{S}(R)$. Indeed, the only difference here from the previous ring R' is that in the coproduct we have additional rings $R_x=F[x]$, $R_y=F[y]$ and $R_\varphi=F\langle u_\varphi,v_\varphi\,:\,u_\varphi v_\varphi=v_\varphi u_\varphi=1\rangle=F[u_\varphi,u_\varphi^{-1}]$ (for each $\varphi\in\Phi$). But the monoids $\mathcal{S}(R_x)$, $\mathcal{S}(R_y)$, and $\mathcal{S}(R_\varphi)$ are each freely generated by the class of a free module of rank 1, so contribute nothing new to the pushout.

With the previous example under our belts, we are ready to prove that even when both idempotents and units lift modulo an ideal, it does not necessarily follow that isomorphic idempotents lift.

Example 3.7. We construct a ring R with an ideal $I\trianglelefteq R$ such that idempotents and units lift modulo I, but isomorphic idempotents do not lift.

Let F be a field and let $S=F\langle a,b\,:\,ab=1\rangle$. In the ring S, the idempotents 1 and ba are isomorphic (via a,b) but not conjugate. Fix an indexing set Λ such that $\{g_\lambda\}_{\lambda\in\Lambda}$ is the set of idempotents in S. Similarly, fix a set Φ such that $\{s_\varphi,s_\varphi^{-1}\}_{\varphi\in\Phi}$ is the set of mutually inverse units in S.

Let R be the ring defined in (3.6). There exists a surjective ring homomorphism $R\to S$ sending $x\mapsto a$, $y\mapsto b$, $u_\varphi\mapsto s_\varphi$, $v_\varphi\mapsto s_\varphi^{-1}$, $e_\lambda\mapsto g_\lambda$. Let I be the kernel of this map.

Clearly, idempotents and units lift modulo I. On the other hand, isomorphic idempotents in R are conjugate, since $\mathcal{S}(R)$ is cancellative by the computation performed in Example 3.5. Thus if 1 and ba lifted to isomorphic idempotents in R, they would lift to conjugate idempotents in R, which would mean (after passing back down to S) they were conjugate in S to begin with. As this is not the case, we see that isomorphic idempotents do not lift modulo I.

For the remainder of this section, we will focus on conjugate idempotent lifting. It almost goes without saying that the following result must hold.

Lemma 3.8. *If idempotents and units lift modulo an ideal, then conjugate idempotents lift.*

Proof. Let R be a ring with an ideal $I \trianglelefteq R$ such that idempotents and units lift modulo I. Further, fix $x, y \in R$ such that $\overline{x}, \overline{y} \in R/I$ are conjugate idempotents. From the idempotent lifting hypothesis, we may as well assume that $x \in \mathrm{idem}(R)$. Write $\overline{y} = \overline{u}^{-1}\overline{xu}$, where we may assume $u \in U(R)$ by unit lifting. Then $f := u^{-1}xu \in \mathrm{idem}(R)$ is conjugate to x and $f \equiv y \pmod{I}$. □

When conjugate idempotents lift we should not expect (in general) that units also lift, since we only need to lift *some* of the units to preserve conjugation. Indeed, conjugate idempotents lift modulo $8\mathbb{Z}$ inside $\mathbb{Z}$, by Example 3.3(B), but units do not lift. The following example shows that the converse of Lemma 3.8 may fail even for regular rings.

Example 3.9. Our ring is a well-known example of Bergman originally published as [13, Example 1], but given a simpler description in [27, Example 3.1]. Let F be a field. We will let R denote the ring of column-finite matrices, with entries from F, that are constant along diagonals, apart from finitely many arbitrary rows at the top (the number of such rows depending upon the given matrix). By a *diagonal* of a matrix $M = (m_{i,j})$, we mean the set of entries $\{m_{i,j}\}_{i-j=n}$ for some fixed constant $n \in \mathbb{Z}$. Thus, the main diagonal is the set of entries where $n = 0$. Given a matrix M in R, after deleting a finite number of rows from the top, it looks like

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

in other words, M is an infinite upper triangular Toeplitz matrix after removing its top few rows.

By work done in [27], it is known that the ring R is a regular ring, and the subset of matrices with only finitely many nonzero rows is an ideal I. Further, the ring R/I is isomorphic to the Laurent power series ring $F(\!(t)\!)$, and thus has only trivial idempotents. In particular, conjugate idempotents in R/I are actually equal, and thus lift to conjugate idempotents.

On the other hand, the matrix $P \in R$ with 1's along the diagonal directly below the main diagonal is a unit modulo I (whose inverse, modulo I, is the matrix with 1's along the diagonal directly above the main diagonal). It is straightforward to check that $P + I$ does not lift to a unit of R, as any lift has a nonzero left annihilator in R.

Remark 3.10. Alternatively, instead of the ring R used in Example 3.9, we could have appealed to the full ring of $\mathbb{N} \times \mathbb{N}$ column-finite matrices over F (i.e. the full endomorphism ring on a countable-dimensional F-vector space V), which we will denote $S := \mathrm{CFM}(F) = \mathrm{End}_F(V)$, and which is a von Neumann regular ring. The set I of matrices with only finitely many nonzero rows (or equivalently with finite dimensional image in V) is an ideal of S (in fact, it is the only nontrivial ideal). For exactly the same reason as in Example 3.9, the matrix P with 1's down the first "subdiagonal" does not lift modulo I to a unit in S. With some effort (say, by using dimension arguments over V) one can show that conjugate idempotents lift modulo I.

There is one further natural situation where conjugate idempotents lift. Following [11], a ring R is said to be *perspective* if whenever A, B are isomorphic direct summands of R_R, then they have a common direct summand complement. This is a left-right symmetric notion, and generalizes the stable range 1 condition. The following result shows that conjugate (and isomorphic) idempotents behave well when pulling up from a perspective ring.

Proposition 3.11. *Let R be a ring, and let $I \trianglelefteq R$. If R/I is a perspective ring and idempotents lift, then isomorphic idempotents and conjugate idempotents lift.*

Proof. Let $x, y \in R$ be isomorphic (equivalently, under the perspectivity hypothesis, conjugate) idempotents modulo I. Without loss of generality, we may as well assume x is already an idempotent, using the fact that idempotents lift modulo I. As R/I is a perspective ring, we know that the idempotents $x+I$ and $y+I$ are perspective (meaning, they generate right ideals with a common direct summand complement in R/I), and hence by [8, Lemma 6.3] they are connected by a chain of length three of left and right associate idempotents. Applying Proposition 3.1 three times we can lift $y+I$ to an idempotent perspective (and hence both isomorphic and conjugate) to x. $\square$

We have yet to provide an example showing isomorphic idempotent lifting does not imply conjugate idempotent lifting. The reason for this lapse is that the examples we can construct with these properties are quite complicated, but some of this complication can be mitigated after introducing a few key facts and ideas that arise naturally in the next two sections. We will thus leave the reader in suspense for the present time.

4. Lifting of regular elements

In the previous section we saw that when idempotents become isomorphic modulo an ideal I, there needn't be isomorphic lifts even when idempotents lift modulo I. In this section we study a stronger form of lifting that does allow for lifting of isomorphic idempotents; namely, lifting of von Neumann regular elements. Proposition 2.4 provides the ansatz for believing such a claim; if we can lift all regular elements, then we should be able to lift the reflexive inverse pairs, and thus lift isomorphisms between direct summands (although two different isomorphisms might lift to isomorphisms between distinct summands). In the remainder of this section we will provide the formal details for these claims.

An element $x \in R$ is *regular modulo I*, where I is a one-sided ideal of R, if there exists some element $y \in R$ such that $x - xyx \in I$. We say that *regular elements lift* (or elements *lift regularly*) modulo I if it happens that whenever x is regular modulo I, then there exists $a \in \mathrm{reg}(R)$ such that $x - a \in I$. It is quite easy to see that regular elements need not lift when idempotents lift. For instance, (isomorphic and conjugate!) idempotents lift modulo the ideal $8\mathbb{Z} \trianglelefteq \mathbb{Z}$, but the regular (indeed, unit) element $5+8\mathbb{Z}$ does not lift (either to a regular element or a unit). Conversely, it turns out that if regular elements lift modulo a two-sided ideal, then idempotents lift. We will prove a strengthening of this fact shortly, so we won't record that result formally here. Moreover, in the next section we will prove a very strong element-wise version of this fact (see Theorem 5.2), showing that if an element lifts regularly and is idempotent modulo the ideal, then it lifts to an idempotent.

Our next goal will be to show that regular lifting modulo (two-sided) ideals implies that *every* inner inverse also lifts. To do this, we first need to understand the set of inner inverses of a regular element $a \in R$. The following result is both well known and has an easy proof, which can be found with additional information in [20, Example 2.6].

Lemma 4.1. *Given any inner inverse b of an element $a \in R$, the set of all inner inverses of a is comprised of the elements $b + (1 - ba)s + t(1 - ab)$ with $s, t \in R$.*

We are now ready to prove that regular lifting is equivalent to the lifting of all inner inverses, and additionally the lifting of reflexive inverses.

Theorem 4.2. *For an ideal $I \trianglelefteq R$ and any element $x \in R$ that is regular modulo I, the following are equivalent:*

1. *The element x lifts regularly modulo I.*
2. *The element x lifts regularly modulo I, and if y is an inner inverse of x modulo I, then $y + I$ lifts to an inner inverse of any given regular lift of $x + I$.*

(3) *If x, y are reflexive inverses modulo I, then they lift modulo I to reflexive inverses.* (*In fact, $y+I$ lifts to a reflexive inverse of any given regular lift of $x+I$.*)

Proof. (1) $\Longrightarrow$ (2): Suppose $xyx - x \in I$, and fix $a \in \text{reg}(R)$ such that $a - x \in I$. If r is an inner inverse of a, then $x \equiv xrx \pmod{I}$. So by applying Lemma 4.1 to the factor ring R/I, we have $s, t \in R$ such that

$$y \equiv r + (1 - rx)s + t(1 - xr) \pmod{I}.$$

If we put

$$b := r + (1 - ra)s + t(1 - ar),$$

then $a = aba$ and $b - y \in I$.

(2) $\Longrightarrow$ (3): Suppose $xyx - x \in I$ and $yxy - y \in I$. Using the first containment, (2) allows us to choose for each regular lift a of $x+I$, some $b \in R$ (depending on a) such that $a = aba$ and $a-x, b-y \in I$. If we put $c := bab$, then a and c are inner inverses of each other in R, and

$$c = bab \equiv yxy \equiv y \pmod{I}.$$

(3) $\Longrightarrow$ (1): Clear. □

Corollary 4.3. *If regular elements lift modulo an ideal I, then isomorphic idempotents lift modulo I.*

Proof. Assume $p, q \in R$ are isomorphic idempotents modulo $I \trianglelefteq R$. By Lemma 2.2(1) applied to the ring R/I, there exist $\overline{a} \in \overline{p}\overline{R}\overline{q}$ and $\overline{b} \in \overline{q}\overline{R}\overline{p}$ with $\overline{a}\overline{b} = \overline{p}$ and $\overline{b}\overline{a} = \overline{q}$. Thus,

$$\overline{a}\overline{b}\overline{a} = \overline{a}\overline{q} = \overline{a}$$

and similarly $\overline{b}\overline{a}\overline{b} = \overline{b}$. Theorem 4.2 tells us that we can lift pairs of reflexive inverses to R, so we may as well assume $aba = a$ and $bab = b$. Thus, $e := ab$ and $f := ba$ are isomorphic idempotents in R. Further, they lift $\overline{p}$ and $\overline{q}$ respectively. □

Remark 4.4. Assume $e, f \in \text{idem}(R)$ are such that they become isomorphic modulo an ideal I of R. Corollary 4.3 does *not* imply that e and f are isomorphic when regular elements lift; rather that there are some idempotents congruent to e and f modulo I that are isomorphic in R. Examples showing this fact are extremely easy to come by. For instance, if R is a nonzero ring, then taking $I = R$ we see that the idempotents 0 and 1 become isomorphic in the factor ring, and the single element in $R/I = (0)$ lifts regularly.

It turns out that if we try to strengthen Corollary 4.3 by working modulo a one-sided ideal, the conclusion may fail even if we weaken the conclusion from isomorphic idempotent lifting to merely idempotent lifting, as the following example demonstrates.

Example 4.5. Let $\mathbb{F}_2$ be the field with two elements, and let $T := \mathbb{F}_2 \times \mathbb{F}_2$. Also take $S := \mathbb{F}_2[z]$. There is a unique unital ring homomorphism $\varphi : S \to T$ induced by sending $z \mapsto (1, 0)$. Thus, we can make $M := T$ into a left S-module via the action $s \cdot m = \varphi(s)m$ for all $s \in S$ and $m \in M$.

Take $R := \left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right)$, and let $I = \left(\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Note that I is a left ideal in R. Given any $r = \left(\begin{smallmatrix} s & m \\ 0 & t \end{smallmatrix}\right) \in R$, we see that $r + I$ lifts to the regular element $a = \left(\begin{smallmatrix} 1 & m \\ 0 & t \end{smallmatrix}\right)$. (Not only is a regular, but $a^3 = a$.)

A quick computation shows that $x := \begin{pmatrix} z & (1,1) \\ 0 & (0,1) \end{pmatrix}$ is idempotent modulo I. However, if $y - x \in I$, then

$$y = \begin{pmatrix} s & (1,1) \\ 0 & (0,1) \end{pmatrix}$$

for some $s \in S$. If $y \in \text{idem}(R)$, it follows that $s \in \text{idem}(S) = \{0, 1\}$. In both cases (whether $s = 0$ or 1) it is easy to directly check that the element y is not an idempotent. Thus idempotents don't lift modulo I, even though all elements lift regularly.

A natural place to use regular lifting is in IC rings. Recall that according to [15] a ring R is called an *IC ring* if the right R-module R_R has internal cancellation, or equivalently if isomorphic idempotents are conjugate. Note that this is a weaker condition than R being a perspective ring.

Proposition 4.6. *If R is an IC ring and isomorphic idempotents lift (in particular, if regular elements lift) modulo an ideal $I \trianglelefteq R$, then R/I is also an IC ring.*

Proof. Given a pair of isomorphic idempotents of R/I, any isomorphic lifts to R will be conjugate from the IC hypothesis. Conjugate idempotents in R push down to conjugate idempotents in R/I. Thus, all isomorphic idempotents of R/I are conjugate. □

There is a well-known and well-used notion that is weaker than a ring being IC, namely that every one-sided unit is a two-sided unit. Rings with the latter property are called *Dedekind-finite* (or directly finite) rings. It can be the case that R/I is not Dedekind-finite, even when R is Dedekind-finite and regular elements lift modulo I. For instance, a homomorphic image of a regular Dedekind-finite ring may not be Dedekind-finite (see [12, Example 5.11]). However, in some situations we can still preserve Dedekind-finiteness in factor rings.

Proposition 4.7. *Let $I \trianglelefteq R$, and suppose that I contains no nonzero idempotents (such as with the Jacobson radical). If isomorphic idempotents lift modulo I (e.g. if regular elements lift), then R is Dedekind-finite if and only if R/I is Dedekind-finite.*

Proof. Suppose first that R is not Dedekind-finite. We can then fix $x, y \in R$ with $xy = 1$ but $yx \neq 1$. It is easy to check that $1 - yx$ is an idempotent. Since I contains no nonzero idempotents, we have $1 - yx \notin I$ and so R/I is also not Dedekind-finite.

Next assume R is Dedekind-finite. The Dedekind-finite property is equivalent to saying that the only idempotent isomorphic to 1 is 1 itself. So assume there is an idempotent $e + I \in \mathrm{idem}(R/I)$ with $e + I \cong_{R/I} 1 + I$; it suffices to show that $e \equiv 1 \pmod{I}$. By hypothesis, there exist two isomorphic idempotents $g, h \in \mathrm{idem}(R)$ such that $g - e, h - 1 \in I$. As $1 - h \in I$ and I does not contain any nonzero idempotents, $h = 1$. So we have $g \cong_R 1$, and as R is Dedekind-finite, we get that $g = 1$. But that means $e \equiv g = 1 \pmod{I}$ as needed. □

Remark 4.8. If we take the free algebra $R = F\langle x, y\rangle$ over a field F and take I to be the ideal generated by $1 - yx$, then R is Dedekind-finite, I does not contain any nonzero idempotent, but R/I is not Dedekind-finite. Thus, the lifting hypothesis in Proposition 4.7 is not superfluous (in one direction).

Our final application of regular lifting will be for exchange rings. These rings were first defined and studied by Warfield, and shown to be exactly those rings for which idempotents lift modulo all one-sided ideals by Nicholson in [23]. Moreover, Nicholson proved that exchange rings possess a more powerful form of lifting called (appropriately) strong lifting.

A one-sided ideal I of a ring R is said to be *strongly lifting* if whenever $x^2 - x \in I$ for some $x \in R$, there is an idempotent $e \in xR$ such that $e - x \in I$. Strong lifting is left-right symmetric, in the sense that we can replace the conclusion $e \in xR$ with $e \in Rx$, or even $e \in xRx$; this non-trivial fact is proven in [25, Lemma 1]. Strongly lifting ideals were studied in depth by Nicholson and Zhou in [25], and further by Alkan, Nicholson, and Özcan in [1]. There is a connection between strong and regular lifting.

Theorem 4.9. *If a one-sided ideal I of a ring R is strongly lifting, then regular elements lift.*

Proof. Without loss of generality, take I to be a right ideal of R. Assume $x - xyx \in I$ for some $x, y \in R$. As xy is an idempotent modulo I and I is strongly lifting, there exists an idempotent $e \in xyR$ such that $e - xy \in I$. Write $e = xyr$ for some $r \in R$. Note that

$$(ex)(yr)(ex) = e^3x = ex$$

and so ex is regular. Finally, $ex \equiv xyx \equiv x \pmod{I}$, so x lifts regularly modulo I. □

This theorem has some immediate consequences for exchange rings.

Corollary 4.10. *Regular elements, and hence isomorphic idempotents, lift modulo every one-sided ideal of an exchange ring.*

Proof. One-sided ideals of exchange rings are strongly lifting, by [25, Theorem 4]. □

We should also point out that the converse of Theorem 4.9 is false. For instance, taking $R = \mathbb{Z}[x]$ and $I = (x) \trianglelefteq R$, we see that regular elements lift through the quotient map $R \to R/I$. However, while $1+x$ is a nonzero idempotent modulo I, there is no nonzero idempotent in $(1+x)R$, so $1+x$ cannot lift strongly modulo I.

We finish this section by discussing a potential strengthening of the conclusion of Corollary 4.3. When $a \in \mathrm{reg}(R)$, there exists some $b \in R$ with $aba = a$ and ab is an idempotent. Thus, one might wonder whether regular lifting implies that whenever a product of two elements is idempotent modulo an ideal, that product can be lifted to an idempotent. The answer is no.

Lemma 4.11. *Let R be a ring, and let $I = (r^2)$ where r is central but neither a unit nor a zero-divisor in R. Then there do not exist $a, b \equiv r \pmod{I}$ such that ab is an idempotent in R (although $\overline{a}\overline{b}$ is the zero idempotent in R/I).*

Proof. Assume instead that a, b exist. Upon writing $a = r(1+rx)$ and $b = r(1+ry)$ (for some $x, y \in R$), $ab \in \mathrm{idem}(R)$ would imply

$$r^2(1+rx)(1+ry) - r^4\left((1+rx)(1+ry)\right)^2 = 0.$$

As r is not a zero-divisor, we may cancel r^2. Expanding the first summand and rearranging terms gives

$$1 = r\left[-(x+y+rxy) + r\left((1+rx)(1+ry)\right)^2\right],$$

which would contradict $r \notin U(R)$. □

Example 4.12. Let R is a commutative domain that is not a field, and fix $r \in R$ that is nonzero and not a unit. Putting $I = (r^2)$ we see that $r^2 + I$ is the zero idempotent in R/I. However, by Lemma 4.11 there do not exist any lifts $a, b \in R$ of $r + I$ such that ab is an idempotent in R. Despite this non-lifting result, by specializing the ring R we can force some significant lifting conditions. For instance, if we further take R to be a discrete valuation ring with nonzero uniformizer r, then we have the additional properties that units lift modulo I, and idempotents lift strongly.

5. Connections between different lifting assumptions

As we mentioned earlier, exchange rings are exactly those rings for which idempotents lift modulo all one-sided ideals. There is an element-wise notion that captures a significant amount of the structure of exchange rings. Following Nicholson [23], given a ring R, an element $a \in R$ is said to be *suitable* (or sometimes, *exchange*) if there is an idempotent $e \in a + R(a - a^2)$. This is a left-right symmetric notion [16], and we'll write $\mathrm{suit}(R)$ for the set of all suitable elements in a ring R. It is well-known that R is an exchange ring if and only if $R = \mathrm{suit}(R)$. We direct the reader to [16] for some equational characterizations of suitable elements. Examples of suitable elements include the following:

- Idempotents and units.
- Quasiregular elements, which include nilpotent elements and elements of the Jacobson radical. (An element $r \in R$ is quasiregular if $1 - r \in U(R)$.)
- Clean elements. (An element is clean if it is the sum of an idempotent and a unit. Incidentally, this class encompasses the two bullet points above.)
- Regular, and even π-regular elements. (An element is π-regular when some positive power of it is regular.)

We will show that lifting idempotents to suitable elements is often equivalent to just lifting idempotents to idempotents. To prove this we first need a lemma characterizing when idempotence passes between elements equal modulo a one-sided ideal.

Lemma 5.1. *Let I be a left ideal, and assume $x \in R$ is idempotent modulo I. Every $y \in R$ that is congruent to x modulo I is idempotent modulo I if and only if $Ix \subseteq I$.*

Proof. Let $y = x + z$ for some $z \in I$. Using the fact I is a left ideal we compute

$$y^2 - y = (x+z)^2 - (x+z) = (x^2 - x) + xz + zx + z^2 - z \equiv zx \pmod{I}.$$

Therefore y is idempotent mod I if and only if $zx \in I$. Ranging over all $z \in I$ gives the lemma. □

With this lemma, we can now give a very strong element-wise connection among different lifting conditions on suitable elements.

Theorem 5.2. *Let I be an left ideal of a ring R, let $x \in R$ be idempotent modulo I, and suppose $Ix \subseteq I$. (This latter condition holds, for instance, if $I \trianglelefteq R$.) The following are equivalent:*

1. *x lifts to an idempotent modulo I.*
2. *x lifts to an element of $\mathrm{suit}(R)$ modulo I.*

In particular, if such an x lifts modulo I to a (π–)regular or clean element, then it lifts to an idempotent.

Proof. We trivially have (1) ⇒ (2), so we need only prove (2) ⇒ (1).

Assume x lifts modulo I to $a \in \mathrm{suit}(R)$, so there is some idempotent of the form $e := a + r(a - a^2)$ for some $r \in R$. By Lemma 5.1 we have $a - a^2 \in I$. Thus,

$$e - x \equiv e - a = r(a - a^2) \equiv 0 \pmod{I}.$$ □

One nice consequence of this theorem is the following new characterizations of exchange rings.

Theorem 5.3. *Given a ring R, the following are equivalent:*

1. *R is an exchange ring.*
2. *Regular elements lift modulo all left ideals of R.*
3. *Idempotents lift to regular elements modulo all left ideals of R.*
4. *Idempotents lift to elements of $\mathrm{suit}(R)$ modulo all left ideals of R.*

Proof. (1) ⇒ (2): This follows from Corollary 4.10.

(2) ⇒ (3): If $x \in R$ is idempotent modulo a left ideal I, then it is regular modulo I, and thus lifts to a regular element modulo I from the assumption.

(3) ⇒ (4): This is clear since $\mathrm{reg}(R) \subseteq \mathrm{suit}(R)$.

(4) ⇒ (1): From the known equivalence for being an exchange ring, it suffices to show that if $x \in R$ is idempotent modulo a left ideal $I \subseteq R$, then x lifts to an idempotent. We may as well replace I by $R(x - x^2)$, and show lifting modulo this smaller left ideal. Notice that after this change we have $Ix \subseteq I$.

Now as x is idempotent modulo I it lifts to an element of $\mathrm{suit}(R)$ by our assumption. Theorem 5.2 applies, so x lifts to an idempotent. □

Our goal for the remainder of this section is to elucidate all interconnections between the various lifting properties introduced thus far. Before doing so, we will use the fact that idempotents lift modulo all one-sided ideals in exchange rings as motivation to define one final lifting property for ideals. The definition is as follows.

Definition 5.4. A left ideal $I \subseteq R$ is *fully lifting* if idempotents lift (to R) modulo every left ideal contained in I. Fully lifting right ideals are defined similarly.

Given $x \in R$, there is a unique smallest left ideal modulo which x is an idempotent, namely $R(x-x^2)$. Thus, an alternate characterization of a left ideal $I \subseteq R$ being fully lifting is:

(5.5) For any $x \in R$, if $R(x-x^2) \subseteq I$, then there is some $e \in \text{idem}(R)$ with $e - x \in R(x-x^2)$.

This condition is quite powerful as it implies all the other types of idempotent lifting we have defined thus far.

Proposition 5.6. *Any fully lifting left ideal $I \subseteq R$ is strongly lifting.*

Proof. By (5.5), using the notation given there, $e - x \in R(x-x^2)$. Thus, $e \in x + R(x-x^2) \subseteq Rx$. □

Corollary 5.7. *Given a left ideal $I \subseteq R$, then I is fully lifting if and only if regular elements lift modulo all left ideals contained in I.*

Proof. ($\Rightarrow$): Any left ideal $J \subseteq I$ is fully lifting, hence strongly lifting, and thus regular elements lift modulo J by Theorem 4.9.

($\Leftarrow$): Suppose $J := R(x-x^2) \subseteq I$ for some $x \in R$. By hypothesis, x lifts to a regular element. Thus, x lifts modulo J to an idempotent by Theorem 5.2, and in particular (5.5) holds. □

There is a natural connection between fully lifting ideals and exchange rings. Indeed, the conclusion of (5.5) is precisely that $x \in \text{suit}(R)$. Thus, we have the following characterization, which is also a restatement of [14, Lemma 2.3].

Lemma 5.8. *Let $I \subseteq R$ be a left ideal. The following are equivalent:*

1. *I is a fully lifting left ideal of R.*
2. *For each $x \in R$, if $x - x^2 \in I$, then $x \in \text{suit}(R)$.*

As a first consequence of this lemma, we see that condition (2) gives a single criterion which can be used to define fully lifting for any one-sided ideal (whether left or right) since the definition of $\text{suit}(R)$ is left-right symmetric. Second, it follows immediately that any two-sided ideal is fully lifting as a left ideal if and only if it is fully lifting as a right ideal, thus recovering the left-right symmetry proved in [15, p. 228]. (In that paper, a fully lifting ideal is just called a lifting ideal; see also [14, Proposition 4.3]. We believe that the name "fully lifting" is more expressive.)

As a third consequence, we have a deeper connection to exchange rings, as given in the following theorem. Before we prove that result we need two further definitions, both based on the work in [2], particularly Lemma 1.1 from that paper. Let I be a non-unital ring. First, we say x is *suitable* in I if it is suitable is *some* (equivalently, *every*) unital ring R containing I as a two-sided ideal. Write $\text{suit}(I)$ for the set of suitable elements of I. The definition of suitable elements in non-unital rings can thus be expressed as

$$\text{if } x \in I \trianglelefteq R, \text{ then } x \in \text{suit}(I) \text{ if and only if } x \in \text{suit}(R). \tag{5.9}$$

Second, we say I is an *exchange ring* if $I = \text{suit}(I)$.

Theorem 5.10. *Any fully lifting left ideal $I \subseteq R$ is an exchange ring (as a possibly non-unital ring).*

Proof. Given $x \in I$, then automatically $x^2 - x^4 \in I$. Thus Lemma 5.8 tells us that $x^2 \in \text{suit}(R)$. Fix an idempotent $e = x^2 + r(x^2 - x^4) \in I$ for some $r \in R$. Letting I' be the subring of R generated by I and 1, we have

$$e - x = x^2 - x + r(x^2 - x^4) = (-1 + r(1+x)x)\,(x - x^2) \in I'(x - x^2) \tag{5.11}$$

where we used the fact that $r(1+x)x \in I$, since $x \in I$ and I is a left ideal of R. Notice that (5.11) is precisely the statement $x \in \text{suit}(I')$. Since $I \trianglelefteq I'$ it follows from (5.9) that $x \in \text{suit}(I)$. Thus $I = \text{suit}(I)$, and hence I is an exchange ring. □

Example 1 of [2] says "If I is an ideal of a unital exchange ring, then I is an exchange ring." Modifying the previous proof only slightly, we have a two-fold improvement: (1) we can drop all unital assumptions, and (2) I need only be a one-sided ideal.

Theorem 5.12. *If J is a possibly non-unital exchange ring, so is every left ideal $I \subseteq J$.*

Proof. Given $x \in I$, then $x^2 \in J = \mathrm{suit}(J)$. Letting R be any unital ring with $J \trianglelefteq R$, then (5.9) gives $x^2 \in \mathrm{suit}(R)$. The remainder of the proof is exactly the same as the one given for Theorem 5.10. □

Remark 5.13. Theorem 5.12 fails on the level of suitable elements. For example, take $R = \mathbb{M}_2(\mathbb{Z})$ and let I be the set of matrices with zero second column, which is a left ideal in R. A routine computation shows that the element $x = \left(\begin{smallmatrix} 3 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in I$ is regular in R with inner inverse $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Hence $x \in \mathrm{suit}(R)$. However, the non-unital ring homomorphism

$$\varphi : I \to \mathbb{Z}, \qquad \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mapsto a$$

sends $x \mapsto 3 \notin \mathrm{suit}(\mathbb{Z})$ and hence $x \notin \mathrm{suit}(I)$.

Returning now to the topic of fully lifting ideals, we note that the converse of Theorem 5.10 fails in general, since an ideal which is an exchange ring (again, possibly non-unitally) may not even be idempotent lifting. For instance, take the Jacobson radical of any ring where idempotents do not lift modulo that radical, such as in the ring $S^{-1}\mathbb{Z}$ where S is the set of integers relatively prime to 6.

Another way to think about the previous paragraph is as follows. Let $I \trianglelefteq R$, for a unital ring R, and assume x is idempotent modulo I. Moreover, assume that x lifts to an idempotent modulo I. By Theorem 5.2 we know that $x + I$ lifts to $a \in \mathrm{suit}(R)$. Thus, by the proof of that theorem, $x + I$ lifts to an idempotent $e \in a + R(a - a^2)$. By the example given in the previous paragraph, this does not mean that x is suitable in R, or in other words we cannot guarantee the stronger conclusion that $e \in x + R(x - x^2)$.

As mentioned earlier, it is well-known that π-regular elements are suitable, and hence π-regular rings are exchange rings; this continues to hold true even in the non-unital setting (see [2, Example 3]). Since it is more difficult to be a fully lifting one-sided ideal than to merely be a non-unital exchange ring, the following lemma strengthens the fact that π-regular rings are exchange.

Lemma 5.14. *In any unital ring R, a left ideal $I \subseteq R$ is fully lifting if I is π-regular as a (possibly non-unital) ring.*

Proof. Let $x \in R$ with $x - x^2 \in I$. As $x - x^2$ is π-regular in I, a straightforward computation (as done in the proof of [10, Lemma 1]) shows that x is π-regular in R, and in particular x is suitable in R. By Lemma 5.8, I is fully lifting in R. □

We are now prepared to give the promised example of an ideal (in a unital ring) modulo which isomorphic idempotents lift, but not conjugate idempotents.

Example 5.15. For simplicity, let $F = \mathbb{F}_2$ be the field with two elements. Set

$$R_0 := F\langle u,\, u^{-1},\, e \,:\, e^2 = e,\, uu^{-1} = u^{-1}u = 1\rangle.$$

Clearly the idempotents e and $u^{-1}eu$ are conjugate in R_0, via the unit u. We will show this is the only such unit, so assume that $v^{-1}ev = u^{-1}eu$ for some unit $v \in U(R_0)$. We then have $(uv^{-1})e = e(uv^{-1})$ and hence $w := uv^{-1} \in U(R_0)$ commutes with e. In particular

$$ewew^{-1}e = e. \tag{5.16}$$

We claim that the only way (5.16) can hold is if $ewe = ew^{-1}e = e$. To see this, note that elements of eR_0e can all be written uniquely as sums of distinct monomials of the form

$$m(k) = m(k, n_1, n_2, \ldots, n_k) := eu^{n_1}eu^{n_2}e \cdots eu^{n_k}e \tag{5.17}$$

with $n_1, \dots, n_k \in \mathbb{Z} \setminus \{0\}$ and $k \geq 0$. (This is where we are using the fact that F is the field with two elements, in order to avoid working with nontrivial coefficients.) Letting $m(k)$ be any monomial in the support of ewe with k maximal, and letting $m'(\ell)$ be any monomial in the support of $ew^{-1}e$ with ℓ maximal, we see that the monomial mm' occurs in the support of $ewew^{-1}e = e$ (since no other monomial in the product can match it, from the maximality condition). Thus $k = \ell = 0$, and hence $ewe = ew^{-1}e = e$.

By a symmetric argument (replacing e by $1 - e$) we see that $(1 - e)w(1 - e) = 1 - e$. But as e and w commute, we must have

$$w = ew + (1 - e)w = ewe + (1 - e)w(1 - e) = e + (1 - e) = 1$$

and hence $u = v$. Summing up,

$$e \text{ and } u^{-1}eu \text{ are conjugate in } R_0\text{, but } \textit{only} \text{ via the unit } u. \tag{5.18}$$

Our next goal is to embed R_0 in a certain matrix ring. This will require a small amount of computation. Let $x_0 = 1$ and let $\{x_i\}_{i \in \mathbb{Z} \setminus \{0\}}$ be a collection of non-commuting variables. Given any $n \geq 0$, let M_n to be the $(2n + 1) \times (2n + 1)$ matrix with zero rows except that the $(n + 1)$st row (i.e. the middle row) is $(x_{-n}, x_{-n+1}, \dots, x_{n-1}, x_n)$. Thus, for instance

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_{-2} & x_{-1} & 1 & x_1 & x_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Each M_n is idempotent. Moreover, let $S(i)$ be an operator that takes all the entries of any square matrix and pushes them down i rows, replacing the top i rows with 0. (When $i < 0$, this operator pushes entries upwards, forcing the bottom $-i$ rows to be zero.) It is straightforward to compute that

$$M_n \cdot (S(i)M_n) = x_i M_n \tag{5.19}$$

as long as $-n \leq i \leq n$, otherwise the product is 0.

Now let D be any division ring containing $F\langle x_i \ (i \in \mathbb{Z}) : x_0 = 1\rangle$. Take S to be the ring of column-finite matrices over D, and let I be the ideal consisting of matrices with finitely many nonzero rows. Since S is a regular ring, I is a regular (non-unital) ring (since any reflexive inverses of elements in I will still be in I). As in Example 3.9 we let P be the matrix in S with 1's along the "sub-diagonal" and zeros elsewhere. Similarly, let Q be the matrix with 1's along the "super-diagonal." Finally, let E be the block diagonal matrix, with the blocks $M_1, M_2, \dots$ along the main diagonal.

We claim that R_0 embeds in S/I via the map ψ induced by $u \mapsto P$, $u^{-1} \mapsto Q$, and $e \mapsto E$ (all modulo I). First, it should be noted that $E^2 = E$, and modulo I the matrices P and Q are unit inverses, thus ψ exists (due to the fact R_0 is freely generated by those relations). So it suffices to prove that ψ is an injection.

Suppose $r \in R_0$. We can write r uniquely as a sum of distinct monomials of the form u^{i_1} or $u^{i_1}m(k)u^{i_2}$ where $m(k) = eu^{n_1}eu^{n_2}e \cdots eu^{n_k}e$ is as in (5.17) and $i_1, i_2 \in \mathbb{Z}$. The matrix P acts as a down-shift operator while Q acts as an upshift, via left multiplication. Thus, since we can remove the first n_j blocks from E when working modulo I, the computation in (5.19) tells us that $\varphi(eu^{n_j}e) = x_{n_j}E + I$. Therefore, $\varphi(u^{i_1}m(k)u^{i_2})$ is the element $x_{n_1}x_{n_2} \cdots x_{n_k}E + I$ shifted down i_1 rows and to the right i_2 columns. Similarly, $\varphi(u^{i_1})$ is the identity matrix, modulo I, shifted down i_1 rows. No nontrivial sum of such matrices is zero modulo I, and hence $\varphi(r) = 0$ implies $r = 0$.

Finally, let R be the unital subring of S generated by P, Q, E and the ideal I. Clearly $I \trianglelefteq R$. As I is a regular ring, Lemma 5.14 tells us that I is fully lifting (and in particular isomorphic idempotents lift). On the other hand P does not lift modulo I to any unit in S, let alone R, and so the conjugate idempotents $E+I$ and $QEP+I$ do not lift to conjugate idempotents in R by (5.18).

While this example demonstrates a strong independence between some of the lifting properties, there is one more point of contact among them, which is most easily seen by again using the ansatz provided by Proposition 2.4. That result tells us that an isomorphism between direct summands of R_R (along with its inverse) is induced by some reflexive inverse pair. Thus, it was not surprising to find that regular lifting (i.e. the ability to lift *all* reflexive inverse pairs) is stronger than the ability to lift isomorphic idempotents (i.e. the ability to lift *some* reflexive inverse pair, for each pair of isomorphic summands). But just how much do two different isomorphisms $\psi, \psi' : M_R \to N_R$ differ? A moment's thought reveals that they differ merely by precomposition with an automorphism of M (or equivalently, by postcomposition with an automorphism of N). When M_R is a direct summand of R_R, such an automorphism can be extended to a unit in R. Thus, one should expect, and we can indeed prove:

Proposition 5.20. *If units and isomorphic idempotents lift modulo $I \trianglelefteq R$, then regular elements lift.*

Proof. Assume $x, y \in R$ are reflexive inverses modulo I. Since xy and yx are isomorphic idempotents modulo I, then by the second lifting hypothesis we can find isomorphic idempotents $e, f \in R$ with $e - xy, f - yx \in I$. Further, by Lemma 2.2(1)(c) there exist elements $s \in eRf$, $t \in fRe$, $st = e$, $ts = f$.

Next, we claim that $p := tx + (1-f)$ and $q := ys + (1-f)$ are inverse units modulo I. First,

$$\begin{aligned}
pq &= \big(tx + (1-f)\big)\big(ys + (1-f)\big) \\
&= txys + tx(1-f) + (1-f)ys + (1-f) \\
&\equiv tes + tx(1-yx) + (1-yx)ys + (1-f) && \text{(using } e - xy, f - yx \in I\text{)} \\
&\equiv ts + (1-f) = f + (1-f) = 1 \pmod{I} && \text{(using } x - xyx, y - yxy \in I \text{ and } t \in Re\text{)}.
\end{aligned}$$

We also find $qp \equiv 1 \pmod{I}$ by a similar computation (using the fact $s \in eRf$ so $s(1-f) = 0$, and similarly $(1-f)t = 0$). Since units lift modulo I, let u be a unit lift of $p + I$.

We then see that $(su)(u^{-1}t)(su) = stsu = esu = su$ so that $su \in \operatorname{reg}(R)$. Also,

$$su \equiv sp = s\big(tx + (1-f)\big) = stx = ex \equiv xyx \equiv x \pmod{I},$$

and so x lifts modulo I to the regular element su as desired. □

Many authors have studied lifting of units; see [22], [26], and [28] for just three such papers. As Example 3.9 (or Remark 3.10) demonstrate, units do not even need to lift modulo ideals in regular rings. However, if we make the assumption that there are no nontrivial idempotents in the ideal (as we did in Proposition 4.7), then unit lifting does follow from regular lifting, and even more is true.

Theorem 5.21. *Let $I \trianglelefteq R$ be such that* $\operatorname{idem}(I) = \{0\}$, *and let $\overline{R} = R/I$. If x is π-regular in R, then $x \in U(R)$ if and only if $\overline{x} \in U(\overline{R})$. In particular, if regular elements lift modulo an ideal containing no nonzero idempotents, so do units.*

Proof. The forward direction is obvious, so we now prove the backwards direction. After replacing x by a suitable power x^n (for some $n \geq 1$), we may assume $x \in \operatorname{reg}(R)$, say $x = xyx$ for some $y \in R$. Since $\overline{x} \in U(\overline{R})$, we have $\overline{xy} = \overline{yx} = \overline{1}$. Thus $1 - xy, 1 - yx \in \operatorname{idem}(I) = \{0\}$. Hence $xy = yx = 1$, and therefore $x \in U(R)$.

To show the last sentence, suppose $\overline{x} \in U(\overline{R})$. In this case we also have $\overline{x} \in \operatorname{reg}(\overline{R})$. By regular lifting, we may as well assume $x \in \operatorname{reg}(R)$. But the conclusion $x \in U(R)$ now follows directly from the work done above. □

Remark 5.22. (i) Theorem 5.21 does not hold if we replace regular lifting with idempotent lifting. For example if we take $I = 8\mathbb{Z}$ (which has no nonzero idempotents) and $R = \mathbb{Z}$, then isomorphic idempotents lift modulo I, but $5 + I$ is a unit in $\mathbb{Z}/8\mathbb{Z}$ that does not lift to a unit in $\mathbb{Z}$.

(ii) The converse of the last sentence of Theorem 5.21 does not hold in general. For instance units always lift modulo the Jacobson radical of a ring, and the Jacobson radical contains no nonzero idempotents. However, regular elements may not lift.

We end this section with a full classification theorem for the lifting conditions for ideals. The interrelationships between them are visually displayed in Figure 5.23.

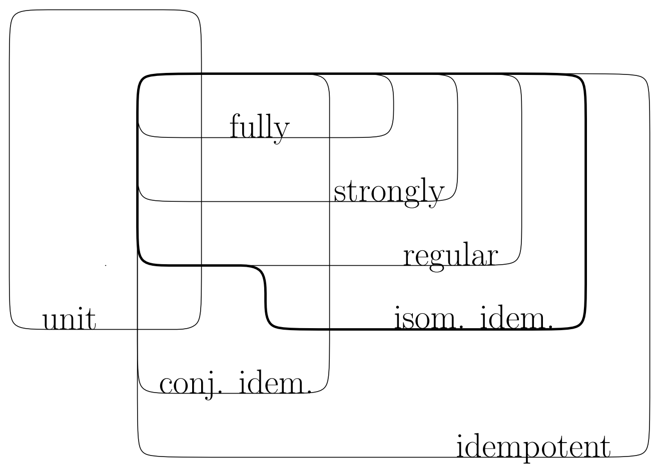

FIGURE 5.23. Euler diagram of the interrelations between lifting conditions for ideals.

Theorem 5.24. *The relationships between unit, idempotent, conjugate idempotent, isomorphic idempotent, regular, strongly, and fully lifting modulo an ideal are given by* Figure 5.23*. If the ideal is contained in the Jacobson radical, then unit lifting is automatic, and all other conditions are equivalent.*

Proof sketch. The various relations between the properties are either obvious or consequences of Lemma 3.8, Corollary 4.3, Theorem 4.9, Proposition 5.6, and Proposition 5.20. All that remains is to construct, for every region in the figure, a ring R and an ideal $I \trianglelefteq R$ such that I has exactly those properties of the given region. To do this, we make use of some important examples.

(1) For an example of an ideal that is fully and unit lifting, we can take I to be the zero ideal in any ring R.
(2) For a unit and strongly lifting ideal that is not fully lifting, we can take I to be the entire ring R for any ring where idempotents do not lift modulo some ideal. For instance, we can take $I = R = \mathbb{Z}$.
(3) For a unit and regular lifting but not strongly lifting example, let $R = \mathbb{Z}[x]$ and $I = (x)$. The element $1+x \in R$ is a lift of the idempotent $\overline{1}$, but no nonzero multiple of $1+x$ is an idempotent.
(4) For a unit and idempotent lifting but not regular lifting example, use Example 3.7.
(5) For a unit lifting but not idempotent lifting example, use the pair $6\mathbb{Z} \trianglelefteq \mathbb{Z}$.
(6) For a conjugate idempotent and isomorphic idempotent lifting but not regular lifting example, use $8\mathbb{Z} \trianglelefteq \mathbb{Z}$.
(7) For a conjugate and fully lifting ideal that is not unit lifting, use Example 3.9 or Remark 3.10.
(8) For a fully lifting ideal that is not conjugate idempotent lifting, use Example 5.15.

All other needed examples can be built using direct products of these main cases. We will demonstrate this idea with one example, as all other instances are similar. To construct an ideal that is regular lifting but not strongly nor conjugate idempotent lifting, we can take $R = R_3 \times R_8$ with $I = I_3 \times I_8$, where $I_3 \trianglelefteq R_3$ is example (3) above, and similarly $I_8 \trianglelefteq R_8$ is example (8) above.

Finally, the statement about ideals in the Jacobson radical follows from Lemma 5 of [25]. □

6. Other forms of lifting

A natural generalization of regularity is the unit-regularity condition introduced by Ehrlich in [9]. A regular element is *unit-regular* if it has a unit inner inverse. These elements are of the form $a = eu$ for some $e \in \mathrm{idem}(R)$ and $u \in U(R)$ by [18, Exercise 4.14B].

Given this connection between unit-regular elements, units, and idempotents, the following result should come as no surprise.

Lemma 6.1. *Let I be an ideal of a ring R, and let $x \in R$ be a unit modulo I. If x lifts to a unit-regular element modulo I, then x lifts to a unit modulo I.*

Proof. Let $\overline{R} = R/I$. If a unit-regular element $a = eu$ (with $e \in \mathrm{idem}(R)$, $u \in U(R)$) lifts $\overline{x}$, then $\overline{e} = \overline{au^{-1}} = \overline{xu^{-1}} \in U(\overline{R})$ implies that $\overline{e} = \overline{1}$. So $\overline{x} = \overline{u}$. □

With this result in place, we are now able to place unit-regular lifting in Figure 5.23.

Theorem 6.2. *An ideal is unit-regular lifting if and only if it is both unit lifting and idempotent lifting.*

Proof. ($\Rightarrow$): Note that units and idempotents are both always unit-regular. Thus, unit lifting follows from unit-regular lifting by Lemma 6.1. Idempotent lifting similarly follows by Theorem 5.2.

($\Leftarrow$): This implication follows from the fact that unit-regular elements are precisely the products of idempotents and units. □

It is well known (see, for instance, [15]) that the regular elements and unit-regular elements of a ring coincide if and only if the ring is IC. Thus, we have the following immediate corollary to Theorem 6.2.

Corollary 6.3. *Let I be an ideal of an IC ring R. If regular elements lift modulo I, then units lift modulo I.*

While unit-regularity deals with multiplying units and idempotents, there is an additive analog of this condition. Recall that an element $a \in R$ is said to be *clean* if $a = e + u$ where $e \in \mathrm{idem}(R)$ and $u \in U(R)$; and a ring is clean when all its elements are clean. Šter shows in [28, Proposition 3.1] that if I is an ideal of a clean ring such that R/I *is local*, then units lift modulo I. The regular endomorphism ring $R = \mathrm{End}(V)$ of a countable-dimensional vector space V is clean by [24] (but also see [5] for a significant strengthening of this result). Thus, from Remark 3.10 we find that the local assumption in Šter's result is not superfluous. Hence, clean lifting is not sufficient for unit lifting, but we know that it is sufficient for idempotent lifting, even on the level of elements, by Theorem 5.2.

Remark 6.4. Assume $I \subseteq R$ is a left ideal, that $x \in R$ is idempotent modulo I, that $Ix \subseteq I$, and that $x + I$ lifts to a clean element $a = e + u$ for some $e \in \mathrm{idem}(R)$ and $u \in U(R)$. We have seen in Theorem 5.2 that $x + I$ lifts to an idempotent in R. However, we can say more. In Nicholson's proof of [23, Proposition 1.8(1)], he derived the equality

$$u(a - u^{-1}(1-e)u) = a^2 - a. \tag{6.5}$$

Since $a - x \in I$, we have $a^2 - a \in I$ by Lemma 5.1, so (6.5) shows $x + I$ lifts to $u^{-1}(1-e)u \in \mathrm{idem}(R)$.

7. Final thoughts and questions

Recall that if e and f are idempotents in a ring R, then any isomorphism $eR \to fR$ is given (not necessarily uniquely) as left multiplication by some element $a \in R$. Thus, it is straightforward to define what it means to lift isomorphisms between idempotents modulo an ideal.

Definition 7.1. Let I be an ideal of a ring R, and put $\overline{R} = R/I$. Let $e, f \in \text{idem}(R)$. We say that an isomorphism $\varphi : \overline{eR} \to \overline{fR}$ of right $\overline{R}$-modules *lifts modulo I to an isomorphism between the idempotents e and f*, if there exists some $a \in R$ such that φ is left multiplication by $a + I$ restricted to $\overline{eR}$, and left multiplication by a induces an isomorphism $a\cdot : eR \to fR$ of right R-modules. In this case we say that $a \in R$ is a *lift* of the isomorphism φ between the idempotents e, f.

There are some natural limitations to the assumption that "isomorphic idempotents lift modulo an ideal $I \trianglelefteq R$." For instance, given two elements $x, y \in R$ that are isomorphic idempotents modulo I, we can only guarantee that at least *one* (among many) of the isomorphisms between them lifts. We have seen that we can work around this problem, by additionally positing that units lift (as any two such isomorphisms differ by, say, a precomposition with a unit). Moreover, if we want to guarantee that for *any* idempotent lifts e and f, of $x + I$ and $y + I$ respectively, all isomorphisms between $x + I$ and $y + I$ lift to isomorphisms between e and f, we must *require* that units lift. (Just take $x = y = e = f = 1$.)

Regular lifting gives more than isomorphic idempotent lifting does, even when just talking about isomorphic idempotents, and without unit lifting; namely, each isomorphism between $x + I$ and $y + I$ lifts (see Theorem 4.2). Nevertheless, there is still a problem, even when working with a *fully lifting* ideal in a regular ring, since these lifted isomorphisms are not necessarily between the same idempotent lifts of $x + I$ and $y + I$. Indeed, consider the case when S is the endomorphism ring of a countable-dimensional vector space V over a division ring, as in Remark 3.10 or Example 5.15. Let I be the ideal of matrices with finitely many nonzero rows, and let P and Q be the matrices with 1's along the diagonals below and above the main diagonal, respectively, as before. We have that P and Q are inverse units modulo I. However, taking $X = Y = \mathbb{1}$ (the identity matrix), we claim that there are no idempotent lifts E and F (of $X + I$ and $Y + I$) for which *both* P and Q lift to isomorphisms $ES \to FS$.

Proof of claim. Let V have an ordered basis $\mathscr{B} = \{b_0, b_1, b_2, \ldots\}$. We view S as a ring of $\mathbb{N} \times \mathbb{N}$ column-finite matrices, via this basis.

Let $E = \begin{pmatrix} E_0 & E_1 \\ 0 & \mathbb{1} \end{pmatrix}$ and $F = \begin{pmatrix} F_0 & F_1 \\ 0 & \mathbb{1} \end{pmatrix}$ be idempotents that agree with the identity matrix modulo I; so E_0 and F_0 are $n \times n$ idempotent matrices (for some $n \geq 1$) and $E_0E_1 = F_0F_1 = 0$. After increasing n by 1 if necessary, we can assume that the bottom row of E_0 consists of zeros except for the last entry, which is 1. In particular, there is some subset $Z \subseteq \{1, 2, \ldots, n-1\}$ such that $\{Eb_i\}_{i \in Z} \cup \{Eb_j\}_{j \geq n}$ is a basis for ES, and is a disjoint union.

Let P' and Q' agree with P and Q modulo I (respectively), and further assume by way of contradiction that multiplication by P' and Q' restrict to isomorphisms $ES \to FS$. We may as well replace P' by $FP'E$ and Q' by $FQ'E$, so that we then have $P' = FP'E$ and $Q' = FQ'E$. After increasing n if necessary, we may assume that P' is of the form

$$\begin{pmatrix} A_0 & A_2 \\ A_1 & 0 \\ 0 & \mathbb{1} \end{pmatrix},$$

where A_0 is an $n \times n$ matrix, A_1 is a $1 \times n$ row where all entries are zero except that the last entry is 1, and the other entries have the appropriate sizes to make this a block decomposition.

As multiplication by $P' = FP'E$ is an isomorphism $EV \to FV$, we see that $\{P'b_i\}_{i \in Z} \cup \{P'b_j\}_{j \geq n}$ is a basis for FV. Looking at the last nonzero entries of these columns, we see that the column span of F_0 is exactly the span of $\{P'b_i\}_{i \in Z}$. Hence $\text{rank}(F_0) = \text{rank}(E_0) - 1 = |Z|$.

By a similar computation, after replacing P' with Q', we obtain $\operatorname{rank}(F_0) \geq \operatorname{rank}(E_0)$. This gives us the necessary contradiction to finish our claim. $\square$

The problems expressed in the example above disappear if we do not fix the idempotent lifts of $x + I$ and $y + I$, but rather let them vary with each lift of an isomorphism. More precisely, we *can* fix idempotent lifts, but we may need to then modify these idempotents slightly, by passing to a "smaller" idempotent, as in the following definition.

Definition 7.2. Given idempotents $e, e' \in R$, we say that e' is a *subidempotent* of e if $e' \in eRe$. Putting $e'' := e - e'$, we may say that e'' is the *complementary subidempotent* to e' in e, and we write $e = e' \oplus e''$.

The following lemma illustrates how isomorphisms restrict to subidempotents.

Lemma 7.3. *Let I be an ideal of a ring R. Let $e, f \in \operatorname{idem}(R)$ and assume that left multiplication by $a \in R$ induces an isomorphism $eR \to fR$. If $f = f_1 \oplus f_2$ for some subidempotents $f_1, f_2 \in R$ of f, then $e = e_1 \oplus e_2$ for some subidempotents $e_1, e_2 \in R$ such that left multiplication by a induces isomorphisms $e_1R \to f_1R$ and $e_2R \to f_2R$. Moreover, if $f_2 \in I$, then $e_2 \in I$, and we have $e \equiv e_1, f \equiv f_1 \pmod{I}$.*

Proof. Assume $f = f_1 \oplus f_2$. We then have $fR = f_1R \oplus f_2R$. Left multiplication by a gives an isomorphism $eR \to fR$, and so taking A to be the inverse image of f_1R, and taking B to be the inverse image of f_2R, we have $eR = A \oplus B$ as right R-modules. Let e_1 be the idempotent that projects to A with kernel $B \oplus (1 - e)R$, and similarly let e_2 be the idempotent that projects to B with kernel $A \oplus (1 - e)R$.

As $R = eR \oplus (1 - e)R = A \oplus B \oplus (1 - e)R$, we can uniquely write $1 = a + b + c$ for some $a \in A$, $b \in B$ and $c \in (1 - e)R$. It is a standard computation to show that $a = e_1$, $b = e_2$, and $c = 1 - e$ are orthogonal idempotents. Note in particular that

$$e = 1 - (1 - e) = 1 - c = a + b = e_1 + e_2.$$

Thus, $ee_1 = (e_1 + e_2)e_1 = e_1$, and similarly $e_1e = e_1$, so that $e_1 \in eRe$. Hence $e = e_1 \oplus e_2$ as desired. By construction, left multiplication by a induces isomorphisms $e_1R \to f_1R$ and $e_2R \to f_2R$.

Next, assume that $f_2 \in I$. As f_2 is isomorphic to e_2, there exist $s, t \in R$ with $e_2 = st$ and $f_2 = ts$. Thus

$$e_2 = e_2^2 = stst = sf_2t \in I.$$

The last statement is clear since $e = e_1 + e_2 \equiv e_1 \pmod{I}$, and $f = f_1 + f_2 \equiv f_1 \pmod{I}$. $\square$

We need one more preliminary result, which shows that under strong lifting we can refine *arbitrary* idempotent lifts of isomorphic idempotents, to isomorphic idempotents.

Proposition 7.4. *Let I be a strongly lifting ideal in a ring R. Let $e, f \in \operatorname{idem}(R)$ and $a \in R$. Assume that left multiplication by $\overline{a}$ induces an isomorphism $\overline{eR} \to \overline{fR}$ in $\overline{R} = R/I$. There exist $e_1, e_2, f_1, f_2 \in \operatorname{idem}(R)$ and $a_1 \in R$ satisfying*

- *$e = e_1 \oplus e_2$ and $f = f_1 \oplus f_2$,*
- *$e_2, f_2 \in I$, or equivalently $e \equiv e_1, f \equiv f_1 \pmod{I}$,*
- *$a \equiv a_1 \pmod{I}$, and*
- *left multiplication by a_1 induces an isomorphism $e_1R \to f_1R$.*

Proof. As left multiplication by $\overline{a}$ induces an isomorphism $\overline{eR} \to \overline{fR}$, we may fix some $b \in R$ so that left multiplication by $\overline{b}$ induces the inverse isomorphism. In particular, $\overline{e} = \overline{ebfae}$ and $\overline{f} = \overline{faebf}$.

Now, since $ebfae$ is an idempotent modulo I, by strong lifting (using, [25, Lemma 1(2)]) we find an idempotent

$$e_1 \in (ebfae)R(ebfae) \subseteq eRebfae \tag{7.5}$$

with $e_1 \equiv ebfae \equiv e \pmod{I}$.

Define $e_2 := e - e_1 \in I$, and since $e_1 \in eRe$, we see that $e = e_1 \oplus e_2$. By (7.5), fix some $t \in R$ so that

$$e_1 = etebfae. \tag{7.6}$$

Whenever a product xy is an idempotent, then it is straightforward to check that $yxyx$ is also an (isomorphic) idempotent. Taking $x := etebf$ and $y := fae$, it follows from (7.6) that $e_1 = xy$ is an idempotent, and thus from the previous sentence $f_1 := yxyx \in fRf$ is an isomorphic idempotent.

Setting $f_2 := f - f_1$, clearly $f = f_1 \oplus f_2$. Our next goal is to prove that $f - f_1$ is an element of I. We first find

$$e \equiv e_1 \overset{\text{by (7.6)}}{=\!=} et(ebfae) \equiv ete \pmod{I}. \tag{7.7}$$

Using the definitions of x and y, we then have

$$f_1 = y(xy)x = (fae)(xy)(etebf) = (fae)e_1(etebf) \overset{\text{by (7.7)}}{\equiv} faebf \equiv f \pmod{I}.$$

Finally, set

$$a_1 := f_1ae_1 + a(1 - e_1). \tag{7.8}$$

Since left multiplying by $\overline{a}$ takes $\overline{eR}$ to $\overline{fR}$, we have that $\overline{(1 - f)ae} = \overline{0}$. Hence,

$$a_1 = f_1ae_1 + a(1 - e_1) \equiv fae + a(1 - e) \equiv (1 - f)ae + fae + a(1 - e) = ae + a(1 - e) = a \pmod{I}.$$

All that remains is to show that left multiplication by a_1 induces an isomorphism $e_1R \to f_1R$. Equivalently, by (7.8) we need to show that left multiplication by f_1ae_1 is such an isomorphism. This will follow once we show e_1tebf_1 is the inverse map. We compute

$$(e_1tebf_1)(f_1ae_1) = e_1(etebf)(f_1)(fae)e_1 = e_1(x)(yxyx)(y)e_1 = e_1^5 = e_1.$$

Similarly $(f_1ae_1)(e_1tebf_1) = f_1$, and so the induced maps are indeed inverse isomorphisms. □

We can now put the previous lemma and proposition together, to show that compatible systems of isomorphisms lift compatibly. In light of the example at the beginning of this section, this is the best one could hope for without additional hypotheses (such as unit lifting).

Theorem 7.9. *Let I be a strong lifting ideal of a ring R. Let $e_1, e_2, \ldots, e_n \in \operatorname{idem}(R)$ for some $n \geq 2$, and assume that there are isomorphisms $\overline{e_iR} \to \overline{e_{i+1}R}$ for each $1 \leq i \leq n - 1$. Those isomorphisms lift to isomorphisms $e_i'R \to e_{i+1}'R$, where e_j' is a subidempotent of e_j, for each $1 \leq j \leq n$.*

Proof. We work by induction on n, with the base case done by Proposition 7.4.

For the inductive step, after passing to subidempotents and renaming if necessary, we may as well assume that the isomorphisms $\overline{e_iR} \to \overline{e_{i+1}R}$ lift to isomorphisms $e_iR \to e_{i+1}R$, for $1 \leq i \leq n - 2$. By appealing to Proposition 7.4 once more, we can lift the isomorphism $\overline{e_{n-1}R} \to \overline{e_nR}$ to an isomorphism $e_{n-1}'R \to e_n'R$ for some subidempotents e_{n-1}' of e_{n-1}, and e_n' of e_n. Applying Lemma 7.3 recursively, we can construct subidempotents $e_{n-2}', e_{n-3}', \ldots, e_1'$ (in that order) such that each isomorphism $e_iR \to e_{i+1}R$ restricts to an isomorphisms $e_i'R \to e_{i+1}'R$ (for $1 \leq i \leq n-2$), which still lifts the given $\overline{R}$-module isomorphism $\overline{e_iR} \to \overline{e_{i+1}R}$. □

We end this paper by mentioning one final question that we were unable to answer, concerning the extent to which Example 5.15 can be improved, and a remark about extending results in this paper to non-unital rings.

Question 7.10. Does there exist an ideal in a **regular** ring, such that conjugate idempotents do not lift modulo that ideal?

Remark 7.11. Several concepts in this paper require rings to have 1; such as unit lifting, conjugate idempotent lifting, unit-regular lifting, and clean lifting. However, the other lifting properties, and many of the facts about them, do not require rings to have 1. Often, the proofs we've given can be modified to avoid using units; sometimes simply by expanding products. We leave it to the interested reader to make such changes as needed, and we note that some of this non-unital theory will be explored further in [19].

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