RADICALS IN SKEW POLYNOMIAL AND SKEW LAURENT POLYNOMIAL RINGS

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Abstract. We provide a general procedure for characterizing radical-like functions of skew polynomial and skew Laurent polynomial rings under grading hypotheses. In particular, we are able to completely characterize the Wedderburn and Levitzki radicals of skew polynomial and skew Laurent polynomial rings in terms of ideals in the coefficient ring. We also introduce the T-nilpotent radideals, and perform similar characterizations.

Introduction

Pearson and Stephenson [19] characterized the prime radical of a skew polynomial ring as $P(R[x;\sigma]) = (P(R)\cap P_\sigma(R))+P_\sigma(R)xR[x;\sigma]$ where $P_\sigma(R)$ is the so-called *σ-prime radical* of R, which is a σ-invariant ideal of R. Similarly, Bedi and Ram [1] characterized the Jacobson radical of a skew polynomial ring as $J(R[x;\sigma]) = (J(R)\cap J_\sigma(R)) + J_\sigma(R)xR[x;\sigma]$ where $J_\sigma(R)$ is a σ-invariant ideal of R. (This notation should not be confused with that utilized in [20].) Similar formulas hold for a great number of other radicals and radical-like constructions. See, for example, the papers [7, 8, 18].

In this paper we find that the similarities among these formulas arise, primarily, due to the fact that these radicals preserve $\mathbb{N}$-grading (in the skew polynomial case) and $\mathbb{Z}$-grading (in the skew Laurent polynomial case). In specific situations, we can do even better, by giving element-wise characterizations of these σ-skewed radicals. This is especially true when working with nilpotence properties. In particular we completely characterize the Levitzki radical and Wedderburn radical of a skew (Laurent) polynomial ring in terms of conditions in the base ring. We also introduce new radical-like ideals connected to T-nilpotence, and similarly characterize these ideals over skew (Laurent) polynomial rings.

Throughout, all rings are assumed to be associative with 1. We let R denote an arbitrary ring, and let σ be any automorphism of R. By $R[x;\sigma]$ we mean the skew polynomial ring over R, subject to the (left) skewing condition $xr = \sigma(r)x$ for each $r \in R$. The ring of skew Laurent polynomials will be written as $R[x, x^{-1};\sigma]$. If I is an ideal in R, we will write $I \trianglelefteq R$. For other standard notations or definitions, we refer the reader to [15].

There is one graph theoretic tool that we will use throughout the paper, called *König's tree lemma*, which we now recall.

König's tree lemma. *If G is a connected graph with infinitely many vertices and every vertex is connected to only finitely many other vertices, then there is an infinite path with no repeated vertices.*

We will apply this result with regards to zero products. To give an example, consider the collection of *nonzero* products of the form $a_1a_2\cdots a_n$ with each a_i coming from some subset S of R. We may describe this product as the ordered n-tuple $(a_1, a_2, \ldots, a_n)$. We think of these tuples as vertices of a graph, with $(a_1, a_2, \ldots, a_{i-1})$ connected by an edge to $(a_1, a_2, \ldots, a_i)$. In particular, this graph is connected (through the empty tuple). There is an infinite path in this graph if and only if we can form a sequence of elements $a_1, a_2, \ldots$ from S with $a_1a_2\cdots a_m \neq 0$ for each $m \geq 1$.

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There are two ways we apply König's tree lemma. First, if each a_i comes from a finite set S_i (possibly even depending on $a_1, a_2, \ldots, a_{i-1}$), then each vertex is connected to only finite many others. If we further have some hypothesis on the sets S_i that prevents the existence of infinite paths, then by the contrapositive of the lemma we can conclude there is an absolute bound on the length of nonzero products from S. The second way we apply the lemma is to assume (perhaps by way of contradiction) that the products under consideration can get arbitrarily long (and are finitely connected); so we can reduce to a single infinite sequence of nonzero products.

1. Definition and notation chart

In this paper we use quite a few new definitions and notations. For the convenience of the reader, we include a brief chart which collects the notations and where they are introduced in the paper.

Table 1: List of Definitions and Notations

Name	**Notation**	**Place Introduced**	**Original Source**
ideal function	$\mathfrak{F}$	Before Lemma 2.1	New
σ-Laurent $\mathfrak{F}$-function	$\mathfrak{F}_{\sigma,\sigma^{-1}}$	Definition 2.2	New
Jacobson radical	$\mathfrak{J}(R)$	Example 2.3	Classical
σ-Laurent Jacobson radical	$\mathfrak{J}_{\sigma,\sigma^{-1}}(R)$	Example 2.3	New
upper nilradical	$\mathfrak{N}(R)$	Example 2.4	Classical
σ-Laurent upper nilradical	$\mathfrak{N}_{\sigma,\sigma^{-1}}(R)$	Example 2.4	New
bounded nilradical	$\mathfrak{B}(R)$	Example 2.5	Classical
σ-Laurent bounded nilradical	$\mathfrak{B}_{\sigma,\sigma^{-1}}(R)$	Example 2.5	New
prime radical	$\mathfrak{P}(R)$	Example 2.6	Classical
σ-Laurent prime radical	$\mathfrak{P}_{\sigma,\sigma^{-1}}(R)$	Example 2.6	New
σ-prime radical	$P_\sigma(R)$	Example 2.6	[19]
strongly σ-prime ideal		Example 2.6	[19]
Wedderburn radical	$\mathfrak{W}(R)$	Example 2.8	Classical
σ-Laurent Wedderburn radical	$\mathfrak{W}_{\sigma,\sigma^{-1}}(R)$	Example 2.8	New
σ-Wedderburn radical	$W_\sigma(R)$	Example 2.8	[7]
σ-nilpotent set		Example 2.8	[13]
totally σ-nilpotent set		Before Theorem 2.11	New
Levitzki radical	$\mathfrak{L}(R)$	Example 2.12	Classical
σ-Laurent Levitzki radical	$\mathfrak{L}_{\sigma,\sigma^{-1}}(R)$	Example 2.12	New
difference bounded sequences		Before Theorem 2.13	New
σ-Levitzki radical	$L_\sigma(R)$	Equation 2.15	[7]
locally σ-nilpotent set		After Equation 2.15	[7]
σ-polynomial $\mathfrak{F}$-function	$\mathfrak{F}_\sigma$	Definition 3.3	New
σ-polynomial Jacobson radical	$\mathfrak{J}_\sigma(R)$	Example 3.4	New
σ-polynomial upper nilradical	$\mathfrak{N}_\sigma(R)$	Example 3.5	New
σ-polynomial bounded nilradical	$\mathfrak{B}_\sigma(R)$	Example 3.7	New
σ-polynomial prime radical	$\mathfrak{P}_\sigma(R)$	Example 3.8	New
σ-polynomial Wedderburn radical	$\mathfrak{W}_\sigma(R)$	Example 3.10	New
increasingly σ-nilpotent		Before Theorem 3.11	New
σ-polynomial Levitzki radical	$\mathfrak{L}_\sigma(R)$	Example 3.14	New
(left, right) T-nilpotent set		Definition 4.1	Classical
left and right T-nilpotent radideal	$\mathfrak{T}_\ell(R), \mathfrak{T}_r(R)$	Definition 4.4	New
higher Wedderburn radicals	$\mathfrak{W}^{(\alpha)}(R)$	Before Theorem 4.8	Classical

Continued on next page

Table 1 – continued from previous page

Name	Notation	Place Introduced	Original Source
σ-Laurent left T-nilpotent radideal	$\mathfrak{T}_{\ell,\sigma,\sigma^{-1}}(R)$	Theorem 4.12	New
σ-polynomial left T-nilpotent radideal	$\mathfrak{T}_{\ell,\sigma}(R)$	Theorem 4.14	New
higher left T-nilpotent radideals	$\mathfrak{T}_{\ell}^{(\alpha)}(R)$	Before Question 5.2	New

2. Radicals of skew Laurent polynomial rings

Throughout, $\mathfrak{F}$ will be a function which sends a ring to an ideal in that ring, i.e. $\mathfrak{F}(R) \leq R$. We call such maps *ideal functions*. Typical examples of such functions include the usual radicals, such as the Jacobson radical or the prime radical. We begin with the following straightforward, but enlightening, exercise.

Lemma 2.1. *Assume $\mathfrak{F}$ evaluates to a homogeneous ideal on $\mathbb{Z}$-graded rings. We have an equality $\mathfrak{F}(R[x,x^{-1};\sigma]) = I[x,x^{-1};\sigma]$ where $I \leq R$ is σ-invariant.*

Proof. The ring $R[x,x^{-1};\sigma]$ is a $\mathbb{Z}$-graded ring, graded by powers of the variable x. Thus we can write $J := \mathfrak{F}(R[x,x^{-1};\sigma]) = \sum_{n\in\mathbb{Z}} I_n x^n$ where $I_n \subseteq R$. As $J \leq R[x,x^{-1};\sigma]$, we know that J is closed under multiplication on the right by both x and x^{-1}. Thus $I_m = I_n$ for each $m,n \in \mathbb{Z}$. It is straightforward to show that I_0 is closed under addition, and multiplication from R on the left and the right. Thus, we take $I = I_0$. The σ-invariance follows from the fact that $\mathfrak{F}(R[x,x^{-1};\sigma])$ is an ideal, hence closed under left multiplication by integer powers of x. □

Definition 2.2. Suppose we are given an ideal function $\mathfrak{F}$ which evaluates to a homogeneous ideal on any $\mathbb{Z}$-graded ring, so that we can write $\mathfrak{F}(R[x,x^{-1};\sigma]) = I[x,x^{-1};\sigma]$ for some ideal $I \leq R$. We let $\mathfrak{F}_{\sigma,\sigma^{-1}}$ be the ideal function satisfying

$$\mathfrak{F}_{\sigma,\sigma^{-1}}(R) := \{a \in R \ : \ a \in \mathfrak{F}(R[x,x^{-1};\sigma])\} = I,$$

which we call the *σ-Laurent $\mathfrak{F}$-function*.

We can now characterize numerous radicals of skew Laurent polynomial rings, often clarifying or generalizing the literature in the process.

Example 2.3. Suppose $\mathfrak{J}(R)$ is the Jacobson radical. We know this is a homogeneous ideal in any $\mathbb{Z}$-graded ring, by a theorem of Bergman (see [16, Exercise 5.8]). The behavior of the Jacobson radical under more general (semigroup) gradings has been studied in a large number of papers. For a taste see [4, 9, 10]. Our lemma above recaptures the second main result of [1]. At present it is difficult to describe $\mathfrak{J}_{\sigma,\sigma^{-1}}(R)$ solely in terms of conditions in R, even when $\sigma = \mathrm{id}$.

Example 2.4. Suppose $\mathfrak{N}(R)$ is the upper nilradical. By a recent result of Smoktunowicz [22], this radical is $\mathbb{Z}$-graded when R is, and thus our lemma applies here as well, proving the existence of $\mathfrak{N}_{\sigma,\sigma^{-1}}(R)$.

The same is true of the Brown-McCoy radical, even when graded by a free group, according to [12].

Example 2.5. Suppose $\mathfrak{B}(R) = \{a \in R \ : \ aR \text{ is nil of bounded index}\}$, which is called the bounded nilradical although it is technically not a radical. This is an ideal due to a theorem of Amitsur, see [21, Theorem 2.6.27] for the clever proof. By [18, Theorem 6] this ideal is homogeneous in any $\mathbb{Z}$-graded ring. One can describe $\mathfrak{B}_{\sigma,\sigma^{-1}}(R)$ explicitly as an ideal satisfying infinitely many identities over R (see [18, Theorem 9]).

Example 2.6. Let $\mathfrak{P}(R)$ be the prime radical (also called the lower nilradical). This is a homogeneous ideal for $\mathbb{Z}$-gradings, but more generally for gradings by u.p. monoids [11, Corollary 1.3]. According to [3, Theorem 3.11], we have

$$\mathfrak{P}_{\sigma,\sigma^{-1}}(R) = \mathfrak{P}(R) \cap P_\sigma(R) \cap P_{\sigma^{-1}}(R), \tag{2.7}$$

where $P_\sigma(R)$ is the σ-prime radical of Pearson and Stephenson [19]. We recall that technical definition now. The σ-prime radical is the intersection

$$P_\sigma(R) = \bigcap \{I \ : \ I \text{ is a strongly } \sigma\text{-prime ideal of } R\}$$

where an ideal $I \leq R$ is *strongly σ-prime* if $I \neq R$, $\sigma(I) = I$, and whenever there are two ideals $A, B \leq R$ with $AB \subseteq I$ and $\sigma(B) \subseteq B$ then either $A \subseteq I$ or $B \subseteq I$. Other characterizations of the prime radical in connection with general endomorphisms, for Ore skew-differential polynomial extensions, are given in [17].

Example 2.8. Let

$$\mathfrak{W}(R) = \{a \in R \ : \ aR \text{ is nilpotent}\} \tag{2.9}$$

be the Wedderburn radical (which is, again, not properly a radical). It is straightforward to show that this ideal is homogeneous in a $\mathbb{Z}$-grading, and we have $\mathfrak{W}_{\sigma,\sigma^{-1}}(R) = W_\sigma(R)$ where

$$W_\sigma(R) = \sum \{I \leq R \ : \ I \text{ is a } \sigma\text{-nilpotent } \sigma\text{-ideal of } R\} \tag{2.10}$$

is the so-called σ-Wedderburn radical of [7, Theorem 3.3]. The notion of σ-nilpotence was introduced in [13]; a subset $S \subseteq R$ is σ-nilpotent if for each integer $\ell \geq 1$ there exists an integer $k \geq 1$ (possibly depending on ℓ) so that $S\sigma^{\ell}(S)\sigma^{2\ell}(S)\cdots\sigma^{k\ell}(S) = 0$.

We can provide a self-contained alternate characterization of $\mathfrak{W}_{\sigma,\sigma^{-1}}(R)$. Call a subset $S \subseteq R$ *totally σ-nilpotent* if there exists an integer $k \geq 1$ so that for every sequence of integers $n_1, n_2, \ldots \in \mathbb{Z}$ we have $S\sigma^{n_1}(S)\sigma^{n_2}(S)\cdots\sigma^{n_k}(S) = 0$. We call k the *index* of total σ-nilpotence.

Theorem 2.11. *$\mathfrak{W}_{\sigma,\sigma^{-1}}(R) = \{a \in R \ : \ aR$ is totally σ-nilpotent$\}$.*

Proof. $\subseteq$: Fix $a \in \mathfrak{W}_{\sigma,\sigma^{-1}}(R)$ so $aR[x, x^{-1}; \sigma]$ is a nilpotent right ideal, say of index $k \geq 1$. Fix a sequence of integers $n_1, n_2, \ldots \in \mathbb{Z}$ and fix arbitrary elements $r_0, r_1, \ldots \in R$. We have

$$ar_0x^{n_1}ar_1x^{n_2-n_1}ar_2x^{n_3-n_2}\cdots x^{n_k-n_{k-1}}ar_k = 0.$$

Hence $ar_0\sigma^{n_1}(ar_1)\cdots\sigma^{n_k}(ar_k) = 0$. This exhibits the fact that aR is totally σ-nilpotent.

$\supseteq$: Fix $a \in R$ such that aR is totally σ-nilpotent, say of index $k \geq 1$. Fix $f_1, f_2, \ldots \in R[x, x^{-1}; \sigma]$. We will prove that $af_1af_2\cdots af_kaf_{k+1} = 0$, thus demonstrating that $a \in \mathfrak{W}_{\sigma,\sigma^{-1}}(R)$. If we expand the product, each term looks like

$$ar_1x^{n_1}ar_2x^{n_2}\cdots ar_{k+1}x^{n_{k+1}}$$

where $r_ix^{n_i}$ is an arbitrary summand from f_i. Moving all powers of x to the right, the product equals

$$\left[ar_1\sigma^{n_1}(ar_2)\sigma^{n_1+n_2}(ar_3)\cdots\sigma^{n_1+n_2+\cdots+n_k}(ar_{k+1})\right]x^{n_1+n_2+\cdots+n_{k+1}}.$$

The quantity in the square brackets is zero, by the assumption of total σ-nilpotence, proving the claim. □

Example 2.12. Recall that a set $S \subseteq R$ is said to be locally nilpotent if every finite subset of S is nilpotent. The Levitzki radical $\mathfrak{L}(R)$ is the sum of all locally nilpotent ideals of R. It is known that $\mathfrak{L}(R)$ is homogeneous for u.p. monoid-gradings by [11, Theorem 2.2], and thus $\mathfrak{L}_{\sigma,\sigma^{-1}}$ exists.

Similar to the work done above, we can describe $\mathfrak{L}_{\sigma,\sigma^{-1}}(R)$ explicitly in terms of conditions on elements in R. We call a sequence of integers $n_1, n_2, \ldots \in \mathbb{Z}$ *difference bounded* if there exists an integer $N \geq 1$ such that $|n_i - n_{i-1}| < N$ for all $i \geq 1$. (Here, and throughout the paper, $n_0 = 0$.)

Theorem 2.13. *$\mathfrak{L}_{\sigma,\sigma^{-1}}(R)$ consists of the elements $a \in R$ with the property that for every finite subset $S \subseteq aR$ and every integer $N \geq 1$ there exists an integer $k \geq 1$ so that for every difference bounded sequence $n_1, n_2, \ldots \in \mathbb{Z}$ with differences bounded by N, we have $S\sigma^{n_1}(S)\sigma^{n_2}(S)\cdots\sigma^{n_k}(S) = 0$.*

Proof. $\subseteq$: Fix $a \in \mathfrak{L}_{\sigma,\sigma^{-1}}(R)$. Given an integer $N \geq 1$ and a finite set $T = \{r_1, \ldots, r_\ell\} \subseteq R$, we have that $T' = \{ar_i x^j\}_{1\leq i\leq \ell,\ -N\leq j\leq N}$ is nilpotent, say of index $k \geq 1$. Fix a difference bounded sequence $n_1, n_2, \ldots \in \mathbb{Z}$ with differences bounded by $N \geq 1$. Writing $m_i = n_i - n_{i-1}$ we thus have

$$ar_{i_0}x^{m_1}ar_{i_1}x^{m_2}\cdots ar_{i_k}x^{m_{k+1}} \in (T')^{k+1} = 0$$

where $r_{i_0}, r_{i_1}, \ldots, r_{i_k}$ are arbitrary elements of T. Hence,

$$ar_{i_0}\sigma^{n_1}(ar_{i_1})\sigma^{n_2}(ar_{i_2})\cdots\sigma^{n_k}(ar_{i_k}) = 0.$$

As $ar_{i_0}, \ldots, ar_{i_k}$ are arbitrary elements in aT, and as $S = aT = \{ar_1, \ldots, ar_\ell\}$ is an arbitrary finite subset of aR, this demonstrates that a has the necessary property.

$\supseteq$: Suppose $a \in R$ has the property given in the statement of the proposition; we wish to prove that $a \in \mathfrak{L}_{\sigma,\sigma^{-1}}(R)$. To that end, fix a finite collection of elements $f_1, f_2, \ldots, f_\ell \in R[x, x^{-1}; \sigma]$. We need to show that $S' = \{af_i\}_{1\leq i\leq \ell}$ is nilpotent. Without loss of generality, expanding our set if necessary (by using each of the homogeneous components of each of the f_i), we can assume that $f_i = r_i x^{m_i}$.

Let $N = \max_{1\leq i\leq \ell}|m_i| + 1$ and let $S = \{ar_1, ar_2, \ldots, ar_\ell\} \subseteq aR$. By assumption, there exists some integer $k \geq 1$ so that for every difference bounded sequence $n_1, n_2, \ldots \in \mathbb{Z}$ with differences bounded by N, we have $S\sigma^{n_1}(S)\sigma^{n_2}(S)\cdots\sigma^{n_k}(S) = 0$. Any $(k+1)$-fold product from S' looks like

$$\begin{aligned} af_{i_0}af_{i_1}\cdots af_{i_k} &= ar_{i_0}x^{m_{i_0}}ar_{i_1}x^{m_{i_1}}\cdots ar_{i_k}x^{m_{i_k}} \\ &= \left[ar_{i_0}\sigma^{m_{i_0}}(ar_{i_1})\sigma^{m_{i_0}+m_{i_1}}(ar_{i_2})\cdots\sigma^{m_{i_0}+m_{i_1}+\cdots m_{i_{k-1}}}(ar_{i_k})\right]x^{m_{i_0}+m_{i_1}+\cdots+m_{i_k}} \\ &\in S\sigma^{n_1}(S)\sigma^{n_2}(S)\cdots\sigma^{n_k}(S)x^*, \end{aligned}$$

where each $n_j = \sum_{p=0}^{j-1} m_{i_p}$. As the sequence $n_1, n_2, \ldots$ is difference bounded by N, the product above is zero. Thus S' is nilpotent, of index at most $k+1$. □

We can weaken the statement of the previous theorem, thus giving us an easier criterion for proving containment in the σ-Laurent $\mathfrak{L}$-function.

Corollary 2.14. *$\mathfrak{L}_{\sigma,\sigma^{-1}}(R)$ consists of the elements $a \in R$ with the property that for every finite subset $S \subseteq aR$ and every difference bounded sequence $n_1, n_2, \ldots \in \mathbb{Z}$ there exists an integer $k \geq 1$ so that $S\sigma^{n_1}(S)\sigma^{n_2}(S)\cdots\sigma^{n_k}(S) = 0$.*

Proof. $\subseteq$: The condition given in the statement of this corollary is a tautological weakening of the statement given in Theorem 2.13. (The integer k here now depends not only on S and N, but on the specific difference bounded sequence.) Thus this containment is trivial.

$\supseteq$: Fix an integer $N \geq 1$ and a finite subset $S \subseteq aR$. If a satisfies the condition given in the statement of the corollary, then for each difference bounded sequence $n_1, n_2, \ldots \in \mathbb{Z}$ (say, bounded by N) there is an integer $k \geq 1$ for which $S\sigma^{n_1}(S)\cdots\sigma^{n_k}(S) = 0$. As sequences whose differences are bounded by N have only finitely many choices for each entry, König's tree lemma says that k may be chosen independently of the sequence (but not N), thus finishing the proof. □

Following [7, Lemma 3.9], one defines the σ-Levitzki radical as

$$L_\sigma(R) = \{a \in R\ :\ \sum_{i=0}^{\infty}\sigma^i(a)R \text{ is locally } \sigma\text{-nilpotent}\} \tag{2.15}$$

where "locally σ-nilpotent" means that for every finite subset $S \subseteq \sum_{i=0}^{\infty}\sigma^i(a)R$ and every integer $\ell \geq 1$, there exists an integer $m \geq 1$ for which $S\sigma^\ell(S)\sigma^{2\ell}(S)\cdots\sigma^{m\ell}(S) = 0$. We will see later that this ideal has more to do with skew polynomial rings than skew Laurent polynomial rings. However, we do have the containment $\mathfrak{L}_{\sigma,\sigma^{-1}}(R) \subseteq \mathfrak{L}(R) \cap L_\sigma(R) \cap L_{\sigma^{-1}}(R)$; indeed, by [7, Theorem 3.11] we find $\mathfrak{L}_{\sigma,\sigma^{-1}}(R) \subseteq \mathfrak{L}(R) \cap L_\sigma(R)$, and the rest of the containment follows by symmetry considerations. We do not know if the reverse containment holds, so we ask:

Question 2.16. Is it the case that $\mathfrak{L}_{\sigma,\sigma^{-1}}(R) = \mathfrak{L}(R) \cap L_\sigma(R) \cap L_{\sigma^{-1}}(R)$?

3. Radicals of skew polynomial rings

We now want to prove an analogue of Lemma 2.1 for skew polynomial rings. The statement of this result is complicated by the fact that we no longer can multiply by x^{-1}.

Lemma 3.1. *Assume $\mathfrak{F}$ evaluates to a homogeneous ideal on $\mathbb{N}$-graded rings. We have an equality $\mathfrak{F}(R[x;\sigma]) = I_0 + I_1x + I_2x^2 + \cdots$ where $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ are ideals in R.*

Proof. Let $S = R[x;\sigma]$. From homogeneity, we can write $\mathfrak{F}(S) = I_0 + I_1x + I_2x^2 + \cdots$ where each I_n is a subset of R. As $\mathfrak{F}(S)$ is closed under multiplication by x on the right, we have the containments $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$. Since $\mathfrak{F}(S)$ is an ideal, each I_n is closed under addition and left multiplication from R. Finally, given $r \in R$ and $a \in I_n$ we have $ar \in I_n$ since $ax^n\sigma^{-n}(r) = arx^n$. (Here we are using the fact that σ is an automorphism.) □

Lemma 3.2. *Assume $\mathfrak{F}$ evaluates to a homogeneous ideal on $\mathbb{N}$-graded rings. Also assume the following conditions for any ring R:*

- (i) *There is some property $\mathcal{P}$ on subsets of R, such that $\mathfrak{F}(R) = \{a \in R \,:\, aR$ satisfies $\mathcal{P}\}$ and:*
  - (a) *If I satisfies $\mathcal{P}$ and $J \subseteq I$, then J satisfies $\mathcal{P}$.*
  - (b) *If $(aR)^n$ has $\mathcal{P}$ for some $n \geq 1$, then aR has $\mathcal{P}$.*
- (ii) *If R is $\mathbb{N}$-graded with grade-0 component R_0 and positive grade R_+, and if $a \in R_0$, then the following two implications hold:*
  - (a) *If $a \in \mathfrak{F}(R_0)$ and $aR_+ \subseteq \mathfrak{F}(R)$, then $a \in \mathfrak{F}(R)$.*
  - (b) *If $a \in \mathfrak{F}(R)$, then $a \in \mathfrak{F}(R_0)$.*

We then have $\mathfrak{F}(R[x;\sigma]) = (\mathfrak{F}(R) \cap I) + IxR[x;\sigma]$ for some σ-stable ideal $I \leq R$.

Proof. As in the previous lemma we write $S = R[x;\sigma]$ and we have $\mathfrak{F}(S) = I_0 + I_1x + I_2x^2 + \cdots$ where $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ are ideals in R. Given $a \in I_n$ for some $n \geq 1$, we see that ax^nS satisfies $\mathcal{P}$. Thus $J = (axS)^n \subseteq ax^nS$ satisfies $\mathcal{P}$ by condition (i) part (a). By condition (i) part (b) we know that axS satisfies $\mathcal{P}$, and so $a \in I_1$. This prove $I_n = I_1$ for all $n \geq 1$.

Let $I := I_1$. As $\mathfrak{F}(S)$ is closed under left multiplication by x, we see that I is σ-stable. All that remains is to show that $I_0 = \mathfrak{F}(R) \cap I$. We already have $I_0 \subseteq I_1 = I$, and we also get $I_0 \subseteq \mathfrak{F}(R)$ by part (b) of condition (ii). Finally, fix $a \in \mathfrak{F}(R) \cap I$. By the first implication of condition (ii) we obtain $a \in \mathfrak{F}(S)$, and hence $a \in I_0$. □

A few comments on the previous lemma are in order.

1. If $\mathfrak{F}$ is an ideal function that respects isomorphisms (which will be true of all specific ideal functions we will consider), then $a \in \mathfrak{F}(R)$ if and only if $\sigma(a) \in \mathfrak{F}(R)$. In particular, the ideal I in the previous lemma is σ-invariant in that case.
2. As $\mathfrak{F}(R)$ is an ideal, we see that $aR \subseteq \mathfrak{F}(R)$ if and only if $RaR \subseteq \mathfrak{F}(R)$, if and only if $Ra \subseteq \mathfrak{F}(R)$, if and only if $a \in \mathfrak{F}(R)$.
3. The statement $aR_+ \subseteq \mathfrak{F}(R)$ is not left-right symmetric in general.
4. If we take $\mathcal{P}$ to be the (somewhat trivial) property "the set belongs to $\mathfrak{F}(R)$" then condition (i) amounts to the statement that $R/\mathfrak{F}(R)$ has no nonzero nilpotent right ideals.
5. If $\mathfrak{F}(R)$ is defined in terms of some property $\mathcal{P}$, such as nilpotence, then one can informally think of the second implication in condition (ii) as the statement that $\mathcal{P}$ restricts to constant polynomials.

Definition 3.3. Suppose we are given an ideal function $\mathfrak{F}$ which evaluates to a homogeneous ideal on any $\mathbb{N}$-graded ring, and for which we have an equality $\mathfrak{F}(R[x;\sigma]) = (\mathfrak{F}(R) \cap I) + Ix[x;\sigma]$ for some ideal $I \leq R$. We let $\mathfrak{F}_\sigma$ be the ideal function satisfying

$$\mathfrak{F}_\sigma(R) := \{a \in R \ : \ ax \in \mathfrak{F}(R[x;\sigma])\} = I,$$

which we call the *σ-polynomial $\mathfrak{F}$-function.*

Mirroring the previous section, we can now characterize numerous radicals of skew polynomial rings.

Example 3.4. If $\mathfrak{J}(R)$ is the Jacobson radical, then we have

$$\mathfrak{J}(R) = \{a \in R \ : \ \text{ if } x \in aR \text{ then } 1 - x \text{ is right invertible}\}.$$

It is well known that $R/\mathfrak{J}(R)$ has no nilpotent ideals, and thus condition (i) in Lemma 3.2 holds. (Alternatively, condition (i) is also clear from the definitions.)

Furthermore, suppose R is $\mathbb{N}$-graded; every element $r \in R$ has a decomposition $r = r_0 + r_+$ where $r_0 \in R_0$ has grade 0 and $r_+ \in R_+$ consists only of positive grade. Assume $a = a_0 \in \mathfrak{J}(R_0)$ and $aR_+ \subseteq \mathfrak{J}(R)$. We want to show that for every element $x \in aR$ we can find a right inverse for $1 - x$. By assumption $1 - x_0$ has a right inverse, call it v. Then $(1 - x)v = (1 - x_0 - x_+)v = 1 - x_+v \in 1 - aR_+$ has a right inverse, hence so does $1 - x$. This proves the first implication of condition (ii). The second implication is even easier, since the grade-0 component of a right invertible element (in an $\mathbb{N}$-graded ring) is always right invertible.

As the conditions in Lemma 3.2 are met, $\mathfrak{J}_\sigma(R)$ exists. This recovers [1, Theorem 3.1(i)].

Example 3.5. For the upper nilradical $\mathfrak{N}(R)$, we have the equality

$$\mathfrak{N}(R) = \{a \in R \ : \ RaR \text{ is nil}\}.$$

Condition (i) and the second implication of condition (ii) are easy to verify. We will now give an argument verifying the first implication of condition (ii), thus establishing the existence of $\mathfrak{N}_\sigma(R)$.

Fix $a = a_0 \in R_0$ such that R_0aR_0 is a nil ideal of R_0 and such that $aR_+ \subseteq \mathfrak{N}(R)$. We need to see that RaR is a nil ideal in R, so fix an arbitrary element $\alpha = \sum_{j=1}^{n} r_j a s_j \in RaR$. Expanding, we have $\alpha = \alpha_0 + \alpha_+$ where $\alpha_0 \in R_0aR_0$ and α_+ belongs to the ideal I in R generated by $R' = RaR_+ \cup R_+aR$. We know $RaR_+ \subseteq \mathfrak{N}(R)$, and that $I^2 \subseteq RaR_+$, hence by (i) we must have $I \subseteq \mathfrak{N}(R)$. Furthermore, as R_0aR_0 is nil there exists some $k \geq 1$ so that $\alpha_0^k = 0$. Thus $\alpha^k \in I \subseteq \mathfrak{N}(R)$. So α^k is nilpotent, and hence α is nilpotent.

In parallel with Question 2.16, we ask:

Question 3.6. Is $\mathfrak{N}_{\sigma,\sigma^{-1}}(R) = \mathfrak{N}(R) \cap \mathfrak{N}_\sigma(R) \cap \mathfrak{N}_{\sigma^{-1}}(R)$?

Proving the inclusion $\subseteq$ is not difficult since $R[x;\sigma]$ is a subring of $R[x, x^{-1};\sigma]$, and subsets of nil sets are nil. It is only the other containment which is unclear.

Example 3.7. The bounded radical satisfies the conditions of Lemma 3.2, following the proof of [18, Proposition 7]. In fact, that result is what motivated the lemma. The ideal $\mathfrak{B}_\sigma(R)$ is given another explicit description in Theorem 8 of that paper.

Example 3.8. Following work of Levitzki, we can characterize the prime radical as

$$\mathfrak{P}(R) = \{a \in R \ : \ \text{ every element in } aR \text{ is strongly nilpotent}\}, \tag{3.9}$$

where an element $r \in R$ is strongly nilpotent if every m-sequence starting with r is eventually zero (see [16, Exercise 10.17] for more information). As in the case of the upper nilradical, condition (i) and the second implication of condition (ii) are easy to obtain, while the other implication takes a short argument (which we leave to the dedicated reader). Applying Lemma 3.2, we recover [19, Theorem 1.3]. The ideal $\mathfrak{P}_\sigma(R)$ is given an alternate characterization, akin to (3.9), in both [3] and [17].

Example 3.10. Recall that the Wedderburn radical $\mathfrak{W}(R)$ is defined by (2.9). Both parts of condition (i) are easily verified. For condition (ii), clearly if aR_0 and aR_+ are nilpotent then so is aR, and the second implication is even easier. So once again Lemma 3.2 applies to prove the existence of $\mathfrak{W}_\sigma(R)$.

Let us call a subset $S \subseteq R$ *increasingly σ-nilpotent* if there exists an integer $k \geq 1$ so that for every strictly increasing sequence of positive integers $n_1, n_2, \ldots \in \mathbb{Z}_{>0}$ we have $S\sigma^{n_1}(S)\sigma^{n_2}(S)\cdots\sigma^{n_k}(S) = 0$. This notion allows us to describe $\mathfrak{W}_\sigma(R)$ in a way analogous to Theorem 2.11. The proof is similar, but we include it both for completeness and to demonstrate that the methods for skew polynomial and skew Laurent polynomial rings are complementary.

Theorem 3.11. *$\mathfrak{W}_\sigma(R) = \{a \in R \;:\; aR$ is increasingly σ-nilpotent$\}$.*

Proof. $\subseteq$: Fix $a \in \mathfrak{W}_\sigma(R)$ so $axR[x;\sigma] = aR[x;\sigma]x$ is a nilpotent right ideal, say of index $k \geq 1$. Fix a sequence of increasing positive integers $n_1, n_2, \ldots \in \mathbb{Z}_{>0}$ and fix arbitrary elements $r_0, r_1, \ldots \in R$. We have

$$ar_0x^{n_1}ar_1x^{n_2-n_1}ar_2x^{n_3-n_2}\cdots ar_{k-1}x^{n_k-n_{k-1}}ar_kx = 0.$$

Hence $ar_0\sigma^{n_1}(ar_1)\cdots\sigma^{n_k}(ar_k) = 0$. This exhibits the fact that aR is increasingly σ-nilpotent.

$\supseteq$: Fix $a \in R$ such that aR is increasingly σ-nilpotent, say of index $k \geq 1$. Fix $f_1, f_2, \ldots \in R[x;\sigma]$. We will prove that $af_1xaf_2x\cdots af_kxaf_{k+1}x = 0$, thus demonstrating that $axR[x;\sigma]$ is nilpotent and hence that $a \in \mathfrak{W}_\sigma(R)$. If we expand the product, each term looks like

$$ar_1x^{n_1+1}ar_2x^{n_2+1}\cdots ar_{k+1}x^{n_{k+1}+1}$$

where $r_ix^{n_i}$ is an arbitrary summand from f_i. Moving all powers of x to the right, the product equals

$$\left[ar_1\sigma^{n_1+1}(ar_2)\sigma^{n_1+n_2+2}(ar_3)\cdots\sigma^{n_1+n_2+\cdots+n_k+k}(ar_{k+1})\right]x^{n_1+n_2+\cdots+n_{k+1}+k+1}.$$

The quantity in the square brackets is zero, by the assumption of increasing σ-nilpotence, thus proving the claim. $\square$

It turns out that while the ideal $W_\sigma(R)$ defined by (2.10) is a good measure for $\mathfrak{W}_{\sigma,\sigma^{-1}}(R)$, the same is not true for the $\mathfrak{W}_\sigma(R)$. From the fact that $W_\sigma(R) = \mathfrak{W}_{\sigma,\sigma^{-1}}(R)$ and from the definitions it follows that we have an inclusion $W_\sigma(R) \subseteq \mathfrak{W}_\sigma(R)$ (alternatively, see Theorem 3.13 below). This containment is not an equality in general.

Example 3.12. Let F be a field, and consider the ring $R = F\langle\{a_i\}_{i\in\mathbb{Z}} \;:\; a_ia_j = 0 \text{ if } i < j\rangle$. There exists an automorphism σ on R uniquely defined by acting as the identity on F and sending $a_i \mapsto a_{i+1}$ (for each $i \in \mathbb{Z}$). It is easy to check that $a_0xR[x;\sigma]$ is nilpotent (of index 2). However, $U := \sum_{i=0}^{\infty}\sigma^i(a_0)R$ is not σ-nilpotent since

$$0 \neq a_n^n = a_n\sigma(a_{n-1})\sigma^2(a_{n-2})\cdots\sigma^n(a_0) \in U\sigma(U)\sigma^2(U)\cdots\sigma^n(U)$$

holds for each $n \geq 1$.

We finish our work on the Wedderburn radical by proving an analogue of (2.7).

Theorem 3.13. *$\mathfrak{W}_{\sigma,\sigma^{-1}}(R) = \mathfrak{W}(R) \cap \mathfrak{W}_\sigma(R) \cap \mathfrak{W}_{\sigma^{-1}}(R)$.*

Proof. $\subseteq$: If aR is totally σ-nilpotent, then aR is nilpotent and increasingly σ-nilpotent. Thus we have $\mathfrak{W}_{\sigma,\sigma^{-1}}(R) \subseteq \mathfrak{W}(R) \cap \mathfrak{W}_\sigma(R)$. Symmetry considerations yield the remainder of the containment.

$\supseteq$: Let $a \in \mathfrak{W}(R) \cap \mathfrak{W}_\sigma(R) \cap \mathfrak{W}_{\sigma^{-1}}(R)$. Let $n_1, n_2, \ldots \in \mathbb{Z}$ be an arbitrary sequence. As $a \in \mathfrak{W}(R)$ there exists some integer $k_1 \geq 1$ so that $(aR)^{k_1} = 0$. Similarly, as $a \in \mathfrak{W}_\sigma(R)$ (respectively $a \in \mathfrak{W}_{\sigma^{-1}}(R)$) there exists some integer $k = k_2 \geq 1$ (resp. $k = k_3 \geq 1$) so that

$$aR\sigma^{m_1}(aR)\sigma^{m_2}(aR)\cdots\sigma^{m_k}(aR) = 0$$

for any strictly increasing (resp. decreasing) sequence of exponents.

Let $\ell = k_1(k_2k_3+1)$, and consider $S := aR\sigma^{n_1}(aR)\cdots\sigma^{n_\ell}(aR)$. A sequence of length $(r-1)(s-1)+1$ consisting of distinct integers either has a strictly increasing subsequence of length r or strictly decreasing subsequence of length s, by the Erdős-Szekeres theorem [5]. Thus, among $n_1, n_2, \ldots, n_\ell$ there must either be a constant subsequence of length k_1, a strictly increasing subsequence of length k_2+1, or a strictly decreasing subsequence of length k_3+1. In any of these cases $S = 0$. $\square$

Example 3.14. If $\mathfrak{L}(R)$ is the Levitzki radical, then we have

$$\mathfrak{L}(R) = \{a \in R \ : \ aR \text{ is locally nilpotent}\}$$

by an application of [15, Proposition 10.31], and Lemma 3.2 applies once more to show the existence of $\mathfrak{L}_\sigma(R)$.

Adding a "strictly increasing" assumption to Theorem 2.13 allows us to characterize the σ-polynomial $\mathfrak{L}$-function.

Theorem 3.15. *$\mathfrak{L}_\sigma(R)$ consists of the elements $a \in R$ with the property that for every finite subset $S \subseteq aR$ and every integer $N \geq 1$ there exists an integer $k \geq 1$ so that for every difference bounded sequence of strictly increasing positive integers $n_1, n_2, \ldots \in \mathbb{Z}_{>0}$ with differences bounded by N, we have $S\sigma^{n_1}(S)\sigma^{n_2}(S)\cdots\sigma^{n_k}(S) = 0$.*

Proof. This follows, *mutatis mutandis*, by the proof of Theorem 2.13. $\square$

Recall that the σ-Levitzki radical $L_\sigma(R)$ is defined by (2.15). This ideal gives another description of the σ-polynomial $\mathfrak{L}$-function.

Theorem 3.16. *$\mathfrak{L}_\sigma(R) = L_\sigma(R)$.*

Proof. $\subseteq$: Trivial.

$\supseteq$: Fix $a \in L_\sigma(R)$ and an integer $N \geq 1$. Fix an arbitrary finite set $S = \{ar_1, \ldots, ar_\ell\} \subseteq aR$, and let $T = \{r_1, \ldots, r_\ell\}$. We let S' denote the set of all products of the form

$$ar_{j_0}\sigma^{m_1}(ar_{j_1})\cdots\sigma^{m_t}(ar_{j_t})$$

where $r_{j_0}, r_{j_1}, \ldots, r_{j_t}$ are arbitrary elements of T and where $1 \leq m_1 < m_2 < \ldots < m_t \leq N$ is a strictly increasing sequence of positive integers (bounded above by N). Note that S' is finite, and so $S'' = S' \cup \sigma(S') \cup \cdots \cup \sigma^N(S') \subseteq \sum_{i=0}^{\infty} \sigma^i(a)R$ is also finite. By hypothesis, since $a \in L_\sigma(R)$ there exists an integer $k \geq 1$ so that $S''\sigma^N(S'')\sigma^{2N}(S'')\cdots\sigma^{kN}(S'') = 0$.

Fix a strictly increasing sequence of positive integers $n_1, n_2, \ldots \in \mathbb{Z}_{>0}$, which are difference bounded by N. If we can show that $U := S\sigma^{n_1}(S)\cdots\sigma^{n_{kN}}(S)$ is zero, then we are done. We claim that an arbitrary element in U begins as a product in $S''\sigma^N(S'')\sigma^{2N}(S'')\cdots\sigma^{kN}(S'') = 0$. Indeed, if we partition the sequence $n_1 < n_2 < \ldots$ according to whether $pN \leq n_i < (p+1)N$, say by writing

$$1 \leq n_1 < n_2 < \ldots < n_{\ell_1} < N \leq n_{\ell_1+1} < n_{\ell_1+2} < \ldots < n_{\ell_2} < 2N \leq n_{\ell_2+1} < \ldots < kN \leq n_{\ell_{kN}} < \ldots,$$

then we see that for every integer $p \geq 0$,

$$\sigma^{n_{\ell_p+1}}(S)\sigma^{n_{\ell_p+2}}(S)\cdots\sigma^{n_{\ell_{p+1}}}(S) \subseteq \sigma^{pN}(S'').$$

Thus $S\sigma^{n_1}(S)\sigma^{n_2}(S)\cdots\sigma^{n_{\ell_k}}(S)\sigma^{n_{\ell_k+1}}(S) \subseteq S''\sigma^N(S'')\sigma^{2N}(S'')\cdots\sigma^{kN}(S'') = 0$. Furthermore, since $n_{\ell_k+1} \geq kN$ and the sequence $\{n_i\}$ is strictly increasing, we have $\ell_k + 1 \leq kN$; hence, products in U are zero. $\square$

We can now rephrase Question 2.16 to ask: Does $\mathfrak{L}_{\sigma,\sigma^{-1}}(R) = \mathfrak{L}(R) \cap \mathfrak{L}_\sigma(R) \cap \mathfrak{L}_{\sigma^{-1}}(R)$? There is one special case where this is easy to prove. Recall that σ is said to be *locally of finite order* if for each $r \in R$ there exists some $n \geq 1$ such that $\sigma^n(r) = r$.

Proposition 3.17. *We have $\mathfrak{L}_{\sigma,\sigma^{-1}}(R) = \mathfrak{L}(R) = \mathfrak{L}_\sigma(R)$ when σ is locally of finite order.*

Proof. The second equality follows from Theorem 3.16 above, and [7, Lemma 3.12]. Thus, we restrict ourselves to considering the first equality.

$\subseteq$: This follows from the comments above Question 2.16.

$\supseteq$: Fix $a \in \mathfrak{L}(R)$. We wish to show that $a \in \mathfrak{L}_{\sigma,\sigma^{-1}}(R)$. To that end, fix a finite collection of elements $f_1, f_2, \ldots, f_\ell \in R[x, x^{-1}; \sigma]$. We need to show that $S = \{af_i\}_{1 \leq i \leq \ell}$ is nilpotent. Without loss of generality, we can assume that $f_i = r_i x^{m_i}$ for some $r_i \in R$ and $m_i \in \mathbb{Z}$.

Let $S' = \{\sigma^n(ar_i) \ : \ n \in \mathbb{Z},\ 1 \leq i \leq \ell\}$. Notice that S' is finite since σ is locally of finite order. As $\mathfrak{L}(R)$ is σ-invariant, by [7, Lemma 3.9(1)] we have $S' \subseteq \mathfrak{L}(R)$. Thus S' is nilpotent as $\mathfrak{L}(R)$ is locally nilpotent. This implies that S is also nilpotent. $\square$

4. Radicals and T-nilpotence

In this section we introduce new radical-like ideals related to T-nilpotence, and explore how the results of the previous two sections apply in this situation. First, we recall a standard definition.

Definition 4.1. A set $S \subseteq R$ is *left T-nilpotent* if, for any countable sequence of elements $x_1, x_2, \ldots \in S$ there exists an integer $n \geq 1$ such that $x_1 x_2 \cdots x_n = 0$. Right T-nilpotent sets are defined similarly, and if a set S is both left and right T-nilpotent, we say it is *T-nilpotent*.

Clearly, left T-nilpotence passes to subsets. Every left T-nilpotent, one-sided ideal is contained in the prime radical [15, Proposition 23.15]. One especially interesting paper on the subject of T-nilpotence is [6]. We begin our investigations by studying the addition of two left T-nilpotent sets.

Lemma 4.2. *Let $I \subseteq R$ and $J \leq R$. If I and J are left T-nilpotent, then so is $I + J$.*

Proof. Let $a_1 + b_1, a_2 + b_2, \ldots$ be a sequence in $I + J$, where $a_i \in I$ and $b_i \in J$ for each $i \geq 1$. Since I is left T-nilpotent there exists an integer m_1 such that $a_1 a_2 \cdots a_{m_1} = 0$. Recursively, there exist strictly increasing, positive integers m_n (for $n \geq 1$) such that $a_{m_{n-1}+1} a_{m_{n-1}+2} \cdots a_{m_n} = 0$ (where, for simplicity, $m_0 = 0$).

For each $n \geq 1$, set

$$c_n = (a_{m_{n-1}+1} + b_{m_{n-1}+1})(a_{m_{n-1}+2} + b_{m_{n-1}+2}) \cdots (a_{m_n} + b_{m_n}) \in J.$$

(This is where we use the fact that J is a two-sided ideal.) As J is left T-nilpotent, there exists some integer $\ell \geq 1$ such that $c_1 c_2 \cdots c_\ell = 0$. Thus

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_{m_\ell} + b_{m_\ell}) = c_1 c_2 \cdots c_\ell = 0.$$

This demonstrates that $I + J$ is left T-nilpotent. $\square$

Proposition 4.3. *Let $I \subseteq R$ and let J be a one-sided ideal of R.*

1. *If J is left T-nilpotent, then RJR is left T-nilpotent.*
2. *If I and J are left T-nilpotent, then $I + J$ is left T-nilpotent.*

Proof. (1) Suppose J is a right ideal (the other case is similar). We prove the contrapositive, so assume RJR is not left T-nilpotent. Thus there exists a sequence $\alpha_1, \alpha_2, \ldots \in RJ = RJR$ such that $\alpha_1 \alpha_2 \cdots \alpha_k \neq 0$ for any $k \geq 1$. Each α_i is a finite sum of terms of the form ra with $r \in R$ and $a \in J$. By König's tree lemma, we may as well assume $\alpha_i = r_i a_i$ for each $i \geq 1$. We now have $(r_1 a_1)(r_2 a_2) \cdots (r_k a_k) \neq 0$ for any $k \geq 1$. Hence, $(a_1 r_2)(a_2 r_3) \cdots (a_k r_{k+1}) \neq 0$ for any $k \geq 1$. As the sequence $a_1 r_2, a_2 r_3, \ldots$ belongs to J, this proves that J is not left T-nilpotent.

(2) Assume I and J are left T-nilpotent. By part (1), RJR are left T-nilpotent. Hence, the previous lemma says that $I + RJR$ is left T-nilpotent. The subset $I + J$ is thus also left T-nilpotent. $\square$

Definition 4.4. The *left T-nilpotent radideal* of R, denoted by $\mathfrak{T}_\ell$, is the ideal function given by

$$\mathfrak{T}_\ell(R) = \sum\{I \leq R \ : \ I \text{ is a left } T\text{-nilpotent ideal of } R\}. \tag{4.5}$$

The right T-nilpotent radideal is defined similarly, and denoted $\mathfrak{T}_r$.

By Proposition 4.3, we can recharacterize $\mathfrak{T}_\ell$ as the ideal function given by the rules

$$\begin{aligned}\mathfrak{T}_\ell(R) &= \{a \in R \ : \ aR \text{ is left } T\text{-nilpotent}\} = \{a \in R \ : \ Ra \text{ is left } T\text{-nilpotent}\}\\ &= \{a \in R \ : \ RaR \text{ is left } T\text{-nilpotent}\}.\end{aligned}$$

We will see shortly that $\mathfrak{T}_\ell(R) \neq \mathfrak{T}_r(R)$. We will also prove that neither of these ideals are Kurosh-Amitsur radicals, for $R/\mathfrak{T}_\ell(R)$ can contain nonzero left T-nilpotent ideals.

It turns out that $\mathfrak{T}_\ell(R)$ is not always left T-nilpotent itself. For example, let

$$R = F[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$$

where F is a field. Then each ideal (x_i) is nilpotent, and hence (left and right) T-nilpotent. Thus $I = \sum_{i=0}^\infty (x_i) \subseteq \mathfrak{T}_\ell(R)$ (and the reverse containment holds, but we won't need it). We find $x_1 x_2 \cdots x_n \neq 0$ for each $n \geq 1$.

The following proposition is easy. We leave the proof as an exercise.

Proposition 4.6. *We have $\mathfrak{W}(R) \subseteq \mathfrak{T}_\ell(R) \subseteq \mathfrak{P}(R)$.*

Example 4.7. Each of the containments above can be proper. We will borrow some examples constructed in [8, Example 11]. First, let $R = \mathbb{Q}\langle x_1, x_2, \ldots\rangle$ with relations $x_1^2 = 0$, $x_i x_j = 0$ when $i, j > 1$, and $x_m x_1 x_n = 0$ whenever $m \leq n$. The ideal $I = x_1 R$ is left T-nilpotent, but not right T-nilpotent and hence not nilpotent. In particular, $x_1 \in \mathfrak{T}_\ell(R) \setminus \mathfrak{T}_r(R) \subseteq \mathfrak{T}_\ell(R) \setminus \mathfrak{W}(R)$. We note in passing that I is not nil of bounded index.

To show that the second containment in the previous proposition can be proper, consider the ring $R = \mathbb{F}_2\langle x_1, x_2, \ldots\rangle$ subject to the relations $x_1^2 = 0$, $x_i x_j = 0$ when $i, j > 1$, $x_i x_1 x_i = 0$ for all $i > 1$, and $x_i x_1 x_j = x_j x_1 x_i$ when $i, j > 1$. The ideal $I = x_1 R$ is nil of bounded index, and so $x_1 \in \mathfrak{B}(R) \subseteq \mathfrak{P}(R)$. But I is not left T-nilpotent, nor right T-nilpotent. Thus $x_1 \in \mathfrak{P}(R) \setminus (\mathfrak{T}_\ell(R) \cap \mathfrak{T}_r(R))$.

It is well known that there are higher versions of the Wedderburn radical. Indeed, let $\mathfrak{W}^{(0)}(R) = (0)$, and given an ordinal α, recursively set

$$\mathfrak{W}^{(\alpha)}(R) = \{a \in R \ : \ a + \mathfrak{W}^{(\beta)}(R) \in \mathfrak{W}(R/\mathfrak{W}^{(\beta)}(R))\}$$

if α is the successor of β, and if α is a limit ordinal put

$$\mathfrak{W}^{(\alpha)}(R) = \bigcup_{\beta < \alpha} \mathfrak{W}^{(\beta)}(R).$$

Levitzki proved that these radicals stabilize to the prime radical; see [16, Exercise 10.11] for a proof. Furthermore, we have the interesting containment $\mathfrak{W}^{(1)}(R) \subseteq \mathfrak{B}(R) \subseteq \mathfrak{W}^{(2)}(R)$ due to Klein [14]. Thus, we can similarly define higher versions of the bounded nilradical, and these too stabilize to the prime radical (at a rate at most twice as fast as the Wedderburn radical).

Unlike the bounded nilradical, the left T-nilpotent radideal does not fit inside any fixed higher Wedderburn radical.

Theorem 4.8. *Let α be any ordinal. There exists a ring R for which $\mathfrak{T}_\ell(R)$ is not a subset of $\mathfrak{W}^{(\alpha)}(R)$.*

Proof. Let F be a field. Let $R = F\langle b_{n,\beta}$ ($n \in \mathbb{Z}_{>0}$, $\beta \leq \alpha$ is an ordinal)$\rangle$ where the letters are subject to relations which we now describe. For simplicity, let $R_{\geq\gamma}$ be the subring of R generated by the variables $b_{n,\beta}$ with $\gamma \leq \beta \leq \alpha$. Our relations say:

- Any monomial in $R_{\geq\beta}$ containing $b_{n,\beta}$ and of total degree $> n$ is zero.

First, we claim that $b_{1,\alpha} \in \mathfrak{T}_\ell(R)$. Fix a sequence of elements $f_1, f_2, \ldots$ from $b_{1,\alpha}R$; we wish to find some $k \geq 1$ so that $f_1 f_2 \cdots f_k = 0$. By König's tree lemma, it suffices to consider the case where each f_i is a monomial. Among all ordinals β which occur as subscripts of letters in the f_i, there exists some minimal such ordinal, call it β_0. Fix an index j so that b_{n_0,β_0} occurs as a letter in f_j. From the relations, we can take $k = \max(n_0 + 1, j)$.

Finally, we prove that $b_{1,\alpha} \notin \mathfrak{W}^{(\alpha)}(R)$. More specifically, we will prove by transfinite induction on $\beta \leq \alpha$ that $a \notin \mathfrak{W}^{(\beta)}(R)$ whenever a is a nonzero monomial in $R_{\geq \beta}$. The base case, when $\beta = 0$, is easy since $a \neq 0$ but $\mathfrak{W}^{(0)}(R) = (0)$. Now, assume by way of induction that the claim holds for all ordinals $\gamma < \beta$, and fix a nonzero monomial $a \in R_{\geq \beta}$. There are two cases to consider.

Case 1: Suppose β is a limit ordinal. For any $\gamma < \beta$ we have $a \in R_{\geq \gamma}$ is nonzero, and so the inductive assumption implies that $a \notin \mathfrak{W}^{(\gamma)}(R)$. Thus $a \notin \bigcup_{\gamma < \beta} \mathfrak{W}^{(\gamma)}(R) = \mathfrak{W}^{(\beta)}(R)$.

Case 2: Suppose β is a successor ordinal, say $\beta = \gamma + 1$. Let $n \geq 1$ be an integer, and let m be the total degree of a. Define $a_n = ab_{(m+1)n,\gamma}$. As $a \in R_{\geq \beta}$, we see that $a_n^n \neq 0$ and $a_n^n \in R_{\geq \gamma}$. Thus, by our inductive assumption, $a_n^n \notin \mathfrak{W}^{(\gamma)}(R)$ for any $n \geq 1$. In particular $(aR)^n \not\subseteq \mathfrak{W}^{(\gamma)}(R)$ for any $n \geq 1$, and hence $a \notin \mathfrak{W}^{(\gamma+1)}(R) = \mathfrak{W}^{(\beta)}(R)$ as desired. $\square$

We finish this section by describing the left T-nilpotent radideal for skew (Laurent) polynomial rings. To begin, it turns out that in the non-skew case this ideal is very easy to describe.

Lemma 4.9. *Let R be a ring. Given a right ideal $I \leq R[x]$, let A_I be the right ideal of coefficients in I. If I is left T-nilpotent, then A_I is left T-nilpotent.*

Proof. This is [2, Theorem 18]. $\square$

Proposition 4.10. *We have $\mathfrak{T}_\ell(R[x]) = \mathfrak{T}_\ell(R)[x]$.*

Proof. The containment $\subseteq$ follows from the previous lemma. For the reverse direction, let $f(x) = \sum_{i=0}^{m} a_i x^i \in \mathfrak{T}_\ell(R)[x]$. We know $J := a_0R + a_1R + \cdots + a_mR$ is left T-nilpotent by Proposition 4.3. Thus, for any sequence $\{f(x)g_i(x)\}_{i=1}^{\infty}$ there exists an integer $n \geq 1$ with $f(x)g_1(x)f(x)g_2(x)\cdots f(x)g_n(x) = 0$. This prove that $f(x) \in \mathfrak{T}_\ell(R[x])$. $\square$

A similar statement and proof holds true for the usual Laurent polynomial ring, *mutatis mutandis.* As one might expect, power series rings do not turn out the same way.

Example 4.11. Let $R = \mathbb{Q}[x_1, x_2, \ldots]$ with relations $x_n \prod_{j=1}^{n} x_{i_j} = 0$ for all $n \geq 1$. Letting $I = (x_1, x_2, \ldots)$ we see that I is (both left and right) T-nilpotent. However, $I[\![t]\!]$ is not even nil. Indeed, putting $f(t) = \sum_{i=1}^{\infty} x_i t^i$ we find that $f(t)^n = x_n^n t^{n^2} + \cdots \neq 0$.

Modeling the proof of Theorem 2.11, we have the following:

Theorem 4.12. $\mathfrak{T}_{\ell,\sigma,\sigma^{-1}}(R) = \{a \in R \ : \ \sum_{i=-\infty}^{\infty} \sigma^i(a)R$ *is left T-nilpotent*$\}$.

Proof. Let $J := \{a \in R \ : \ \sum_{i=-\infty}^{\infty} \sigma^i(a)R$ is left T-nilpotent$\}$, and fix $a \in J$. Let $af_1(x), af_2(x), \ldots$ be a sequence from $aR[x, x^{-1}; \sigma]$; we wish to show that there exists some index $k \geq 1$ such that $af_1(x)af_2(x)\cdots af_k(x) = 0$. Without loss of generality (applying König's tree lemma), we may write $f_i = r_i x^{m_i}$ for elements $r_i \in R$ and integers $m_i \in \mathbb{Z}$ (for each $i \geq 1$). The sequence $ar_1, \sigma^{m_1}(ar_2), \ldots$ occurs in the left T-nilpotent set $\sum_{i=-\infty}^{\infty} \sigma^i(a)R$. Thus, we may fix some $k \geq 1$ such that

$$ar_1\sigma^{m_1}(ar_2)\sigma^{m_1+m_2}(ar_3)\cdots\sigma^{m_1+m_2+\cdots+m_{k-1}}(ar_k) = 0.$$

We find

$$\begin{aligned} af_1(x)af_2(x)\cdots af_j(x) &= ar_1x^{m_1}ar_2x^{m_2}\cdots ar_jx^{m_j} \\ &= ar_1\sigma^{m_1}(ar_2)\sigma^{m_1+m_2}(ar_3)\cdots\sigma^{m_1+m_2+\cdots+m_{j-1}}(ar_j)x^{m_1+m_2+\cdots+m_j} = 0. \end{aligned}$$

Hence $a \in \mathfrak{T}_{\ell,\sigma,\sigma^{-1}}(R)$.

Conversely, fix $a \in \mathfrak{T}_{\ell,\sigma,\sigma^{-1}}(R)$. Set $I := \sum_{i=-\infty}^{\infty} \sigma^i(a)R$; we wish to show that I is left T-nilpotent. So fix an arbitrary sequence $\sigma^{k_1}(ar_1), \sigma^{k_2}(ar_2), \ldots$ from I. Consider the new sequence $ar_1x^{k_2-k_1}, ar_2x^{k_3-k_2}, \ldots$ from $aR[x, x^{-1}; \sigma]$. Since $a \in \mathfrak{T}_{\ell,\sigma,\sigma^{-1}}(R)$, we know that there exists some index $j \geq 2$ such that

$$0 = ar_1x^{k_2-k_1}ar_2x^{k_3-k_2} \cdots ar_jx^{k_j-k_{j-1}}.$$

Expanding, and applying σ^{k_1}, we have $\sigma^{k_1}(ar_1)\sigma^{k_2}(ar_2) \cdots \sigma^{k_{j-1}}(ar_{j-1}) = 0$. This proves that I is left T-nilpotent as desired. □

Corollary 4.13. *$\mathfrak{T}_{\ell,\sigma,\sigma^{-1}}(R)$ consists of the elements $a \in R$ such that for every sequence of integers $n_1, n_2, \ldots \in \mathbb{Z}$ and every sequence of elements $ar_0, ar_1, \ldots \in aR$ there exists an integer $k \geq 1$ such that $ar_0\sigma^{n_1}(ar_1) \cdots \sigma^{n_k}(ar_k) = 0$.*

Proof. The condition described in this corollary is tautologically weaker than the definition of the set J (in the proof above). The converse holds by an easy application of König's tree lemma. □

Our next result now easily follows by using the same methods by which we obtained Theorem 3.11.

Theorem 4.14. *$\mathfrak{T}_{\ell,\sigma}(R)$ consists of the elements $a \in R$ such that for every sequence of strictly increasing positive integers $n_1, n_2, \ldots \in \mathbb{Z}_{>0}$ and every sequence of elements $ar_0, ar_1, \ldots \in aR$ there exists an integer $k \geq 1$ such that $ar_0\sigma^{n_1}(ar_1) \cdots \sigma^{n_k}(ar_k) = 0$.*

Proof. We first prove that Lemma 3.2 applies. Condition (i) in that lemma holds, by taking $\mathcal{P}$ to be the property "is left T-nilpotent." Condition (ii), part (b), follows easily by considering what happens to the grade-0 portion of a left T-nilpotent set inside an $\mathbb{N}$-graded ring. Part (a) follows from Proposition 4.3, since if aR_0 and aR_+ are both left T-nilpotent, then so is $aR_0 + aR_+ = aR$.

The remainder of the proof consists of modifying Theorem 4.12 to the skew polynomial case (just as we modified Theorem 2.11 to obtain Theorem 3.11, and similarly with the Levitzki radical). □

5. Final Questions

We finish the paper by raising a few questions that future researchers on this topic may wish to address. We found in a few cases that $\mathfrak{F}_{\sigma,\sigma^{-1}}(R) = \mathfrak{F}(R) \cap \mathfrak{F}_\sigma(R) \cap \mathfrak{F}_{\sigma^{-1}}(R)$. In even more cases, we found that at least one of the inclusions holds. Thus, we ask:

Question 5.1. Are there simple sufficient conditions on the ideal function $\mathfrak{F}$ which imply that we have an equality $\mathfrak{F}_{\sigma,\sigma^{-1}}(R) = \mathfrak{F}(R) \cap \mathfrak{F}_\sigma(R) \cap \mathfrak{F}_{\sigma^{-1}}(R)$? More generally, are there simple sufficient conditions which imply the containment $\subseteq$?

As far as we have ascertained, the equality above is open for the Jacobson radical, the Brown-McCoy radical, the upper nilradical, the Levitzki radical, and the (left) T-nilpotent radideal.

Recall that we proved that $\mathfrak{T}_\ell(R)$ is in general not a subset of $\mathfrak{W}^{(\alpha)}(R)$ for any fixed ordinal α. Just as with the Wedderburn radicals, one can define higher order versions of the left T-nilpotent radideal, recursively. Let $\mathfrak{T}_\ell^{(0)}(R) = (0)$, and given an ordinal α, recursively set

$$\mathfrak{T}_\ell^{(\alpha)}(R) = \{a \in R \ : \ a + \mathfrak{T}_\ell^{(\beta)}(R) \in \mathfrak{T}_\ell(R/\mathfrak{T}_\ell^{(\beta)}(R))\}$$

if α is the successor of β, and if α is a limit ordinal put

$$\mathfrak{T}_\ell^{(\alpha)}(R) = \bigcup_{\beta<\alpha} \mathfrak{T}_\ell^{(\beta)}(R).$$

We know that these higher radideals stabilize to the prime radical because of four facts: (1) the higher Wedderburn ideals stabilize to the prime radical, (2) $\mathfrak{W}(R) \subseteq \mathfrak{T}_\ell(R)$, (3) $\mathfrak{T}_\ell(R) \subseteq \mathfrak{P}(R)$, and (4)

$\mathfrak{P}(R/\mathfrak{P}(R)) = 0$. Further, fact (2) implies that if $\mathfrak{T}_\ell^{(\alpha)}(R) \supseteq \mathfrak{W}^{(\beta)}(R)$, then $\mathfrak{T}_\ell^{(\alpha+1)}(R) \supseteq \mathfrak{W}^{(\beta+1)}(R)$. Are any other containments necessary? More formally, we ask:

Question 5.2. Let α be an ordinal. Let $\{\beta_\gamma\}_{\gamma\leq\alpha}$ be any sequence (indexed by ordinals up to α) of strictly increasing ordinals. Does there exists a ring R such that $\mathfrak{W}^{(\beta_\gamma)}(R) \subseteq \mathfrak{T}_\ell^{(\gamma)}(R) \subsetneq \mathfrak{W}^{(\beta_\gamma+1)}(R)$ for all $\gamma \leq \alpha$?

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