# DEDEKIND-FINITE STRONGLY CLEAN RINGS

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Abstract. In this paper we partially answer two open questions concerning clean rings. First, we demonstrate that if a quasi-continuous module is strongly clean then it is Dedekind-finite. Second, we prove a partial converse. We also prove that all clean decompositions on submodules of continuous modules extend to the entire module.

## 1. Introduction

W. K. Nicholson first introduced clean rings as a class of examples in his investigations on exchange rings [9]. Subsequently, these rings have become objects of interest in their own right. The relevant definitions are as follows:

**Definition 1.** Let $k$ and $R$ be rings. An element $r \in R$ is *clean* if we can write $r = u + e$, where $u \in U(R)$ is a unit and $e \in R$ is an idempotent. If we can pick $u$ and $e$ so that they commute, we say that $r$ is *strongly clean*. If all the elements of a ring are (strongly) clean, we say the ring is a (*strongly*) *clean ring*. If $M_k$ is a right $k$-module, with a (strongly) clean endomorphism ring, then we call $M_k$ a (*strongly*) *clean module*.

Nicholson in [10] asked the following five questions:

1. Does every strongly clean ring $R$ have stable range 1?
2. Is every strongly clean ring directly finite?
3. Is the property of being strongly clean a Morita invariant?
4. Is every unit regular ring strongly clean?
5. Is every semiperfect ring strongly clean?

Questions 3 and 5 were answered in the negative in [13], which showed that a direct sum of two strongly clean local modules need not be strongly clean. More examples of similar phenomena, and generalizations, are given in [1]. On the other hand, it was shown in [3] that if $R$ is a strongly clean ring and $e^2 = e \in R$ then $eRe$ is also strongly clean, which answers the other half of the Morita invariance problem in the positive.

In this paper we will show that a large class of unit regular rings always have a property implied by strongly clean rings, thus making progress on question 4. We also answer question 2 in the positive for endomorphism rings of quasi-continuous modules. We finish the paper by answering a question left open in [2] concerning lifting clean decompositions for continuous modules.

Throughout this paper, by "ring" we will mean an associative ring with 1. All modules are unital. If $M_k$ is a module, $\varphi \in \mathrm{End}(M_k)$, and $N \subseteq M$, one says that $N$ is *$\varphi$-invariant* if $\varphi(N) \subseteq N$. We direct the reader to [6], [7], and [8] for further information on many of the definitions and elementary concepts used herein.

## 2. Clean Decompositions

Let $\varphi \in S = \mathrm{End}(M_k)$. Recently, an alternate characterization of when $\varphi$ is clean was provided in [2]. Specifically, $\varphi$ is clean if and only if there are right $k$-module decompositions $M = A \oplus B = C \oplus D$

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such that $\varphi|_A$ maps $A$ isomorphically to $C$ and $(1-\varphi)|_B$ maps $B$ isomorphically to $D$. Pictorially

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & {\scriptstyle\varphi}\downarrow{\scriptstyle\sim} & & {\scriptstyle(1-\varphi)}\downarrow{\scriptstyle\sim} \\ M & = & C & \oplus & D. \end{array}$$

If such a diagram holds we call this an $ABCD$-decomposition for $\varphi$. It turns out by a result of Nicholson in [10] that $\varphi$ is strongly clean if and only if we can find an $ABCD$-decomposition where $A = C$ and $B = D$.

There are some interesting consequences to assuming $A = C$ or $B = D$. Suppose we have an $ABCD$-decomposition for $\varphi$. If $A = C$, this means that $A$ is $\varphi$-invariant. Further, $\varphi(A) = A$, and hence $\varphi^n(A) = A$ for each $n \geqslant 1$. This implies $A \subseteq \varphi^n(M)$ for each $n \geqslant 1$. So, if we define $I_\varphi = \bigcap_{n\geqslant 1} \varphi^n(M)$ then we have $A \subseteq I_\varphi$.

Similarly, if $B = D$ this means $B$ is $(1-\varphi)$-invariant, and hence $\varphi$-invariant. We claim that $\ker(\varphi) \subseteq B$. To prove this, let $x \in \ker(\varphi)$, and write $x = a + b$ with $a \in A$ and $b \in B$. We have $0 = \varphi(x) = \varphi(a) + \varphi(b)$, so $\varphi(a) = -\varphi(b) \in C \cap B = (0)$. But $\varphi$ is an isomorphism from $A$ to $C$, hence $a = 0$ and $x = b \in B$.

We further claim that $\ker(\varphi^m) \subseteq B$ for each $m \geqslant 1$. We proceed by induction, the base case being the substance of the previous paragraph. Suppose that for all $m < n$ we have $\ker(\varphi^m) \subseteq B$. Let $x \in \ker(\varphi^n)$, and write $x = a + b$ with $a \in A$ and $b \in B$. We have $\varphi(x) = \varphi(a) + \varphi(b)$ and so $\varphi(a) = \varphi(x) - \varphi(b) \in C \cap B = (0)$ (since $\varphi(x) \in \ker(\varphi^{n-1}) \subseteq B$). As before, this means $a = 0$, and so $x \in B$. Defining $K_\varphi = \bigcup_{n\geqslant 1} \ker(\varphi^n)$, we have $K_\varphi \subseteq B$.

By the above, and symmetry, we have just proven:

**Lemma 2.** *Let $M_k$ be a right $k$-module, $S = \mathrm{End}(M_k)$, $\varphi \in S$ a clean element, and $\varphi' = 1 - \varphi$. Fix an $ABCD$-decomposition for $\varphi$.*

1. *If $A = C$ in the decomposition then $K_{\varphi'} \subseteq A \subseteq I_\varphi$.*
2. *If $B = D$ in the decomposition then $K_\varphi \subseteq B \subseteq I_{\varphi'}$.*

*In particular, both of these conclusions hold for any strongly clean decomposition.*

When one of the two conditions in the Lemma holds, we have a condition stronger than "clean" but weaker than "strongly clean." We use the following terminology:

**Definition 3.** Let $\varphi \in S = \mathrm{End}(M_k)$. We say $\varphi$ is *left capably clean* if there is an $ABCD$-decomposition for $\varphi$ with $A = C$. Similarly, $\varphi$ is *right capably clean* if there is an $ABCD$-decomposition with $B = D$. For left $k$-modules, we switch these two definitions, for reasons to be explained later.

Just as in [2] and [10], we have alternate characterizations of these properties. For notational ease in the next proposition, and in the discussion following, we let $e' = 1 - e$ and $\varphi' = 1 - \varphi$.

**Proposition 4.** *Let $M_k$ be a right $k$-module, and $S = \mathrm{End}(M_k)$. The following are equivalent:*

1. *$\varphi \in S$ is left capably clean.*
2. *$\varphi' \in S$ is right capably clean.*
3. *$\varphi = u + e$ with $u \in U(S)$, $e^2 = e \in S$, $e\varphi e' = 0$, and $e'\varphi e' \in U(e'Se')$.*
4. *$\varphi' = -u + e'$ with $u \in U(S)$, $e^2 = e \in S$, $e\varphi' e' = 0$, and $e'\varphi e' \in U(e'Se')$.*

*Proof.* (1) $\Leftrightarrow$ (2): Clear by definition.

(1) $\Rightarrow$ (3): Suppose we have an $ABCD$-decomposition for $\varphi$ with $A = C$. Let $e \in S$ be an idempotent with $e'(M) = A$ and $e(M) = B$. We define $u$ to equal $\varphi$ on $A$, and to equal $-\varphi'$ on $B$. Pictorially

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & {\scriptstyle u}\big\downarrow\wr & & {\scriptstyle -u}\big\downarrow\wr \\ M & = & A & \oplus & D. \end{array}$$

So $u \in U(S)$. It is easy to check $\varphi = u + e$. Now $eue(a) = 0 = eu(a)$, and $eue(b) = eu(b)$, for $a \in A$ and $b \in B$. Therefore, $eue'(m) = 0$ on all $m \in M$ and hence $e\varphi e' = 0$. Finally, $e'\varphi e' = u|_A \in U(e'Se')$.

(3) $\Rightarrow$ (1): Write $\varphi = u + e$ with $u \in U(S)$, $e^2 = e \in S$, $e\varphi e' = 0$, and $e'\varphi e' \in U(e'Se')$. Set $A = e'(M)$, $B = e(M)$, $C = A$, and $D = \varphi'(B)$. Then one simply checks that this is an $ABCD$-decomposition for $\varphi$. Clearly $C = A$, and so $\varphi$ is left capably clean.

(3) $\Leftrightarrow$ (4): This follows either from the above work, or one sees directly that the two equations $\varphi = u+e$ and $e\varphi e' = 0$ give the same information as the two equations $\varphi' = -u+e'$ and $e\varphi' e' = 0$. $\square$

**Remark 5.** Consider the diagram

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & {\scriptstyle u}\big\downarrow\wr & & {\scriptstyle -u}\big\downarrow\wr \\ M & = & A & \oplus & D \end{array}$$

which was used in the proof above. Recall that the idempotent $e \in R$ was chosen so that $e(M) = B$ and $e(A) = 0$. Since $D$ is a direct sum complement to $A$ we have $e|_D : D \to B$ is an isomorphism. On the other hand, we defined $u$ so that $u|_B = -\varphi'|_B$, and in particular $\varphi' e|_B : B \to D$ is an isomorphism. Therefore, the map $e\varphi' e|_B : B \to B$ is an isomorphism. In other words, $e\varphi' e \in U(eSe)$.

There is another way to see this. Let $\varphi = u+e$ be a left capably clean decomposition. Then $eue' = 0$, and $e'ue' = e'\varphi e' \in U(e'Se')$. Also notice that $e\varphi' e = -eue$. Therefore, the Peirce decomposition for $u$ with respect to $e$ and $e' = 1 - e$ is

$$u = \begin{pmatrix} eue & eue' \\ e'ue & e'ue' \end{pmatrix} = \begin{pmatrix} -e\varphi' e & 0 \\ * & e'\varphi e' \end{pmatrix}.$$

Since $u$ is a unit, the matrix is lower-triangular, and the south-east corner is a unit, this implies that the north-west corner is a unit. Therefore, $e\varphi' e \in U(eSe)$.

We say a right $k$-module, $M_k$, is *left capably clean* if every endomorphism is left capably clean, while *right capably clean* modules are defined similarly. Let $\varphi \in S = \mathrm{End}(M)$ and assume $M$ is left capably clean. The equivalence (1) $\Leftrightarrow$ (2) above means $\varphi'$ is right capably clean. But since $\varphi'$ is an arbitrary element of $S$, this means $M_k$ is also right capably clean. Conversely, right capably clean modules are left capably clean, so we will drop the left and right and just call such modules capably clean.

We say a ring, $R$, is *capably clean* if $R_R$ is capably clean as a module. Proposition 4 then implies that a module $M_k$ is capably clean if and only if $S = \mathrm{End}(M_k)$ is capably clean; so capable cleanness is an endomorphism ring invariant. We say a ring element $r \in R$ is *left capably clean* if $r$ is left capably clean as an endomorphism of the module $R_R$; or equivalently, by Proposition 4, $r = u + e$ with $u \in U(R)$, $e^2 = e \in R$, $e're' \in U(e'Se')$, and $ere' = 0$. Right capably clean elements are defined analogously.

For a left $k$-module, ${}_kM$, recall that we switched the two definitions given in Definition 3. We do this so that $\varphi \in S = \mathrm{End}({}_kM)$ is left (resp. right) capably clean as an endomorphism of ${}_kM$ if and only if it is left (resp. right) capably clean as an endomorphism of ${}_SS$, if and only if it is left (resp. right) capably clean as an endomorphism of $S_S$. In this way, there is no confusion when we talk of left (or right) capably clean ring elements.

## 3. Examples and Counter-examples

**Example 6.** Let $k = \mathbb{Q}$, and let $M_k$ be a countably infinite-dimensional vector space over $k$, say with basis $\{m_1, m_2, \ldots\}$. Let $\varphi$ be the right shift operator, given by $\varphi(m_i) = m_{i+1}$. We know $\varphi$ is clean, and in fact $S = \mathrm{End}(M_k)$ is clean, by [11]. However, we will show that $\varphi$ is neither left nor right capably clean.

Notice that $I_\varphi = (0)$ and $I_{1-\varphi} = (0)$. So, in particular, $\varphi$ cannot possibly be strongly clean by Lemma 2. Suppose there is an $ABCD$-decomposition for $\varphi$ with $A = C$. Then $A \subseteq I_\varphi = (0)$ and so $M = B = D$, showing that $\varphi$ is strongly clean, a contradiction. We reach a similar contradiction if we suppose that $\varphi$ is right capably clean. Thus, $\varphi$ gives us an example of an element which is clean but neither left nor right capably clean. Further, $S = \mathrm{End}(M_k)$ gives an example of a clean ring which is not capably clean.

**Example 7.** Consider the matrix $\varphi = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z})$. By direct calculation one sees that

$$\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a left capably clean decomposition, while

$$\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix}$$

is a right capably clean decomposition. If we had an $ABAB$-decomposition for $\varphi$ then $A \oplus B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 1 \\ 3 \end{pmatrix} \mathbb{Z}$ since $A$ and $B$ would be spanned by eigenvectors of $\varphi$ and $1 - \varphi$. But then $A \oplus B \neq M$, a contradiction. Thus $\varphi$ is not strongly clean.

**Example 8.** A direct product ring $\prod_{i \in I} R_i$ is capably clean if and only if each $R_i$ is capably clean.

**Example 9.** Let $k = \mathbb{Q}$ and $M$ be a countably infinite-dimensional vector space. We write $M = V_1 \oplus V_2$ with $V_1 = \bigoplus_{i \in \mathbb{Z}} x_i \mathbb{Q}$ and $V_2 = \bigoplus_{1 \leqslant i \leqslant j} y_{i,j} \mathbb{Q}$. For later notation, we set $y_{i,j} = 0$ if $i \leqslant 0$. Also let $\pi$ be the projection to $V_1$ with kernel $V_2$.

We define $\varphi \in \mathrm{End}(M_k)$ by $\varphi(y_{i,j}) = y_{i-1,j}$ and $\varphi(x_i) = x_{i-1} + y_{i,i}$. Notice that $K_\varphi = V_2$. We claim that $I_{1-\varphi} \cap V_1 = (0)$. In fact, let $\alpha = \sum_{i=k_1}^{k_2} a_i x_i \in V_1$, where $a_{k_1} \neq 0$ and $a_{k_2} \neq 0$. Then an easy calculation shows $\pi(1-\varphi)^n \alpha = \sum_{i=k_1-n}^{k_2} b_i x_i$ where $b_{k_2} = a_{k_2}$ and $b_{k_1-n} = (-1)^n a_{k_1}$. Thus, $(1-\varphi)^n$ "spreads out" the smallest and largest non-zero supports in $\alpha$ by $n$. So, in particular, $\alpha \in \mathrm{im}((1-\varphi)^n)$ only if $k_2 - k_1 \geqslant n$. Taking $n$ large enough, $\alpha \notin \mathrm{im}((1-\varphi)^n) \supseteq I_{1-\varphi}$. This proves our claim, and it is easy to check by similar arguments that $V_2 = K_\varphi = I_{1-\varphi}$.

A straightforward computation demonstrates that $I_\varphi = (0)$. Thus, if we have an $ABCD$-decomposition with $A = C$ then $A = C = (0)$, and $B = D = M$. But as $\varphi$ is not an isomorphism, this is impossible. Hence $\varphi$ is not left capably clean. However, taking $A = V_1$, $C = \varphi(V_1)$, and $B = D = V_2$ yields a decomposition for $\varphi$ showing it is right capably clean.

**Remark 10.** Using the techniques of [12] it turns out that all capably clean modules have the countable exchange property. If a capably clean module has $(C_2)$ then it has the full exchange property.

**Remark 11.** If $M$ is strongly clean then any summand is strongly clean by [3]. This statement also holds true for capably clean modules. If fact, suppose $M = N \oplus N'$ is capably clean. Fix $\varphi \in \mathrm{End}(N)$ and extend $\varphi$ to an endomorphism $\psi$ on $M$ by setting $\psi(N') = (0)$. Since $M$ is capably clean there is an $ABCD$-decomposition with $B = D$ and in particular, by Lemma 2, $\ker(\psi) \subseteq B$. Write $\psi = u + e$, where this clean decomposition is with respect to the $ABCD$-decomposition above. Then $e(B) = B$ and in particular $eN' = N'$. Since $N'$ is $\psi$-invariant, this means it is also $u$-invariant. So, looking at $\mathrm{End}(M/N')$, we have a capably clean decomposition $\overline{\psi} = \overline{e} + \overline{u}$. But $M/N' \cong N$, and under this isomorphism there is a natural identification of $\overline{\psi}$ with $\varphi$. In particular, $\varphi$ is capably clean. So we see that $N$ is capably clean.

**Example 12.** While it is known that a finite direct sum of clean modules is clean, in [13] it was shown that the same result does not hold for strongly clean modules. In particular, they showed $\mathbb{Z}_{(2)}^2$ is not strongly clean (where $\mathbb{Z}_{(2)}$ is the ring of integers localized at the prime ideal $(2)$).

We will now show that $M_{\mathbb{Z}_{(p)}} = \mathbb{Z}_{(p)}^2$ is not even capably clean, for any prime $p$. Consider the matrix $\varphi = \begin{pmatrix} 1 & -p \\ 1 & 0 \end{pmatrix} \in S = \mathrm{End}(M)$. One easily sees that neither $\varphi$ nor $1-\varphi$ is a unit. So, given any $ABCD$-decomposition for $\varphi$, we cannot have $A = M$ or $B = M$. Therefore, we must have $A \cong Z_{(p)} \cong B$ (and of course $A \cong C$ and $B \cong D$).

Suppose that $\varphi$ is left capably clean so that $A = C$. Tensoring up to $\mathbb{Q}$ (which is flat over $\mathbb{Z}_{(p)}$) this means that $\varphi$ has an eigenvector. However, the discriminant of the characteristic polynomial of $\varphi$ is $1 - 4p < 0$ and so the characteristic polynomial does not split over $\mathbb{Q}$. This gives a contradiction. Therefore $\varphi$ cannot be left capably clean, and so $M$ is not capably clean.

Denoting the Jacobson radical by $J(S)$, also notice that $S/J(S) \cong \mathbb{M}_2(\mathbb{Z}/p\mathbb{Z})$ is strongly clean. So this example also shows that capable cleanness does not lift modulo the radical even if idempotents do.

We finish this section with an open question:

**Question:** Are there capably clean rings which are not strongly clean?

## 4. Continuous, Dedekind-finite Modules

We begin this section by introducing a few standard definitions and notations. We let $B \subseteq A$ be modules. If $B$ is a summand of $A$ we write $B \subseteq^{\oplus} A$. If $B$ is *essential* in $A$, meaning that for any non-zero submodule $C \subseteq A$ we have $C \cap B \neq (0)$, then we write $B \subseteq_e A$. A module $M_k$ is said to be *nonsingular* if for each $m \in M$, the annihilator $\mathrm{ann}_k(m) = \{r \in k \mid mr = 0\}$ is never essential in $k_k$. A ring $R$ is said to be *right nonsingular* if $R_R$ is nonsingular.

A module $M$ is said to be *Dedekind-finite* (or sometimes, *directly finite*) if $M \cong M \oplus X$ implies $X = (0)$. A ring is said to be *Dedekind-finite* if $xy = 1$ implies $yx = 1$; that is, all one-sided units are units. It is an easy exercise to prove that $M$ is Dedekind-finite as a module if and only if $\mathrm{End}(M)$ is Dedekind-finite as a ring. A ring $R$ is said to be *(von Neumann) regular* if for each $x \in R$ there exists $y \in R$ with $xyx = x$, and is said to be *unit regular* if further $y$ can be chosen to be a unit. Unit regular rings are always Dedekind-finite. These concepts are explored further in [5] and [6].

The following three conditions on a module $M$ are also standard:

($C_1$) If $A \subseteq M$ then there exists $N \subseteq^{\oplus} M$ with $A \subseteq_e N$.
($C_2$) If $A \subseteq M$ and $A \cong B \subseteq^{\oplus} M$ then $A \subseteq^{\oplus} M$.
($C_3$) If $A, B \subseteq^{\oplus} M$ and $A \cap B = (0)$ then $A \oplus B \subseteq^{\oplus} M$.

The property ($C_2$) implies ($C_3$) by [8, Proposition 2.2]. A module $M$ is said to be *(quasi-)continuous* if it satisfies conditions ($C_1$) and ($C_2$) (respectively, ($C_1$) and ($C_3$)). Each of the three properties above is inherited by summands, and so summands of (quasi-)continuous modules are (quasi-)continuous. We assume that the reader is familiar with the results from [8]. Continuous and quasi-continuous modules were developed as generalizations of injective modules. A ring is said to be *right continuous* if $R_R$ is continuous, and two-sided continuous if both $R_R$ and ${}_RR$ are continuous. In [2], it is proved that all continuous modules are clean. The proof begins by considering the case of a nonsingular continuous module $M$. We claim that if one also assumes $M$ is Dedekind-finite then $M$ is capably clean. But before we prove this result we need a few simple lemmas.

**Lemma 13.** *Let $M_k$ be a nonsingular module. Let $\varphi \in \mathrm{End}(M_k)$, and let $A'$ be a $\varphi$-invariant submodule of $M$. If we have $A' \subseteq_e A \subseteq^{\oplus} M$ then $A$ is $\varphi$-invariant.*

*Proof.* This is proven in [2], but for completeness we reproduce the argument here. Let $M = A \oplus X$. Given $a \in A$ write $\varphi(a) = a_1 + x$ with $a_1 \in A$ and $x \in X$. Consider the map $k_k \to A_k$ given by $r \mapsto ar$. Set $I = \{r \in k : ar \in A'\}$. Since $A' \subseteq_e A$ this means $I_k \subseteq_e k_k$. For each $r \in I$, $\varphi(ar) = a_1 r + xr$. Therefore, since $ar \in A'$ which is $\varphi$-invariant, $xr = \varphi(ar) - a_1 r \in X \cap A = (0)$. This means $I \subseteq \mathrm{ann}_k(x)$, which implies $\mathrm{ann}_k(x) \subseteq_e k_k$. By nonsingularity, $x = 0$. Thus $\varphi(a) = a_1 \in A$. □

**Lemma 14.** *Let $M_k$ be a module with $(C_2)$. If $\varphi \in \mathrm{End}(M_k)$ is a monomorphism with* $\mathrm{im}(\varphi) \subseteq_e M$ *then $\varphi$ is an isomorphism.*

*Proof.* We know $\mathrm{im}(\varphi) \cong M \subseteq^{\oplus} M$. So, from $(C_2)$ we have $\mathrm{im}(\varphi) \subseteq^{\oplus} M$. But the only essential summand of $M$ is $M$ itself. Thus $\mathrm{im}(\varphi) = M$, which yields surjectivity of $\varphi$. □

We are now ready to prove:

**Theorem 15.** *If $M_k$ is a nonsingular, Dedekind-finite, continuous module then $M$ is capably clean*

*Proof.* Let $\varphi \in \mathrm{End}(M_k)$. Consider the submodule $K_\varphi$. Since $M$ has $(C_1)$, there is some module $B$ such that $K_\varphi \subseteq_e B \subseteq^{\oplus} M$. Since $K_\varphi$ is $\varphi$-invariant, so is $B$ by Lemma 13. We can write $M = A \oplus B$. Let $C = \varphi(A)$. Then $\varphi|_A$ maps $A$ isomorphically to $C$ (since $\ker(\varphi) \subseteq K_\varphi \subseteq B$). Further, $B \cap C = (0)$ as $K_\varphi \cap C = (0)$ and $K_\varphi \subseteq_e B$. Moreover, $C$ is a direct summand of $M$ by $(C_2)$, and so $C \oplus B$ is a direct summand of $M$ by $(C_3)$. But $M = A \oplus B \cong C \oplus B$, and thus $C \oplus B = M$ as $M$ is Dedekind-finite.

Set $\varphi' = 1 - \varphi$. The only thing left to show is that $\varphi'|_B$ maps $B$ to itself isomorphically. By Lemma 13, since $K_\varphi$ is $\varphi'$-invariant, so is $B$. Also, since $\varphi'|_{K_\varphi}$ is an isomorphism we see that $\varphi'|_B$ is a monomorphism with essential image. Therefore, by the previous lemma, $\varphi'|_B$ is an isomorphism. □

**Example 16.** All Dedekind-finite (von Neumann) regular rings that are one-sided continuous are capably clean, using the previous theorem. This follows because nonsingularity is automatic for all regular rings [5, Corollary 7.7]. In particular, regular rings which are two-sided continuous are capably clean, as two-sided continuity forces Dedekind-finiteness [5, Corollary 6.49].

**Definition 17.** A ring $R$ is said to be an *exchange ring* if for each $a \in R$ there exists an idempotent $e^2 = e \in R$ with $a \in Re$ and $1 - a \in R(1 - e)$. A module $M$ is said to have the *finite exchange property* if $\mathrm{End}(M_k)$ is an exchange ring. It is well known that clean rings are always exchange rings, but not conversely.

**Lemma 18.** *Let $M_k$ be a quasi-continuous module with the finite exchange property. Set $S = \mathrm{End}(M_k)$ and let $J(S)$ denote the Jacobson radical of $S$. We can write $S/J(S) = S_1 \times S_2$ where $S_1$ is an Abelian (all idempotents lie in the center of the ring) reduced ring, which is clean, and $S_2$ is right-continuous and regular. Furthermore, if $M$ is nonsingular, then $S$ has such a product representation.*

*Proof.* This lemma is a combination of [8, Corollary 3.13] and the first remark immediately following it. □

With this result under our belts, we can extend the previous theorem to:

**Theorem 19.** *If $M_k$ is a nonsingular, Dedekind-finite, quasi-continuous module with the finite exchange property then $M$ is capably clean.*

*Proof.* We know that $S = \mathrm{End}(M) \cong S_1 \times S_2$ where $S_1$ is an Abelian exchange ring and $S_2$ is a right-continuous, regular (hence right nonsingular) ring by Lemma 18. Therefore $S_1$ is strongly clean, and $S_2$ is capably clean from our previous result. Thus $S$ is capably clean since the property in question passes to direct products. □

Nicholson has asked whether all unit-regular rings are strongly clean. One can view our previous result as a step towards answering this question, due to the following proposition. We thank T.Y. Lam and Dinesh Khurana for pointing out that we can add condition 3 below.

**Proposition 20.** *The following statements are equivalent:*

1. *Every nonsingular, Dedekind-finite, quasi-continuous module with the finite exchange property is strongly clean.*
2. *Every right continuous, unit-regular ring is strongly clean.*
3. *Every right-self-injective, unit-regular ring is strongly clean.*

*Proof.* The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1): Let $M$ be a nonsingular, Dedekind-finite, quasi-continuous module with the finite exchange property. By Lemma 18, $S = \mathrm{End}(M) \cong S_1 \times S_2$ where $S_1$ is an Abelian exchange ring (hence strongly clean) and $S_2$ is right-self-injective, regular ring. As $M$ is Dedekind-finite, so is $S_2$. Since $(S_2)_{S_2}$ is quasi-continuous and Dedekind-finite, [8, Theorem 2.33] implies $(S_2)_{S_2}$ has internal cancelation. Therefore, $S_2$ is a unit-regular ring. By hypothesis, $S_2$ is then strongly clean. Thus $S \cong S_1 \times S_2$ is strongly clean. □

We have the following result in the other direction, which answers Nicholson's question 2 in [10] for a large class of rings:

**Theorem 21.** *If a module is strongly clean and quasi-continuous then it is Dedekind-finite.*

*Proof.* Let $M_R$ be strongly clean and quasi-continuous. Letting $E = \mathrm{End}(M_R)$, we know that $M$ is Dedekind-finite if and only if $E/J(E)$ is Dedekind-finite (as a ring). By Lemma 18, $E/J \cong S_1 \times S_2$ with $S_1$ reduced (hence Dedekind-finite) and $S_2$ right injective and regular. Hence, replacing $M_R$ by $(S_2)_{S_2}$ if necessary, we may assume $M_R$ is injective (and nonsingular).

Suppose by way of contradiction $M \cong M \oplus X$ with $X \neq 0$. Then there is a submodule $N \subseteq M$ with $N = X_1 \oplus X_2 \oplus \cdots$ and $X_i \cong X$ for each $i \geqslant 1$. Let $f|_N$ be a shift operator sending $X_i$ isomorphically to $X_{i+1}$ for each $i \geqslant 1$. Since $M$ is injective, $N$ is essential in a summand $W \subseteq^{\oplus} M$. By injectivity, we can extend $f$ to an endomorphism on $W$. Also, since the property of being strongly clean passes to summands, we may as well assume $W = M$.

Since $M$ is strongly clean we have an $ABAB$-decomposition for $f$. Suppose for a moment $A \neq 0$, and fix $0 \neq a \in A$. Then, by essentiality, there is some $r \in R$ with $0 \neq ar \in N$. Write $ar = \sum_{i=1}^{n} x_i$ with $x_i \in X_i$. Since $f|_A$ is an isomorphism, and $A$ is $f$-invariant, there is some element $a' \in A$ with $f^{n+1}(a') = ar$. Clearly $a' \neq 0$. Again, by essentiality, there is some $s \in R$ with $0 \neq a's \in N$. But then $0 \neq f^{n+1}(a's) = ars$ lies both in $\bigoplus_{i=1}^{n} X_i$ and in $\bigoplus_{i=n+1}^{\infty} X_i$, which is a contradiction. Hence $A = 0$.

This means $1-f$ is an isomorphism on $M$. A similar argument as in the previous paragraph produces a contradiction, as follows. Fix $0 \neq x \in X_1$, and let $x' \in M$ be such that $(1-f)(x') = x$. Then there is some $t \in R$ with $0 \neq x't \in N$ so we can write $x't = \sum_{i=1}^{n'} x'_i$ with $x'_i \in X_i$ for each $i$, and $n'$ minimal. But then $xt = (1-f)(x't)$ has non-zero support in $X_{n'+1}$, a contradiction. Thus, by assuming $M$ is not Dedekind-finite we obtain a contradiction in all cases. □

**Remark 22.** This theorem remains true if we weaken "strongly clean" to "capably clean."

## 5. Extending the Results of [2]

The proof that a direct sum of clean modules is still clean contains some information that will be useful in answering a question left open in [2]. We collect that information here.

**Lemma 23.** *Let $R$ be a ring, $e^2 = e \in R$ and $e' = 1 - e$. Fix $\varphi \in R$. Suppose we have a decomposition $e\varphi e = p + u$ with $p^2 = p \in eRe$ and $u \in U(eRe)$. Also suppose we have a decomposition $e'(\varphi - \varphi u^{-1}\varphi)e' =$*

*$q + v$ with $q^2 = q \in e'Re'$ and $v \in U(e'Re')$. (Note that if $eRe$ and $e'Re'$ are clean then such decompositions must exist.) Then $\varphi = (p + q) + w$ where $w \in U(R)$, and $p + q$ is an idempotent.*

*Proof.* This is a restatement of the proof of the main result of [4]. □

For a module $M_k$, and an element $\varphi \in \mathrm{End}(M_k)$, let $\mathcal{E}_\varphi$ be the set of pairs $(W, e)$ where $W$ is $\varphi$-invariant submodule of $M$, $e$ is an idempotent on $W$, and $(\varphi - e)|_W$ is an automorphism of $W$. There is a natural ordering on $\mathcal{E}_\varphi$ given by $(W_1, e_1) \leqslant (W_2, e_2)$ if and only if $W_1 \subseteq W_2$ and $e_2|_{W_1} = e_1$. In [2] it is shown that for nonsingular, continuous modules any pair $(W, e) \in \mathcal{E}_\varphi$ can be extended to a pair $(M, \widetilde{e}) \in \mathcal{E}_\varphi$. However, it is left unanswered whether the same property holds for arbitrary continuous modules. The answer is yes.

**Theorem 24.** *Let $M_k$ be a continuous module, $\varphi \in S = \mathrm{End}(M_k)$, and let $\mathcal{E}_\varphi$ be as above. If $(W, e) \in \mathcal{E}_\varphi$ then this pair extends to a pair $(M, \widetilde{e}) \in \mathcal{E}_\varphi$.*

*Proof.* By $(C_1)$ we have $W \subseteq_e N \subseteq^\oplus M$. By quasi-continuity, all idempotents on submodules extend to summands, and so we can extend $e$ to $p$ on $N$. Fix $\pi^2 = \pi \in S$ with $\pi(M) = N$. Notice that $(\pi\varphi\pi - p)|_W = (\varphi - e)_W$ is an automorphism of $W$. Therefore, one easily checks that $\pi\varphi\pi - p$ is an essential monomorphism on $\pi(M)$. But since $M$ is continuous we know all monomorphisms with essential image are isomorphisms, and so $\pi\varphi\pi - p$ is a unit on $\pi(M)$. Hence, we can write $\pi\varphi\pi = p + u$ where $u \in U(\pi S \pi)$. Now, as $(1 - \pi)M$ is continuous, $(1 - \pi)S(1 - \pi)$ is clean. Therefore, by Lemma 23 we have $\varphi = (p + q) + w$ where $q^2 = q \in (1 - \pi)S(1 - \pi)$ and $w \in U(S)$. Setting $\widetilde{e} = p + q$ we see that $(M, \widetilde{e})$ is the pair we set out to construct. □

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