SUMS OF UNITS IN SELF-INJECTIVE RINGS

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Abstract. We prove that if no field of order less than $n+2$ is a homomorphic image of a right self-injective ring R, then for any element $a \in R$ and central units $u_1, u_2, \ldots, u_n \in U(R)$ there exists a unit $u \in U(R)$ such that $a + u_i u \in U(R)$ for each $i \geq 1$.

Many authors have studied rings whose elements can be written as sums of units. For instance see [2, 5, 6, 10–14]. For a quick survey of this field we refer the reader to [11]. In [11] a ring is called n-good if every element can be written as a sum of n units. A classical result of Wolfson [13] and Zelinsky [14] is that for a vector space V_D over a division ring D, the endomorphism ring $\mathrm{End}(V)$ is 2-good except when $\dim(V_D) = 1$ and D has two elements. As a generalization of the result of Wolfson and Zelinsky it is proved in [5] that a right self-injective ring is 2-good if it does not have a field of order two as its homomorphic image. Following [10], a ring R is called *twin-good* if for every $a \in R$ there exists a unit $u \in U(R)$ such that both $a+u$ and $a-u$ are units. In [10, Theorem 9] it is proved that a right self-injective ring is twin-good if no field of order less than 4 is its homomorphic image.

In view of the above results, it is natural to ask what happens if no field of order less than $n+2$ is a homomorphic image of a right self-injective ring R. To state our answer to this question concisely we introduce a new definition. For a positive integer $n \geq 1$ we say that an element $a \in R$ is *n-tuplet-good* if for any set $u_1, \ldots, u_n$ of central units in R, there exists a unit $u \in U(R)$ such that $a + u_i u \in U(R)$ for each $i \geq 1$. For example, every nilpotent element and every element in the Jacobson radical of a ring is *n-tuplet-good*, by taking $u = 1$. A ring will be called *n-tuplet-good* if every element is n-tuplet-good. For instance, a field is n-tuplet-good precisely when it has more than $n+1$ elements. Note that 1-tuplet-good rings are precisely the 2-good rings. It is clear that if $R/J(R)$ is n-tuplet-good, then so is R. Given an indexing set I and a direct product of rings $R = \prod_{i \in I} R_i$, the ring R is n-tuplet-good if and only if R_i is n-tuplet-good for each $i \in I$. We will tacitly use these facts below. In this terminology the following is our main result.

Theorem 1. *A right self-injective ring R is n-tuplet-good if no homomorphic image of R is a field of order less than $n+2$.*

By taking $n = 1$ we get [5, Theorem 1] which says that a right self-injective ring is 2-good if it does not have a field of order two as a homomorphic image. By taking $n = 2$ we get that a right self-injective ring is 2-tuplet good if no homomorphic image is a field of order less than 4. This, in view of Example 6 below, improves upon the main result [10, Theorem 9] where it is proved that such a ring is twin-good. Also, as a corollary we get that for a vector space V_D over a division ring D, the endomorphism ring $\mathrm{End}(V)$ is n-tuplet-good except when $\dim(V_D) = 1$ and D has less than $n+2$ elements.

To prove our main result, we will first prove two lemmas which are interesting in their own right. Recall that a ring R is said to be an *elementary divisor ring* if for any $A \in \mathbb{M}_n(R)$ there exist two units $P, Q \in \mathrm{GL}_n(R)$ such that PAQ is a diagonal matrix. The following result strengthens [4, Lemma1] and [10, Lemma 5].

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Lemma 2. *Let R be a ring and let $n \geq 1$ be a positive integer. Any diagonal matrix in $\mathbb{M}_{n+1}(R)$ is n-tuplet-good. In particular, if R is an elementary divisor ring, then $\mathbb{M}_{n+1}(R)$ is n-tuplet-good.*

Proof. Suppose $A = \mathrm{diag}(a_1, a_2, \ldots, a_{n+1})$ and suppose $U_1, U_2, \ldots, U_n$ are central units in $\mathbb{M}_{n+1}(R)$. We know that the center of $\mathbb{M}_{n+1}(R)$ consists of constant diagonal matrices whose diagonal entry is central in R (see [7, Exercise 1.9]), and so for each $i \geq 1$ there exists a central unit $u_i \in R$ such that $U_i = u_i I$. Let

$$U = \mathrm{diag}(-u_1^{-1}a_1, -u_2^{-1}a_2, \ldots, -u_n^{-1}a_n, 0) + E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} + E_{n,1},$$

where the $E_{i,j}$ are the standard matrix units.

We first claim that U is a unit. Indeed, it is of the form

$$U = \begin{pmatrix} * & 1 & & & \\ & * & 1 & & \\ & & \ddots & \ddots & \\ & & & * & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By performing suitable elementary row operations this matrix reduces to the identity matrix. A similar computation shows that $A + u_i U$ (for each $i \geq 1$) similarly reduces to the identity matrix, and is thus also a unit. □

In general, a diagonal matrix in $\mathbb{M}_n(R)$ may not be n-tuplet-good. For example it is easy to see that the diagonal matrix $A = \mathrm{diag}(2, 3) \in \mathbb{M}_2(\mathbb{Z})$ is not even twin-good. For completeness we include a proof. Let $A = \mathrm{diag}(2, 3) \in \mathbb{M}_2(\mathbb{Z})$ and $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$. Suppose both $A - U$ and $A + U$ are invertible. Then

$$\det(A - U) = 6 - 2d - 3a + \det(U) = \pm 1$$

and

$$\det(A + U) = 6 + 2d + 3a + \det(U) = \pm 1.$$

Adding the two equations we get that $12 + 2\det(U) \in \{0, 2, -2\}$. But as $\det(U) = \pm 1$, this is not possible.

It was proved by Kaplansky (see [4, Lemma 2]) that for any $A \in \mathbb{M}_n(R)$ with $n > 1$, and R any ring, there exists a unit U such that $A - U$ is a diagonal matrix. Further it was proved by Henriksen [4, Lemma 1] that every diagonal matrix is a sum of two units. Thus, it follows that every matrix in $\mathbb{M}_n(R)$, for $n > 1$, is a sum of three units [4, Theorem 3]. In view of Lemma 2 above and [4, Lemma 2] we get that for any $A \in \mathbb{M}_{n+1}(R)$, there exists a unit $U \in \mathbb{M}_{n+1}(R)$ such that $A - U$ is n-tuplet-good.

Henriksen [4, Corollary 7] has given an example of a ring such that for any $n > 1$ there exists a matrix $A \in \mathbb{M}_n(R)$ which is not a sum of two units. In a recent paper [12], Vámos and Wiegand studied the problem of writing matrices over Prüfer domains as sums of two units. In our next result, we give a class of rings over which every matrix is n-tuplet-good for every positive integer $n \geq 1$.

Lemma 3. *Let $n \geq 1$ be a positive integer. Suppose R is a ring for which any J-semisimple indecomposable factor ring is simple artinian. (In particular, this holds if R is an abelian exchange ring.) Then $S = \mathbb{M}_m(R)$ is n-tuplet-good if either $m > 1$ or no field of order less than $n+2$ is a homomorphic image of R.*

Proof. Suppose $S = \mathbb{M}_m(R)$, $A \in S$, and U_i are central units in S for $i \geq 1$. We may again write $U_i = u_i I$ where $u_i \in Z(R)$. For ideals I of R consider the following property:

(∗) $\exists\, U \in \mathrm{GL}_m(R/I)$ such that the image of $A + U_i U$ is a unit in $\mathbb{M}_m(R/I)$ for each $i \geq 1$.

We will work contrapositively, and so assume that $(*)$ does not hold for the ideal (0). We now apply a standard Zorn's Lemma argument (as in the proof of [5, Theorem 1]). Let $\mathcal{T}$ be the set of ideals $I \subseteq R$ for which $(*)$ fails, which we have assumed is nonempty. After partially ordering $\mathcal{T}$ by inclusion, it is straightforward to show that chains in $\mathcal{T}$ have upper bounds (given, as usual, by the union of the chain). Thus, we may fix an ideal $I_0 \in \mathcal{T}$ which is maximal with respect to the partial ordering. Replacing R by R/I_0, we may as well assume $I_0 = (0)$. In particular, every proper factor ring of R satisfies $(*)$.

If $\overline{R} = R/J(R)$ is a proper factor ring then there exists a unit $\overline{U} \in \mathrm{GL}_m(\overline{R})$ such that $\overline{A + U_i U}$ is a unit for each i. As units lift modulo the Jacobson radical, this would say that R also satisfies $(*)$, a contradiction. Thus $J(R) = 0$ and hence R is J-semisimple. Similarly, if $R = R_1 \times R_2$ is properly decomposable (i.e. $R_1, R_2 \neq 0$) then there exists a unit $V \in \mathrm{GL}_m(R_1) = \mathrm{GL}_m(R/R_2)$ and a unit $W \in \mathrm{GL}_m(R_2) = \mathrm{GL}_m(R/R_1)$ exhibiting $(*)$. Putting $U = (V, W) \in \mathrm{GL}_m(R_1 \times R_2)$, we see that R also satisfies $(*)$, a contradiction. Thus, R is indecomposable. By our assumption in the statement of the lemma, we have that R is simple artinian.

Thus, without loss of generality (increasing m if necessary) we may assume that R is a division ring. It now suffices to show the forward direction of the lemma holds in this case. If $m = 1$ then $R = S$. Furthermore, if R has more than $n + 1$ elements then we can fix an element $u \in R \setminus \{0, -u_1^{-1}A, -u_2^{-1}A, \dots, -u_n^{-1}A\}$. As R is a division ring and $u \neq 0$ we know u is a unit. Similarly, $A + u_i u \neq 0$ (for each i) are units, hence R satisfies $(*)$.

Now suppose $m > 1$. In this case, after multiplying on the left and right by units, we may take $A = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$. If we can prove the result for $m = 2$ and $m = 3$, then the result will follow for arbitrary $m > 1$.

First suppose that $m = 2$. If $A = 0$ or $A = E_{11}$, then we let $U = E_{12} + E_{21}$. If $A = I$ and $u_i^2 \neq -1$ for any i, then we let $U = E_{12} - E_{21}$. If $A = I$ and $u_i^2 = -1$ for some i, then pick v from $u_1, \dots, u_n$ such that $v \neq (u_i^2 + 1)/u_i$ for any i and let $U = -vE_{11} - E_{12} + E_{21}$. In each case U and $A + u_i U$ are units for all $i \geq 1$.

Lastly suppose that $m = 3$. If $A = 0$, $A = E_{11}$, or $A = E_{11} + E_{22}$, then we let $U = E_{13} + E_{21} + E_{32}$. If $A = I$ and $u_i^3 \neq -1$ for any i, then again let $U = E_{13} + E_{21} + E_{32}$. If $A = I$ and $u_i^3 = -1$ for some i, then pick v from $u_1, \dots, u_n$ such that $v \neq (u_i^3 + 1)/u_i$ for any i, and let $U = -vE_{11} + E_{13} + E_{21} + E_{32}$. In each case U is a unit, and $A + u_i U \in U(S)$ for each $i \geq 1$. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. By working modulo the Jacobson radical, we may assume that R is (von Neumann) regular and right self-injective. Using Type Theory as in the proof of [5, Theorem 1],

$$R \cong P \times S \times T,$$

where P is purely infinite, S is of type I_f and T is of type II_f (see [1, Chapter 10]). Also by Theorem 10.16, Proposition 10.28, and Theorem 10.24 in [1],

$$P \cong M_{n+1}(P), \;\; T \cong M_{n+1}(T'), \;\text{ and }\; S \cong \prod_{i=1}^{\infty} \mathbb{M}_i(S_i),$$

where T' is a corner ring of T, and each S_i is abelian. As regular, right self-injective rings are elementary divisor rings, it follows from Lemma 2 that P and T are n-tuplet-good. By Lemma 3, each term in the product defining S is n-tuplet-good. So R is n-tuplet-good. □

An idempotent e in a regular ring R is said to be abelian if eRe is an abelian ring. It is clear from the above proof that R is n-tuplet-good if $S_1 = 0$. This is precisely when $R/J(R)$ does not have an abelian central idempotent. Thus we have the following result which generalizes a classical result of Zelinsky [14] and Wolfson [13].

Corollary 4. *Let R be a right self-injective ring. If $R/J(R)$ does not have an abelian central idempotent, then R is n-tuplet-good for every positive integer n. In particular, for a vector space V_D over a division ring D, the endomorphism ring* $\mathrm{End}(V)$ *is n-tuplet-good except when* $\dim(V_D) = 1$ *and* $|D| < n + 2$.

Remark 5. Let R be any ring such that R/J is right self-injective for some ideal $J \subseteq J(R)$. If no homomorphic image of R is a field of order less than $n+2$, then by Theorem 1 we know that R/J, and thus R, is n-tuplet good. We list below three classes of such rings.

(1) Let M_Λ be a quasi-continuous module with the finite exchange property, let $R = \mathrm{End(M)}$ and let $J = \Delta = \{f \in R : \ker(f) \leq_e M\}$. By [8, Theorem 3.10] we have that R/Δ is right self-injective, and by [9, Lemma 11] we have $\Delta \subseteq J(R)$.
(2) Let M_Λ be a flat cotorsion module, let $R = \mathrm{End}(M)$, and let $J = J(R)$. Then R/J is right self-injective by [3, Corollary 14].
(3) Let M_Λ be a strongly invariant submodule of an algebraically compact module, let $R = \mathrm{End}(M)$, and let $J = J(R)$. Then R_R is algebraically compact [16, Theorem 11], and so R/J is right self-injective by [15, Theorem 9].

It is clear that any 2-tuplet-good ring is twin-good, but the converse is not true in general.

Example 6. Let $F = \{0, 1, u, 1 + u\}$ be a field with four elements and $R = \mathbb{M}_2(F[x])$. As $F[x]$ is an elementary divisor ring, we know that R is 2-good. But as the characteristic of R is 2, we know further that R is twin-good. We next show that R is not 2-tuplet-good. Let $A = xI$, $u_1 = u$, and $u_2 = 1 + u$. Note that $u_1u_2 = 1$. If A is 2-tuplet-good, then there exists a unit $U = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in R$ such that $u_iA + U$ is a unit for $i = 1, 2$. Then

$$\det(u_iA + U) = u_i^2x^2 + u_ix(f_1 + f_4) + f_1f_4 + f_2f_3 \in F^*.$$

But as U is a unit, $\det(U) = f_1f_4+f_2f_3 \in F^*$. Thus, $u_i^2x^2+u_ix(f_1+f_4) \in F$, and hence $u_ix+f_1+f_4 = 0$. This implies that $u_1x = u_2x$. But as $u_2 = 1 + u_1$, we get $x = 0$, a contradiction.

We finally note that the converse of the main theorem is not true. Let σ denote the non-trivial automorphism of $\mathbb{F}_4$. The skew-polynomial ring $R = \mathbb{F}_4[x; \sigma]$ is left and right self-injective, is n-tuplet good for all $n \geq 1$, but does have $\mathbb{F}_4$ as a homomorphic image.

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