DEDEKIND-FINITE STRONGLY CLEAN RINGS

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Abstract. In this paper we partially answer two open questions concerning clean rings. First, we demonstrate that if a quasi-continuous module is strongly clean then it is Dedekind-finite. Second, we prove a partial converse. We also prove that all clean decompositions on submodules of continuous modules extend to the entire module.

1. Introduction

W. K. Nicholson first introduced clean rings as a class of examples in his investigations on exchange rings [9]. Subsequently, these rings have become objects of interest in their own right. The relevant definitions are as follows:

Definition 1. Let $k$ and $R$ be rings. An element $r \in R$ is clean if we can write $r = u + e$, where $u \in U(R)$ is a unit and $e \in R$ is an idempotent. If we can pick $u$ and $e$ so that they commute, we say that $r$ is strongly clean. If all the elements of a ring are (strongly) clean, we say the ring is a (strongly) clean ring. If $M_k$ is a right $k$-module, with a (strongly) clean endomorphism ring, then we call $M_k$ a (strongly) clean module.

Nicholson in [10] asked the following five questions:

1. Does every strongly clean ring $R$ have stable range 1?
2. Is every strongly clean ring directly finite?
3. Is the property of being strongly clean a Morita invariant?
4. Is every unit regular ring strongly clean?
5. Is every semiperfect ring strongly clean?

Questions 3 and 5 were answered in the negative in [13], which showed that a direct sum of two strongly clean local modules need not be strongly clean. More examples of similar phenomena, and generalizations, are given in [1]. On the other hand, it was shown in [3] that if $R$ is a strongly clean ring and $e^2 = e \in R$ then $eRe$ is also strongly clean, which answers the other half of the Morita invariance problem in the positive.

In this paper we will show that a large class of unit regular rings always have a property implied by strongly clean rings, thus making progress on question 4. We also answer question 2 in the positive for endomorphism rings of quasi-continuous modules. We finish the paper by answering a question left open in [2] concerning lifting clean decompositions for continuous modules.

Throughout this paper, by “ring” we will mean an associative ring with 1. All modules are unital. If $M_k$ is a module, $\varphi \in \text{End}(M_k)$, and $N \subseteq M$, one says that $N$ is $\varphi$-invariant if $\varphi(N) \subseteq N$. We direct the reader to [6], [7], and [8] for further information on many of the definitions and elementary concepts used herein.

2. Clean Decompositions

Let $\varphi \in S = \text{End}(M_k)$. Recently, an alternate characterization of when $\varphi$ is clean was provided in [2]. Specifically, $\varphi$ is clean if and only if there are right $k$-module decompositions $M = A \oplus B = C \oplus D$
such that \(\varphi|_A\) maps \(A\) isomorphically to \(C\) and \((1 - \varphi)|_B\) maps \(B\) isomorphically to \(D\). Pictorially

\[
M = A \oplus B
\]

\[
\varphi \downarrow \sim (1 - \varphi) \downarrow \sim
\]

\[
M = C \oplus D.
\]

If such a diagram holds we call this an \(ABCD\)-decomposition for \(\varphi\). It turns out by a result of Nicholson in [10] that \(\varphi\) is strongly clean if and only if we can find an \(ABCD\)-decomposition where \(A = C\) and \(B = D\).

There are some interesting consequences to assuming \(A = C\) or \(B = D\). Suppose we have an \(ABCD\)-decomposition for \(\varphi\). If \(A = C\), this means that \(A\) is \(\varphi\)-invariant. Further, \(\varphi(A) = A\), and hence \(\varphi^n(A) = A\) for each \(n \geq 1\). This implies \(A \subseteq \varphi^n(M)\) for each \(n \geq 1\). So, if we define \(I_\varphi = \bigcap_{n \geq 1} \varphi^n(M)\) then we have \(A \subseteq I_\varphi\).

Similarly, if \(B = D\) this means \(B\) is \((1 - \varphi)\)-invariant, and hence \(\varphi\)-invariant. We claim that \(\ker(\varphi) \subseteq B\). To prove this, let \(x \in \ker(\varphi)\), and write \(x = a + b\) with \(a \in A\) and \(b \in B\). We have \(0 = \varphi(x) = \varphi(a) + \varphi(b)\), so \(\varphi(a) = -\varphi(b) \in C \cap B = (0)\). But \(\varphi\) is an isomorphism from \(A\) to \(C\), hence \(a = 0\) and \(x = b \in B\).

We further claim that \(\ker(\varphi^n) \subseteq B\) for each \(m \geq 1\). We proceed by induction, the base case being the substance of the previous paragraph. Suppose that for all \(m < n\) we have \(\ker(\varphi^n) \subseteq B\). Let \(x \in \ker(\varphi^n)\), and write \(x = a + b\) with \(a \in A\) and \(b \in B\). We have \(\varphi(x) = \varphi(a) + \varphi(b)\) and so \(\varphi(a) = \varphi(x) - \varphi(b) \in C \cap B = (0)\) (since \(\varphi(x) \in \ker(\varphi^{n-1}) \subseteq B\)). As before, this means \(a = 0\), and so \(x \in B\). Defining \(K_\varphi = \bigcup_{n \geq 1} \ker(\varphi^n)\), we have \(K_\varphi \subseteq B\).

By the above, and symmetry, we have just proven:

**Lemma 2.** Let \(M_k\) be a right \(k\)-module, \(S = \text{End}(M_k)\), \(\varphi \in S\) a clean element, and \(\varphi' = 1 - \varphi\). Fix an \(ABCD\)-decomposition for \(\varphi\).

1. If \(A = C\) in the decomposition then \(K_{\varphi'} \subseteq A \subseteq I_\varphi\).
2. If \(B = D\) in the decomposition then \(K_{\varphi'} \subseteq B \subseteq I_{\varphi'}\).

In particular, both of these conclusions hold for any strongly clean decomposition.

When one of the two conditions in the Lemma holds, we have a condition stronger than “clean” but weaker than “strongly clean.” We use the following terminology:

**Definition 3.** Let \(\varphi \in S = \text{End}(M_k)\). We say \(\varphi\) is left capably clean if there is an \(ABCD\)-decomposition for \(\varphi\) with \(A = C\). Similarly, \(\varphi\) is right capably clean if there is an \(ABCD\)-decomposition with \(B = D\). For left \(k\)-modules, we switch these two definitions, for reasons to be explained later.

Just as in [2] and [10], we have alternate characterizations of these properties. For notational ease in the next proposition, and in the discussion following, we let \(e' = 1 - e\) and \(\varphi' = 1 - \varphi\).

**Proposition 4.** Let \(M_k\) be a right \(k\)-module, and \(S = \text{End}(M_k)\). The following are equivalent:

1. \(\varphi \in S\) is left capably clean.
2. \(\varphi' \in S\) is right capably clean.
3. \(\varphi = u + e\) with \(u \in U(S)\), \(e^2 = e \in S\), \(e\varphi' = 0\), and \(e'\varphi e' \in U(e'Se')\).
4. \(\varphi' = -u + e'\) with \(u \in U(S)\), \(e^2 = e \in S\), \(e\varphi' e' = 0\), and \(e'\varphi e' \in U(e'Se')\).

**Proof.** (1) \(\iff\) (2): Clear by definition.
(1) \Rightarrow (3): Suppose we have an \(ABCD\)-decomposition for \(\varphi\) with \(A = C\). Let \(e \in S\) be an idempotent with \(e'(M) = A\) and \(e(M) = B\). We define \(u\) to equal \(\varphi\) on \(A\), and to equal \(-\varphi\) on \(B\). Pictorially

\[
\begin{align*}
M & = A \oplus B \\
& \frac{u}{\sim} -u \frac{u}{\sim} \\
M & = A \oplus D.
\end{align*}
\]

So \(u \in U(S)\). It is easy to check \(\varphi = u + e\). Now \(eue(a) = 0 = euu(a)\), and \(eue(b) = eu(b)\), for \(a \in A\) and \(b \in B\). Therefore, \(eue'(m) = 0\) on all \(m \in M\) and hence \(e\varphi e' = 0\). Finally, \(e'\varphi e' = u|_A \in U(e'Se')\).

(3) \Rightarrow (1): Write \(\varphi = u + e\) with \(u \in U(S)\), \(e^2 = e \in S\), \(e\varphi e' = 0\), and \(e'\varphi e' \in U(e'Se')\). Set \(A = e'(M)\), \(B = e(M)\), \(C = A\), and \(D = \varphi'(B)\). Then one simply checks that this is an \(ABCD\)-decomposition for \(\varphi\). Clearly \(C = A\), and so \(\varphi\) is left capably clean.

(3) \Leftrightarrow (4): This follows either from the above work, or one sees directly that the two equations \(\varphi = u + e\) and \(e\varphi e' = 0\) give the same information as the two equations \(\varphi' = -u + e'\) and \(e\varphi' e' = 0\). \(\Box\)

**Remark 5.** Consider the diagram

\[
\begin{align*}
M & = A \oplus B \\
& \frac{u}{\sim} - \frac{u}{\sim} \\
M & = A \oplus D
\end{align*}
\]

which was used in the proof above. Recall that the idempotent \(e \in R\) was chosen so that \(e(M) = B\) and \(e(A) = 0\). Since \(D\) is a direct sum complement to \(A\) we have \(e|_B : D \to B\) is an isomorphism. On the other hand, we defined \(u\) so that \(u|_B = -\varphi'|_B\), and in particular \(\varphi'|_B : B \to D\) is an isomorphism. Therefore, the map \(e\varphi'|_B : B \to B\) is an isomorphism. In other words, \(e\varphi' \in U(e'Se)\).

There is another way to see this. Let \(\varphi = u + e\) be a left capably clean decomposition. Then \(eue' = 0\), and \(e'ue' = e'\varphi' e' \in U(e'Se')\). Also notice that \(e\varphi' e = -eue\). Therefore, the Peirce decomposition for \(u\) with respect to \(e\) and \(e' = 1 - e\) is

\[
u = \begin{pmatrix} eue & eue' \\ e'ue & e'ue' \end{pmatrix} = \begin{pmatrix} -e\varphi' e & 0 \\ e\varphi' e' \end{pmatrix}.
\]

Since \(u\) is a unit, the matrix is lower-triangular, and the south-east corner is a unit, this implies that the north-west corner is a unit. Therefore, \(e\varphi' \in U(e'Se)\).

We say a right \(k\)-module, \(M_k\), is left capably clean if every endomorphism is left capably clean, while right capably clean modules are defined similarly. Let \(\varphi \in S = \text{End}(M)\) and assume \(M\) is left capably clean. The equivalence (1) \Leftrightarrow (2) above means \(\varphi'\) is right capably clean. But since \(\varphi'\) is an arbitrary element of \(S\), this means \(M_k\) is also right capably clean. Conversely, right capably clean modules are left capably clean, so we will drop the left and right and just call such modules capably clean.

We say a ring, \(R\), is capably clean if \(R_R\) is capably clean as a module. Proposition 4 then implies that a module \(M_k\) is capably clean if and only if \(S = \text{End}(M_k)\) is capably clean; so capably cleanness is an endomorphism ring invariant. We say a ring element \(r \in R\) is left capably clean if \(r\) is left capably clean as an endomorphism of the module \(R_R\); or equivalently, by Proposition 4, \(r = u + e\) with \(u \in U(R)\), \(e^2 = e \in R\), \(e'\varphi e' \in U(e'Se')\), and \(ere' = 0\). Right capably clean elements are defined analogously.

For a left \(k\)-module, \(kM\), recall that we switched the two definitions given in Definition 3. We do this so that \(\varphi \in S = \text{End}(kM)\) is left (resp. right) capably clean as an endomorphism of \(kM\) if and only if it is left (resp. right) capably clean as an endomorphism of \(kS\), if and only if it is left (resp. right) capably clean as an endomorphism of \(S_k\). In this way, there is no confusion when we talk of left (or right) capably clean ring elements.
Examples and Counter-examples

Example 6. Let $k = \mathbb{Q}$, and let $M_k$ be a countably infinite-dimensional vector space over $k$, say with basis $\{m_1, m_2, \ldots\}$. Let $\varphi$ be the right shift operator, given by $\varphi(m_i) = m_{i+1}$. We know $\varphi$ is clean, and in fact $S = \text{End}(M_k)$ is clean, by [11]. However, we will show that $\varphi$ is neither left nor right capably clean.

Notice that $I_\varphi = (0)$ and $I_{1-\varphi} = (0)$. So, in particular, $\varphi$ cannot possibly be strongly clean by Lemma 2. Suppose there is an ABCD-decomposition for $\varphi$ with $A = C$. Then $A \subseteq I_\varphi = (0)$ and so $M = B = D$, showing that $\varphi$ is strongly clean, a contradiction. We reach a similar contradiction if we suppose that $\varphi$ is right capably clean. Thus, $\varphi$ gives us an example of an element which is clean but neither left nor right capably clean. Further, $S = \text{End}(M_k)$ gives an example of a clean ring which is not capably clean.

Example 7. Consider the matrix $\varphi = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{Z})$. By direct calculation one sees that

$\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

is a left capably clean decomposition, while

$\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix}$

is a right capably clean decomposition. If we had an $ABAB$-decomposition for $\varphi$ then $A \oplus B = \begin{pmatrix} 1 \\ Z \oplus 1 \\ 3 \end{pmatrix} \mathbb{Z}$ since $A$ and $B$ would be spanned by eigenvectors of $\varphi$ and $1 - \varphi$. But then $A \oplus B \neq M$, a contradiction. Thus $\varphi$ is not strongly clean.

Example 8. A direct product ring $\prod_{i \in I} R_i$ is capably clean if and only if each $R_i$ is capably clean.

Example 9. Let $k = \mathbb{Q}$ and $M$ be a countably infinite-dimensional vector space. We write $M = V_1 \oplus V_2$ with $V_1 = \bigoplus_{i \in \mathbb{Z}} x_i \mathbb{Q}$ and $V_2 = \bigoplus_{1 \leq \iota \leq j} y_{i,j} \mathbb{Q}$. For later notation, we set $y_{i,j} = 0$ if $i \leq j$. Also let $\pi$ be the projection to $V_1$ with kernel $V_2$.

We define $\varphi \in \text{End}(M_k)$ by $\varphi(y_{i,j}) = y_{i-1,j}$ and $\varphi(x_i) = x_{i-1} + y_{i,i}$. Notice that $K_\varphi = V_2$. We claim that $I_{1-\varphi} \cap V_1 = (0)$. In fact, let $\alpha = \sum_{k_1} b_{k_1} x_i \in V_1$, where $a_{k_1} \neq 0$ and $a_{k_2} \neq 0$. Then an easy calculation shows $\pi(1 - \varphi)^n \alpha = \sum_{k_2} b_{k_2} x_i$ where $b_{k_2} = a_{k_2}$ and $b_{k_1-n} = (-1)^n a_{k_1}$. Thus, $(1 - \varphi)^n$ “spreads out” the smallest and largest non-zero supports in $\alpha$ by $n$. So, in particular, $\alpha \notin \text{im}((1 - \varphi)^n)$ only if $k_2 - k_1 \geq n$. Taking $n$ large enough, $\alpha \notin \text{im}((1 - \varphi)^n) \supseteq I_{1-\varphi}$. This proves our claim, and it is easy to check by similar arguments that $V_2 = K_\varphi = I_{1-\varphi}$.

A straightforward computation demonstrates that $I_\varphi = (0)$. Thus, if we have an $ABCD$-decomposition with $A = C$ then $A = C = (0)$, and $B = D = M$. But as $\varphi$ is not an isomorphism, this is impossible. Hence $\varphi$ is not left capably clean. However, taking $A = V_1$, $C = \varphi(V_1)$, and $B = D = V_2$ yields a decomposition for $\varphi$ showing it is right capably clean.

Remark 10. Using the techniques of [12] it turns out that all capably clean modules have the countable exchange property. If a capably clean module has $(C_2)$ then it has the full exchange property.

Remark 11. If $M$ is strongly clean then any summand is strongly clean by [3]. This statement also holds true for capably clean modules. If fact, suppose $M = N \oplus N'$ is capably clean. Fix $\varphi \in \text{End}(N)$ and extend $\varphi$ to an endomorphism $\psi$ on $M$ by setting $\psi(N') = (0)$. Since $M$ is capably clean there is an $ABCD$-decomposition with $B = D$ and in particular, by Lemma 2, $\ker(\psi) \subseteq B$. Write $\psi = u + \epsilon$, where this clean decomposition is with respect to the $ABCD$-decomposition above. Then $\epsilon(B) = B$ and in particular $\epsilon N' = N'$. Since $N'$ is $\psi$-invariant, this means it is also $u$-invariant. So, looking
at End($M/N'$), we have a capably clean decomposition $\overline{\psi} = \overline{\sigma} + \overline{\pi}$. But $M/N' \cong N$, and under this isomorphism there is a natural identification of $\overline{\psi}$ with $\varphi$. In particular, $\varphi$ is capably clean. So we see that $N$ is capably clean.

**Example 12.** While it is known that a finite direct sum of clean modules is clean, in [13] it was shown that the same result does not hold for strongly clean modules. In particular, they showed $\mathbb{Z}_2(2)$ is not strongly clean (where $\mathbb{Z}_2(2)$ is the ring of integers localized at the prime ideal (2)).

We will now show that $M_{\mathbb{Z}_2(2)} = \mathbb{Z}_2(2)$ is not even capably clean, for any prime $p$. Consider the matrix $\varphi = \begin{pmatrix} 1 & -p \\ 1 & 0 \end{pmatrix} \in S = \text{End}(M)$. One easily sees that neither $\varphi$ nor $1 - \varphi$ is a unit. So, given any $ABCD$-decomposition for $\varphi$, we cannot have $A = M$ or $B = M$. Therefore, we must have $A \cong \mathbb{Z}_2(2) \cong B$ (and of course $A \cong C$ and $B \cong D$).

Suppose that $\varphi$ is left capably clean so that $A = C$. Tensoring up to $\mathbb{Q}$ (which is flat over $\mathbb{Z}_2(2)$) this means that $\varphi$ has an eigenvector. However, the discriminant of the characteristic polynomial of $\varphi$ is $1 - 4p < 0$ and so the characteristic polynomial does not split over $\mathbb{Q}$. This gives a contradiction. Therefore $\varphi$ cannot be left capably clean, and so $M$ is not capably clean.

Denoting the Jacobson radical by $J(S)$, also notice that $S/J(S) \cong M_2(\mathbb{Z}/p\mathbb{Z})$ is strongly clean. So this example also shows that capable cleanness does not lift modulo the radical even if idempotents do.

We finish this section with an open question:

**Question:** Are there capably clean rings which are not strongly clean?

### 4. Continuous, Dedekind-finite Modules

We begin this section by introducing a few standard definitions and notations. We let $B \subseteq A$ be modules. If $B$ is a summand of $A$ we write $B \subseteq \oplus A$. If $B$ is essential in $A$, meaning that for any non-zero submodule $C \subseteq A$ we have $C \cap B \neq (0)$, then we write $B \subseteq e A$. A module $M_k$ is said to be nonsingular if for each $m \in M$, the annihilator $\text{ann}_k(m) = \{r \in k \mid mr = 0\}$ is never essential in $k_k$. A ring $R$ is said to be right nonsingular if $R_R$ is nonsingular.

A module $M$ is said to be Dedekind-finite (or sometimes, directly finite) if $M \cong M \oplus X$ implies $X = (0)$. A ring is said to be Dedekind-finite if $xy = 1$ implies $yx = 1$; that is, all one-sided units are units. It is an easy exercise to prove that $M$ is Dedekind-finite as a module if and only if $\text{End}(M)$ is Dedekind-finite as a ring. A ring $R$ is said to be (von Neumann) regular if for each $x \in R$ there exists $y \in R$ with $xyx = x$, and is said to be unit regular if further $y$ can be chosen to be a unit. Unit regular rings are always Dedekind-finite. These concepts are explored further in [5] and [6].

The following three conditions on a module $M$ are also standard:

- $(C_1)$ If $A \subseteq M$ then there exists $N \subseteq M$ with $A \subseteq e N$.
- $(C_2)$ If $A \subseteq M$ and $A \cong B \subseteq M$ then $A \cong M$.
- $(C_3)$ If $A, B \subseteq M$ and $A \cap B = (0)$ then $A \oplus B \subseteq M$.

The property $(C_2)$ implies $(C_3)$ by [8, Proposition 2.2]. A module $M$ is said to be (quasi-)continuous if it satisfies conditions $(C_1)$ and $(C_2)$ (respectively, $(C_1)$ and $(C_3)$). Each of the three properties above is inherited by summands, and so summands of (quasi-)continuous modules are (quasi-)continuous. We assume that the reader is familiar with the results from [8]. Continuous and quasi-continuous modules were developed as generalizations of injective modules. A ring is said to be right continuous if $R_R$ is continuous, and two-sided continuous if both $R_R$ and $R_R$ are continuous. In [2], it is proved that all continuous modules are clean. The proof begins by considering the case of a nonsingular continuous module $M$. We claim that if one also assumes $M$ is Dedekind-finite then $M$ is capably clean. But before we prove this result we need a few simple lemmas.
Lemma 13. Let $M_k$ be a nonsingular module. Let $\varphi \in \text{End}(M_k)$, and let $A'$ be a $\varphi$-invariant submodule of $M$. If we have $A' \subseteq A \subseteq \oplus M$ then $A$ is $\varphi$-invariant.

Proof. This is proven in [2], but for completeness we reproduce the argument here. Let $M = A \oplus X$.

Given $a \in A$ write $\varphi(a) = a_1 + x$ with $a_1 \in A$ and $x \in X$. Consider the map $k_k \to A_k$ given by $r \mapsto ar$. Set $I = \{r \in k : ar \in A'\}$. Since $A' \subseteq A$ this means $I_k \subseteq k_k$. For each $r \in I$, $\varphi(ar) = a_1 r + x r$.

Therefore, since $ar \in A'$ which is $\varphi$-invariant, $x r = \varphi(ar) - a_1 r \in X \cap A = (0)$. This means $I \subseteq \text{ann}_k(x)$, which implies $\text{ann}_k(x)$, and thus $\varphi(a) = a_1 \in A$.

Lemma 14. Let $M_k$ be a module with $(C_2)$. If $\varphi \in \text{End}(M_k)$ is a monomorphism with $\text{im}(\varphi) \subseteq M$ then $\varphi$ is an isomorphism.

Proof. We know $\text{im}(\varphi) \cong M \subseteq \oplus M$. So, from $(C_2)$ we have $\text{im}(\varphi) \subseteq \oplus M$. But the only essential summand of $M$ is $M$ itself. Thus $\text{im}(\varphi) = M$, which yields surjectivity of $\varphi$. □

We are now ready to prove:

Theorem 15. If $M_k$ is a nonsingular, Dedekind-finite, continuous module then $M$ is capably clean.

Proof. Let $\varphi \in \text{End}(M_k)$. Consider the submodule $K_{\varphi}$. Since $M$ has $(C_1)$, there is some module $B$ such that $K_{\varphi} \subseteq B \subseteq \oplus M$. Since $K_{\varphi}$ is $\varphi$-invariant, so is $B$ by Lemma 13. We can write $M = A \oplus B$. Let $C = \varphi(A)$. Then $\varphi|_A$ maps $A$ isomorphically to $C$ (since $\ker(\varphi) \subseteq K_{\varphi} \subseteq B$). Further, $B \cap C = (0)$ as $K_{\varphi} \cap C = (0)$ and $K_{\varphi} \subseteq B$. Moreover, $C$ is a direct summand of $M$ by $(C_2)$, and so $C \oplus B$ is a direct summand of $M$ by $(C_3)$. But $M = A \oplus B \cong C \oplus B$, and thus $C \oplus B = M$ as $M$ is Dedekind-finite.

Set $\varphi' = 1 - \varphi$. The only thing left to show is that $\varphi'|_B$ maps $B$ to itself isomorphically. By Lemma 13, since $K_{\varphi}$ is $\varphi'$-invariant, so is $B$. Also, since $\varphi'|_{K_{\varphi}}$ is an isomorphism we see that $\varphi'|_B$ is a monomorphism with essential image. Therefore, by the previous lemma, $\varphi'|_B$ is an isomorphism. □

Example 16. All Dedekind-finite (von Neumann) regular rings that are one-sided continuous are capably clean, using the previous theorem. This follows because nonsingularity is automatic for all regular rings [5, Corollary 7.7]. In particular, regular rings which are two-sided continuous are capably clean, as two-sided continuity forces Dedekind-finiteness [5, Corollary 6.49].

Definition 17. A ring $R$ is said to be an exchange ring if for each $a \in R$ there exists an idempotent $e^2 = e \in R$ with $a \in Re$ and $1 - a \in R(1 - e)$. A module $M$ is said to have the finite exchange property if $\text{End}(M_k)$ is an exchange ring. It is well known that clean rings are always exchange rings, but not conversely.

Lemma 18. Let $M_k$ be a quasi-continuous module with the finite exchange property. Set $S = \text{End}(M_k)$ and let $J(S)$ denote the Jacobson radical of $S$. We can write $S/J(S) = S_1 \times S_2$ where $S_1$ is an Abelian (all idempotents lie in the center of the ring) reduced ring, which is clean, and $S_2$ is right-continuous and regular. Furthermore, if $M$ is nonsingular, then $S$ has such a product representation.

Proof. This lemma is a combination of [8, Corollary 3.13] and the first remark immediately following it. □

With this result under our belts, we can extend the previous theorem to:

Theorem 19. If $M_k$ is a nonsingular, Dedekind-finite, quasi-continuous module with the finite exchange property then $M$ is capably clean.

Proof. We know that $S = \text{End}(M) \cong S_1 \times S_2$ where $S_1$ is an Abelian exchange ring and $S_2$ is a right-continuous, regular (hence right nonsingular) ring by Lemma 18. Therefore $S_1$ is strongly clean, and $S_2$ is capably clean from our previous result. Thus $S$ is capably clean since the property in question passes to direct products. □
Nicholson has asked whether all unit-regular rings are strongly clean. One can view our previous result as a step towards answering this question, due to the following proposition. We thank T.Y. Lam and Dinesh Khurana for pointing out that we can add condition 3 below.

**Proposition 20.** The following statements are equivalent:

1. Every nonsingular, Dedekind-finite, quasi-continuous module with the finite exchange property is strongly clean.
2. Every right continuous, unit-regular ring is strongly clean.
3. Every right-self-injective, unit-regular ring is strongly clean.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1): Let $M$ be a nonsingular, Dedekind-finite, quasi-continuous module with the finite exchange property. By Lemma 18, $S = \text{End}(M) \cong S_1 \times S_2$ where $S_1$ is an Abelian exchange ring (hence strongly clean) and $S_2$ is right-self-injective, regular ring. As $M$ is Dedekind-finite, so is $S_2$. Since $(S_2)_{S_2}$ is quasi-continuous and Dedekind-finite, [8, Theorem 2.33] implies $(S_2)_{S_2}$ has internal cancelation. Therefore, $S_2$ is a unit-regular ring. By hypothesis, $S_2$ is then strongly clean. Thus $S \cong S_1 \times S_2$ is strongly clean.

We have the following result in the other direction, which answers Nicholson’s question 2 in [10] for a large class of rings:

**Theorem 21.** If a module is strongly clean and quasi-continuous then it is Dedekind-finite.

Proof. Let $M_R$ be strongly clean and quasi-continuous. Letting $E = \text{End}(M_R)$, we know that $M$ is Dedekind-finite if and only if $E/J(E)$ is Dedekind-finite (as a ring). By Lemma 18, $E/J \cong S_1 \times S_2$ with $S_1$ reduced (hence Dedekind-finite) and $S_2$ right injective and regular. Hence, replacing $M_R$ by $(S_2)_{S_2}$ if necessary, we may assume $M_R$ is injective (and nonsingular).

Suppose by way of contradiction $M \cong M \oplus X$ with $X \neq 0$. Then there is a submodule $N \subseteq M$ with $N = X_1 \oplus X_2 \oplus \cdots$ and $X_i \cong X$ for each $i \geq 1$. Let $f|_N$ be a shift operator sending $X_i$ isomorphically to $X_{i+1}$ for each $i \geq 1$. Since $M$ is injective, $N$ is essential in a summand $W \subseteq \oplus M$. By injectivity, we can extend $f$ to an endomorphism on $W$. Also, since the property of being strongly clean passes to summands, we may as well assume $W = M$.

Since $M$ is strongly clean we have an ABAB-decomposition for $f$. Suppose for a moment $A \neq 0$, and fix $0 \neq a \in A$. Then, by essentiality, there is some $r \in R$ with $0 \neq ar \in N$. Write $ar = \sum_{i=1}^n x_i$ with $x_i \in X_i$. Since $f|_A$ is an isomorphism, and $A$ is $f$-invariant, there is some element $a' \in A$ with $f^{n+1}(a') = ar$. Clearly $a' \neq 0$. Again, by essentiality, there is some $s \in R$ with $0 \neq a's \in N$. But then $0 \neq f^{n+1}(a's) = ars$ lies both in $\bigoplus_{i=1}^n X_i$ and in $\bigoplus_{i=n+1}^\infty X_i$, which is a contradiction. Hence $A = 0$.

This means $1 - f$ is an isomorphism on $M$. A similar argument as in the previous paragraph produces a contradiction, as follows. Fix $0 \neq x \in X_1$, and let $x' \in M$ be such that $(1 - f)(x') = x$. Then there is some $t \in R$ with $0 \neq x't \in N$ so we can write $x't = \sum_{i=1}^{n'} x'_i$ with $x'_i \in X_i$ for each $i$, and $n'$ minimal. But then $xt = (1 - f)(x't)$ has non-zero support in $X_{n'+1}$, a contradiction. Thus, by assuming $M$ is not Dedekind-finite we obtain a contradiction in all cases.

**Remark 22.** This theorem remains true if we weaken “strongly clean” to “capably clean.”

5. **Extending the Results of [2]**

The proof that a direct sum of clean modules is still clean contains some information that will be useful in answering a question left open in [2]. We collect that information here.

**Lemma 23.** Let $R$ be a ring, $e^2 = e \in R$ and $e' = 1 - e$. Fix $\varphi \in R$. Suppose we have a decomposition $e\varphi e = p + u$ with $p^2 = p \in eRe$ and $u \in U(eRe)$. Also suppose we have a decomposition $e'(\varphi - \varphi u^{-1}\varphi)e' =
For a module $M_k$, and an element $\varphi \in \text{End}(M_k)$, let $\mathcal{E}_\varphi$ be the set of pairs $(W,e)$ where $W$ is $\varphi$-invariant submodule of $M$, $e$ is an idempotent on $W$, and $(\varphi - e)|_W$ is an automorphism of $W$. There is a natural ordering on $\mathcal{E}_\varphi$ given by $(W_1,e_1) \leq (W_2,e_2)$ if and only if $W_1 \subseteq W_2$ and $e_2|_{W_1} = e_1$. In [2] it is shown that for nonsingular, continuous modules any pair $(W,e) \in \mathcal{E}_\varphi$ can be extended to a pair $(M,\tilde{e}) \in \mathcal{E}_\varphi$. However, it is left unanswered whether the same property holds for arbitrary continuous modules. The answer is yes.

**Theorem 24.** Let $M_k$ be a continuous module, $\varphi \in S = \text{End}(M_k)$, and let $\mathcal{E}_\varphi$ be as above. If $(W,e) \in \mathcal{E}_\varphi$ then this pair extends to a pair $(M,\tilde{e}) \in \mathcal{E}_\varphi$.

**Proof.** By $(C_1)$ we have $W \subseteq e \subseteq N \subseteq \oplus M$. By quasi-continuity, all idempotents on submodules extend to summands, and so we can extend $e$ to $p$ on $N$. Fix $\pi^2 = \pi \in S$ with $\pi(M) = N$. Notice that $(\pi \varphi \pi - p)|_W = (\varphi - e)|_W$ is an automorphism of $W$. Therefore, one easily checks that $\pi \varphi \pi - p$ is an essential monomorphism on $\pi(M)$. But since $M$ is continuous we know all monomorphisms with essential image are isomorphisms, and so $\pi \varphi \pi - p$ is a unit on $\pi(M)$. Hence, we can write $\pi \varphi \pi = p + u$ where $u \in U(\pi S\pi)$. Now, as $(1 - \pi)M$ is continuous, $(1 - \pi)S(1 - \pi)$ is clean. Therefore, by Lemma 23 we have $\varphi = (p + q) + w$ where $q^2 = q \in (1 - \pi)S(1 - \pi)$ and $w \in U(S)$. Setting $\tilde{e} = p + q$ we see that $(M,\tilde{e})$ is the pair we set out to construct. □

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**References**

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