Two-Sided Properties of Elements in Exchange Rings

Dinesh Khurana, T. Y. Lam, and Pace P. Nielsen

Abstract

For any element a in an exchange ring R, we show that there is an idempotent $e \in aR \cap Ra$ such that $1-e \in (1-a)\,R \cap R\,(1-a)$. A closely related result is that a ring R is an exchange ring if and only if, for every $a \in R$, there exists an idempotent $e \in Ra$ such that $1-e \in (1-a)\,R$. The Main Theorem of this paper is a general two-sided statement on exchange elements in arbitrary rings which subsumes both of these results. Finally, applications of these results are given to the study of the endomorphism rings of exchange modules.

§1. Introduction

Throughout this paper, R denotes a unital (generally noncommutative) ring. Following [Ni_1], we say that an element $a \in R$ is **right suitable** if it has the following "idempotent lifting" property: for any right ideal $I \subseteq R$ containing $a - a^2$, there exists an idempotent $e \in R$ such that $e - a \in I$. The set of right suitable elements in R is denoted by $\mathrm{suit}_r(R)$, and the set $\mathrm{suit}_\ell(R)$ of left suitable elements in R is defined similarly. In [Ni_2], Nicholson proved that $\mathrm{suit}_\ell(R) = \mathrm{suit}_r(R)$ for any ring R. The rings R with the property that $R = \mathrm{suit}_r(R)$ are called (right) **suitable rings**. In the seminal paper [Ni_1], Nicholson proved that these are precisely Warfield's **exchange rings** in [Wa]; namely, those rings R for which the right module R_R satisfies the (finite) exchange property of Crawley and Jónsson. In our paper, we'll use the terms "suitable rings" and "exchange rings" interchangeably, noting that suitable elements are sometimes also called "exchange elements" in the literature.

Various characterizations for right suitable elements in rings were given in [Ni_1]. Among the most useful ones is the following "Goodearl-Nicholson characterization" (see [GW] and [Ni_1]): $a \in \mathrm{suit}_r(R)$ *if and only if there exists an idempotent* $e \in aR$ *such that* $1 - e \in (1-a)\,R$. (In particular, R is an exchange ring if and only if every element $a \in R$ has the above idempotent property.) However, the prevailing study of the suitability of elements in rings has been mostly carried out "on one side" of the ring at a time. To our best knowledge, there has been almost no work in which the existence of idempotents with "two-sided" (left and right) properties was investigated. Our main results in this paper are intended to be the beginning steps in this direction.

For ease of understanding, we state explicitly below two special cases of the Main Theorem in this paper.

Theorem A. *For every element a in an exchange ring R, there exists an idempotent $e \in aRa$ such that $1 - e \in (1-a)\,R\,(1-a)$.*

Note that this Theorem A is in fact *equivalent* to the first result stated in the Abstract. One simply observes that, if an idempotent e lies in an intersection $xR \cap Ry$, then $e \in xRy$. Indeed, writing $e = xr = sy$ for some $r, s \in R$, we have $e = e^2 = (xr)\,(sy) \in xRy$. And of course, $xRy \subseteq xR \cap Ry$ for any $x, y \in R$.

Theorem A appears to be new even for a von Neumann regular ring (or a π-regular ring) R, as we have not been able to locate such a result in the standard treatises on the subject (e.g. [vN_1], [vN_2], or [Go]). The same result also seems to be unknown for semiperfect rings and C^*-algebras of real rank zero, both of which are well known examples of exchange rings (by [Wa] and [AG]). In the case where R is an exchange ring of stable range one, a result of Ara given in [Ni_2: Corollary] states that, for every $a \in R$, there exists a pair of conjugate idempotents e, e' such that $e \in aR$, $1-e \in (1-a)R$, $e' \in Ra$, and $1 - e' \in R(1-a)$. A version of this result for suitable elements under the assumption that R satisfies internal cancellation appeared in [KG: Proposition 4.2]. Theorem A (or more precisely, Theorem 3.6 in §3) is a substantial improvement of both of these results in that the stable range one assumption and the internal cancellation assumption on R have been dropped, and the conclusion is strengthened to yield a *single* idempotent $e = e'$ with the aforementioned properties.

The second result we want to specifically mention in this Introduction is the following "left-right mixed" characterization for an exchange ring, for which we also know of no explicit reference in the literature.

Theorem B. *A ring R is an exchange ring if and only if, for every $a \in R$, there exists an idempotent $e \in Ra$ such that $1 - e \in (1-a)\,R$.*

For the larger part of this paper, instead of working with exchange (or suitable) rings, we take the more flexible approach of working with suitable elements. From this viewpoint, most of our results are stated in, and applicable to, a more general "element-wise" setting. In particular, Theorem A and Theorem B above are just special cases of the Main Theorem in this paper, which is stated as Theorem 2.5 in §2. Some ramifications and applications of this Main Theorem are given in §3. The paper closes with a final section (§4), in which the two-sided properties obtained for suitable elements in §3 are applied to the study of the endomorphism rings of modules. In some sense, the decomposition results for endomorphisms obtained in §4 are generalizations of the "Fitting decompositions" in Fitting's classical paper [Fi].

The terminology and notations introduced so far in this Introduction will be used freely throughout the paper. In addition, the following useful definitions are needed in §§3-4: a ring element $a \in R$ is said to be *clean* if $a = f + u$ for some idempotent $f \in R$ and some unit $u \in R$, and a is said to be *strongly clean* if there exists such a decomposition $a = f + u$ with $fu = uf$. It is well known (from the proof of [Ni$_1$: Proposition 1.8]) that *clean elements are suitable.* Next, an element $a \in R$ is said to be *strongly π-regular* if there exists an integer $n \geq 1$ such that $a^n \in a^{n+1}R \cap R\,a^{n+1}$. According to Nicholson [Ni$_3$: Theorem 1], *strongly π-regular elements are strongly clean.* As in our companion paper [KLN], we'll denote the set of idempotents in a ring R by $\text{idem}\,(R)$. Other standard terminology and conventions in ring theory follow those in [AF], [Go], and [La]. Whenever it is more convenient, we'll use the widely accepted shorthand "iff" for "if and only if" in the text.

§2. Main Theorem and Its Proof

The goal of this section is to prove the Main Theorem 2.5 below on the existence of idempotents with certain two-sided properties arising from a "commuting unimodular equation" $pq + st = 1$ in a ring. We start with a lemma that will be crucial for the proof of Theorem 2.5.

Lemma 2.1. (A) *Let p, q, c, d be commuting elements in R. If $pq + qcxd \in \text{idem}\,(R)$ for some $x \in R$, then $(pq)^2 + cydq \in \text{idem}\,(R)$ for some $y \in R$.*

(B) *Let p, q, s, t be commuting elements of R with $pq + st = 1$. If there exists an idempotent $e \in pRq$ such that $1 - e \in sRt$, then there exists an idempotent $g \in pqR$ such that $1 - g \in sRt$.*

Proof. A basic fact to be used several times in this proof is that, in any ring R, $ab \in \text{idem}\,(R) \Rightarrow (ba)^2 \in \text{idem}\,(R)$. This is checked by simply noting that $(ba)^4 = b\,(ab)^3\,a = b\,(ab)\,a = (ba)^2$.

(A) Since $q\,(p + cxd) \in \text{idem}\,(R)$, we have $(pq + cxdq)^2 \in \text{idem}\,(R)$. By inspection, $(pq + cxdq)^2 \in (pq)^2 + cRdq$.

(B) To facilitate the proof of (B), we first observe that, for any integer $n \geq 1$,

$$1 - (pq)^n = (1 - pq)\,(1 + \cdots + (pq)^{n-1}) \in sRt. \tag{2.2}$$

Write the given idempotent e in (B) in the form $p\,(1 + x)\,q$ for some $x \in R$. Since $1 - e = 1 - pq - pxq = st - pxq \in sRt$, we have $pxq = syt$ for some $y \in R$. Thus,

$$q\,(syt) = q\,(pxq) = (1 - st)\,xq \;\Rightarrow\; xq \in sR. \tag{2.3}$$

Similarly, $(syt)\,p = (pxq)\,p = px\,(1 - st) \;\Rightarrow\; px \in Rt$. As $e = p\,(1 + x)\,q$ is an idempotent, so is

$$h := (qp\,(1 + x))^2 = (qp)^2 + qpqpx + qpxqp + qpxqpx \in pqR. \tag{2.4}$$

In view of $px \in Rt$ and (2.2) (for $n = 2$), we have $1 - h \in Rt$. Similarly, using $xq \in sR$ and (2.2) (for $n = 3$), we have $1 - hqp \in sR$. Now let $g := h + h\,(qp)\,(1 - h)$, which is easily seen to be an idempotent by a direct calculation. This is the idempotent we want, since $g \in pqR$ by (2.4), and we have also

$$1 - g = (1 - h) - h\,(qp)\,(1 - h) = (1 - hqp)\,(1 - h) \in (sR)\,(Rt) = sRt. \qquad \Box$$

We can now state and prove the following main result of this paper.

Main Theorem 2.5. *Let* p, q, s, t *be commuting elements of* R *with* $pq + st = 1$. *The following three statements are equivalent*:

(1) $pq \in \text{suit}_r(R)$.
(2) *There exists* $e \in \text{idem}\,(R)$ *such that* $e - pq \in psRqt$.
(3) *There exists an idempotent* $e \in pRq$ *such that* $1 - e \in sRt$.

In particular, if R *is an exchange ring, then* (2) *and* (3) *both hold.*

Proof. (2) $\Rightarrow$ (3). Given $e \in \text{idem}\,(R)$ as in (2), we have clearly $e \in pRq$, and

$$1 - e \in 1 - pq - psRqt = st - psRqt \subseteq sRt.$$

(3) $\Rightarrow$ (1). By Lemma 2.1(B), (3) implies that there exists $g \in \text{idem}\,(R) \cap pq\,R$ such that $1 - g \in sRt$. Since $st + (pq) \cdot 1 = 1$, applying Lemma 2.1(B) again yields $f \in \text{idem}\,(R) \cap stR$ such that $1 - f \in pqR$. As $pq + st = 1$, this implies that $pq \in \text{suit}_r(R)$.

(1) $\Rightarrow$ (2). Assume that $a := pq \in \text{suit}_r(R)$. Since a is idempotent modulo $a(1-a)R$, there exists an idempotent $e_1 \in \text{idem}\,(R)$ such that $e_1 - a \in a\,(1-a)\,R = pqstR$. Writing $e_1 = a + qpstx$ (for some $x \in R$) and applying Lemma 2.1(A), we get an idempotent $e_2 = a^2 + pstyq$ for some $y \in R$. Note that

$$e_2 = a - a\,(1-a) + pstyq = a - pqst + pstyq = a + pst\,(y-1)\,q$$

has complementary idempotent $st + tps\,(1-y)\,q$. Thus, applying Lemma 2.1(A) again gives an idempotent $e_3 = (st)^2 + pszqt$ for some $z \in R$. Now

$$e := 1 - e_3 = pq\,(1 + st) - pszqt \in pq + psRqt$$

is the idempotent we want in (2). $\square$

§3. Applications of the Main Theorem

This section is devoted to the statement of various consequences of the Main Theorem 2.5 in §2. To begin with, it should be no surprise that this theorem implies the left-right symmetry of the notion of suitable elements, with a significantly different proof from that given earlier by Nicholson in [Ni_2].

Proposition 3.1. *For any ring* R, $\text{suit}_r(R) = \text{suit}_\ell(R)$.

Proof. It suffices to prove that every $a \in \text{suit}_r\,(R)$ is left suitable. Taking $p = 1$, $q = a$, $s = 1$, and $t = 1 - a$ (for which $pq + st = 1$), and using the assumption that $pq = a \in \text{suit}_r(R)$, we have from Theorem 2.5 an idempotent $e \in R\,a$ such that $1 - e \in R\,(1-a)$. Thus, by (the left version of) the Goodearl-Nicholson criterion, $a \in \text{suit}_\ell(R)$. $\square$

Having proved Proposition 3.1, we will have essentially no further need to distinguish the left suitable elements from the right suitable elements in a ring R. In particular, we will henceforth denote the common set $\text{suit}_\ell(R) = \text{suit}_r(R)$ simply by $\text{suit}\,(R)$.

A second immediate consequence of Theorem 2.5 is part (1) of the result below. The case $n = 1$ in part (1) is already interesting enough to merit a separate statement in part (2), since it amounts to a new "left-right mixed" characterization for suitable elements that has hitherto remained unknown. This characterization result is, in fact, a more precise "element-wise version" of Theorem B stated in the Introduction.

Theorem 3.2. (1) *For any element $a \in R$ and any integer $n \geq 1$, $a^n \in \text{suit}\,(R)$ iff there exists an idempotent $e \in a^{n-1}R\,a$ such that $1 - e \in (1 - a^n)\,R$, iff there exists an idempotent $e' \in aR\,a^{n-1}$ such that $1 - e' \in (1 - a)\,R\,(1 + a + \cdots + a^{n-1})$.*
(2) *An element $a \in R$ is suitable iff there exists an idempotent in $R\,a$ whose complementary idempotent is in $(1 - a)\,R$.*

Proof. (1) follows from Theorem 2.5 by using the commuting unimodular equations:

$$a^{n-1} \cdot a + (1 - a^n) \cdot 1 = 1 = a \cdot a^{n-1} + (1 - a)\,(1 + a + \cdots + a^{n-1}).$$

In the special case $n = 2$, a somewhat weaker statement of the "only if" part of (1) was proved in [NZ: Lemma 1], using the idea of "strong lifting".
(2) is simply a re-statement of the first "iff" conclusion of (1) in the case $n = 1$. □

We should point out that it is possible to prove part (2) of the result above independently of Theorem 2.5. For instance, a direct proof for the "only if" part of Theorem 3.2(2) was recently given in [KLN: Corollary 3.6]. The "if" part of Theorem 3.2(2) can also be proved from the classical viewpoint of [Ni_1] as follows. Assume there exists an idempotent $f \in (1 - a)\,R$ such that $1 - f \in R\,a$. For $b = 1 - a$, write $f = b\,r$ for some $r \in R$. After replacing r by $r\,f$ if necessary, we may assume that $r = r\,f\ (= r\,b\,r)$. Then $g := rb \in \text{idem}\,(R) \cap R\,b$. The left ideal $R\,g + R\,(1 - b)$ is R, since it contains both $1 - f$ and $b\,g + b\,r(1 - b) = b\,r\,b + f - b\,r\,b = f$. With $R\,g + R\,(1 - b) = R$ and $g \in R\,b$, Nicholson's "Criterion (3)" for (left) suitable elements in [Ni_1: Proposition 1.1] implies that $b \in \text{suit}\,(R)$, and hence also $a \in \text{suit}\,(R)$.

For specific types of suitable elements, e.g. π-regular elements, the "only if" part of Theorem 3.2(2) can be somewhat improved, as the following result shows. (Note that this result is nontrivial even for $m = 0$ and $m = 1$.)

Theorem 3.3. *For any π-regular element $a \in R$, there exists for every integer $m \geq 0$ an idempotent $e_m \in R\,a$ such that $1 - e_m \in \bigcap_{n \geq 0}\,[(1 - a)^n R\,(1 - a)^m]$. In particular, if $\bigcap_{n \geq 0}\,[(1 - a)^n R\,(1 - a)^m] = 0$ for some $m \geq 0$, then a is left invertible in R.*

Proof. Since some positive power a^k of a is regular and $Ra^k \subseteq Ra$, $(1 - a^k)^n R \subseteq (1 - a)^n R$, and $R(1 - a^k)^m \subseteq R(1 - a)^m$ (using the standard expansion of 1 minus a power of a), it suffices to deal with the case where a is regular. In this case, the desired conclusion holds even with the additional requirement that $R\,e_m = R\,a$. To begin with, write $a = axa$ for some $x \in R$. Given any $m \geq 0$, let $f_m = (1 - xa)\,(1 - a)^m$. In particular, $f_0 = 1 - xa$, and $f_m = f_0(1 - a)^m$. Since $af_0 = 0$, we have $af_m = 0$, and so $(1 - a)^n f_m = f_m$ for any $n \geq 0$. The latter implies that $f_m^2 = f_0(1 - a)^m f_m = f_0 f_m = f_m$, and that $f_m \in \bigcap_{n\geq 0} [(1 - a)^n R\,(1 - a)^m]$. It remains only to prove that the complementary idempotent $e_m := 1 - f_m$ satisfies $R\,e_m = R\,a$. As we have observed already, $f_0 f_m = f_m$, and it is easy to see that $f_m f_0 = f_0$. These two equations imply that $e_0 e_m = e_0$ and $e_m e_0 = e_m$, so we have $R\,e_m = R\,e_0 = R\,a$, as desired. If $\bigcap_{n\geq 0} [(1 - a)^n R\,(1 - a)^m] = 0$ for some $m \geq 0$, then $e_m = 1$. In this case, $R\,a = R$, so a is left invertible in R. □

Example 3.4. In Theorem 3.3, one may wonder if the conclusion can be improved to the existence of an idempotent $e \in R\,a$ such that $1 - e \in \bigcap_{n\geq 0} [(1 - a)^n R\,(1 - a)^n]$. It turns out that the answer to this question is "no", even in the case where the ring R itself is regular. To see this, we'll construct a counterexample to the stronger statement, *by switching left and right* (for convenience). Let R be the regular ring $\mathrm{End}_k(V)$, where V is a right vector space with a countably infinite basis $\{v_1, v_2, \dots\}$ over a field k. Let $b \in R$ be the shift operator defined by $b\,(v_i) = v_{i+1}$ for all $i \geq 1$, and let $a = 1 - b \in R$. Assume that there exists an idempotent $e \in aR$ such that $1 - e \in \bigcap_{n\geq 0} [(1 - a)^n R\,(1 - a)^n]$. Note that $\bigcap_{n\geq 0} (1 - a)^n R = \bigcap_{n\geq 0} b^n R = 0$, since $\bigcap_{n\geq 0} \mathrm{im}\,(b^n) = 0$. Therefore, $1 - e = 0$, and so $aR = R$. This is impossible.

Returning now to the Main Theorem 2.5, another typical application is the following.

Proposition 3.5. (1) *Let $a, b, c, d \in R$ be commuting elements such that $a^2c + b^2d = 1$. If $a^2c \in \mathrm{suit}\,(R)$ (or equivalently, $b^2d \in \mathrm{suit}\,(R)$), then there exists an idempotent $e \in aR\,a$ with $1 - e \in bR\,b$.*

(2) *Let R be an exchange ring, and let $a, b \in R$ be two commuting elements that are comaximal in the subring $S \subseteq R$ generated by a, b. Then there exists an idempotent $e \in aR\,a$ with $1 - e \in bR\,b$.*

Proof. (1) follows from Theorem 2.5 by taking the commuting elements $p = a$, $q = ac$, $s = b$, and $t = bd$ in the ring R, upon noting that, for these choices, $pRq \subseteq aR\,a$ and $sRt \subseteq bR\,b$.

(2) By assumption, $aS + bS = S$. Since S is a commutative ring, this implies that $a^2S + b^2S = S$, so there exist $c, d \in S$ such that $a^2c + b^2d = 1$. Now apply part (1) above, using the assumption that $R = \mathrm{suit}\,(R)$. □

The exchange ring assumption on R in (2) above was brought into play just to ensure that a^2c (or equivalently, b^2d) is suitable in R. In the cases where these elements can be explicitly computed, all we need to assume is that one of them be suitable in R (instead of R itself being an exchange ring). The following two-sided result, part (1) of which subsumes Theorem A in the Introduction, is a case in point.

Theorem 3.6. (1) *If* $a \in R$ *is such that* $(1-a^2)^2 \in \text{suit}\,(R)$*, or* a *is strongly clean* (*for instance,* a *is strongly* π*-regular*; *see* [Ni$_3$: Theorem 1]), *then there exists an idempotent* $e \in aRa$ *with* $1-e \in (1-a)\,R\,(1-a)$*. In particular, this conclusion holds for every element* a *in an exchange ring.*

(2) *If* $2\,a=0$*, then* $a^2 \in \text{suit}\,(R)$ *iff there exists an idempotent* $e \in aRa$ *with* $1-e \in (1-a)\,R\,(1-a)$*.*

Proof. (1) First assume that $(1-a^2)^2 \in \text{suit}\,(R)$. In the subring S of R generated by a, the comaximality of a and $1-a$ implies the comaximality of a^2 and $(1-a)^2$, with an explicit commuting unimodular equation $a^2(2-a^2)+(1-a)^2(1+a)^2=1$. Since

$$(1-a)^2(1+a)^2=(1-a^2)^2 \in \text{suit}\,(R),$$

Proposition 3.5(1) implies the existence of the idempotent e asserted in part (1).

Next, assume that $a \in R$ is strongly clean, so that $a=f+u$ for some $f \in \text{idem}\,(R)$ and some unit $u \in R$ such that $fu=uf$. Letting $e:=1-f \in \text{idem}\,(R)$, we have $ea=e\,(f+u)=eu$. Thus, $e=e^2=(eau^{-1})^2 \in aRa$ (since a commutes with both e and u). Replacing a, f, u by $1-a$, e and $-u$, we get also $1-e \in (1-a)\,R\,(1-a)$.

(2) If $2\,a=0$, we have a simpler commuting unimodular equation $a\cdot a+(1-a)\,(1-a)=1$. In this case, the conclusion of (2) follows directly from Theorem 2.5. □

The following two observations about part (1) of the above theorem are in order.

Remark 3.7. (A) Note that even if $a \in R$ is both unit-regular and clean, the conclusion in Theorem 3.6(1) may not hold. Indeed, in this case, there may not even exist an idempotent $e \in aRa$ such that $1-e \in (1-a)\,R$. For instance, in the ring $R=\mathbb{M}_2(\mathbb{Z})$, the matrix

$$(3.8)\qquad a=\begin{pmatrix}3&1\\0&0\end{pmatrix}=\begin{pmatrix}2&1\\-2&-1\end{pmatrix}+\begin{pmatrix}1&0\\2&1\end{pmatrix}=\begin{pmatrix}3&1\\0&0\end{pmatrix}\begin{pmatrix}1&-1\\-2&3\end{pmatrix}\begin{pmatrix}3&1\\0&0\end{pmatrix}$$

is clean and unit-regular. A straightforward computation shows that the only idempotent $e \in aRa$ is the zero matrix. But for this e, we have $1-e=1 \notin (1-a)\,R$ as $1-a$ is clearly not right-invertible in R. Here, the matrix $(1-a^2)^2=\begin{pmatrix}64&21\\0&1\end{pmatrix}$ is (necessarily) not suitable. On the other hand, if R is a ring in which $\text{suit}\,(R)$ is *closed under squaring,* then $a \in \text{suit}\,(R) \Rightarrow (1-a^2)^2 \in \text{suit}\,(R)$ (since $\text{suit}\,(R)$ is always closed under the map $x \mapsto 1-x$). In this case, the main conclusion of Theorem 3.6(1) (on the existence of the idempotent e) does hold for every $a \in \text{suit}\,(R)$.

(B) The *converse* of the main conclusion in Theorem 3.6(1) is not true in general. Indeed, let S be an exchange ring with an element b that is not clean, and let $a=(2,b)$ in $R=\mathbb{Z}\times S$. Take an idempotent $e_0 \in bSb$ such that $1_S-e_0 \in (1_S-b)\,S(1_S-b)$. Then $e:=(0,e_0) \in \text{idem}\,(R)$ satisfies $e \in aRa$ and $1-e \in (1-a)\,R\,(1-a)$. However, $(1-a^2)^2=(9,\,(1-b^2)^2) \notin \text{suit}\,(R)$, and $a=(2,b) \in R$ is not (strongly) clean.

§4. Suitable Endomorphisms on Modules

In this final section of the paper, we offer interpretations for some of our results on suitable elements in a module-theoretic setting. Throughout the section, M denotes a right module over some fixed ring k, and we'll study the suitable elements in the endomorphism ring $R := \mathrm{End}_k(M)$ (acting on the left of M) by using the language of modules. To begin with, we have the following easy lemma. (For the notions of quasi-projective and quasi-injective modules used in this lemma, see [AF: p. 191].)

Lemma 4.1. *For any $a, b \in R$, the following holds.*

(1) $b \in aR \Rightarrow \mathrm{im}\,(b) \subseteq \mathrm{im}\,(a)$. *If M_k is a quasi-projective module, the converse holds.*
(2) $b \in R\,a \Rightarrow \ker\,(a) \subseteq \ker\,(b)$. *If M_k is a quasi-injective module, the converse holds.*

Proof. (1) If $b = ar$ for some $r \in R$, then $b\,(m) = a\,(r(m))$ for any $m \in M$. Thus, $\mathrm{im}\,(b) \subseteq \mathrm{im}\,(a)$. Conversely, if $\mathrm{im}\,(b) \subseteq \mathrm{im}\,(a)$, then we have a k-module homomorphism $b : M \to a\,(M)$. If M is quasi-projective, by definition this homomorphism can be "lifted" (through the epimorphism $a : M \to a(M)$) to a k-endomorphism $r : M \to M$, so that $b = ar \in aR$.

(2) If $b \in R\,a$, clearly $a\,(m) = 0 \Rightarrow b\,(m) = 0$ (for any $m \in M$), so $\ker\,(a) \subseteq \ker\,(b)$. If M is assumed to be a quasi-injective module, the converse can be proved by using an argument that is "dual" to that used in the proof of (1). We leave out the details here since this part (2) will not be needed below. □

To study a given endomorphism $a \in \mathrm{End}_k(M)$, we shall use the following fixed notations for the rest of this section:

(4.2) $K = \ker\,(a), \quad I = \mathrm{im}\,(a), \quad K' = \ker\,(1-a), \quad \text{and} \quad I' = \mathrm{im}\,(1-a).$

Note that $K \subseteq I'$, since $m \in \ker\,(a)$ implies that $m = (1-a)\,(m) \in \mathrm{im}\,(1-a)$. Replacing a by $1-a$, we see also that $K' \subseteq I$.

Proposition 4.3. *For $R = \mathrm{End}_k(M)$ and any $a \in \mathrm{suit}\,(R)$, the following holds.*

(1) *M has a direct summand A such that $K' \subseteq A \subseteq I$.*
(2) *M has a direct summand B such that $K \subseteq B \subseteq I'$.*
(3) *There exists a k-decomposition $M = A_1 \oplus B_1$ such that $A_1 \subseteq I$ and $B_1 \subseteq I'$.*
(4) *There exists a k-decomposition $M = A_2 \oplus B_2$ such that $K' \subseteq A_2$ and $K \subseteq B_2$.*

Proof. (2) By Theorem 3.2(2), there exists an idempotent $e \in R\,a$ such that $1 - e \in (1-a)\,R$. Applying Lemma 4.1, we see that the direct summand $B := \ker\,(e) = \mathrm{im}\,(1-e)$ of M is between K and I'.

(1) This can be gotten from (2) above simply by replacing a by $1-a$.

(3) By the (right) version of the Goodearl-Nicholson criterion for $a \in \mathrm{suit}\,(R)$, there exists an idempotent $e_1 \in aR$ such that $1 - e_1 \in (1-a)R$. Applying Lemma 4.1, we see that $A_1 := \mathrm{im}\,(e_1) \subseteq I$, and $B_1 := \mathrm{im}\,(1-e_1) \subseteq I'$, with $M = A_1 \oplus B_1$.

(4) follows similarly (as in the proof of (3)) by using Lemma 4.1 in conjunction with the *left* version of the Goodearl-Nicholson criterion for $a \in \text{suit}\,(R)$. $\square$

Next, assuming M to be a *quasi-projective* module leads to the following new characterization of suitable elements in endomorphism rings.

Proposition 4.4. *For any quasi-projective module M_k, an endomorphism $a \in R = \text{End}_k(M)$ is suitable iff there exists a decomposition $M = A_1 \oplus B_1$ such that $A_1 \subseteq I = \text{im}\,(a)$ and $B_1 \subseteq I' = \text{im}\,(1-a)$.*

Proof. By Proposition 4.3(3), the "only if" part here holds even without M being quasi-projective. Conversely, suppose the decomposition $M = A_1 \oplus B_1$ exists as specified. Let $e_1 \in \text{idem}\,(R)$ be such that $\text{im}\,(e_1) = A_1$ and $\text{im}\,(1-e_1) = B_1$. Applying Lemma 4.1 to the endomorphisms a and e_1 on the quasi-projective module M shows that $e_1 \in aR$ and that $1 - e_1 \in (1-a)\,R$, so we have $a \in \text{suit}\,(R)$. $\square$

There is also an analogue of Proposition 4.4 that holds for endomorphisms on quasi-injective modules M. However, we need not state such an analogue since M in this case must be an exchange module according to a theorem of Fuchs [Fu]. In this case, there is a much stronger result below, the last part of which holds for *any* module with the finite exchange property. This result, conveniently expressed in the notations in (4.2), may be thought of as a nice module-theoretic application of part (1) of Theorem 3.6.

Theorem 4.5. *For any module M_k, let $a \in R = \text{End}_k(M)$ be such that $(1-a^2)^2 \in \text{suit}\,(R)$, or a is strongly clean in R. Then there exists a k-module decomposition $M = A \oplus B$ such that A is between K' and I, and B is between K and I'. In particular, this conclusion holds for every $a \in R$ if M_k is a module with finite exchange.*

Proof. By Theorem 3.6(1), there exists an idempotent $e \in aR \cap R\,a$ such that $1 - e \in (1-a)\,R \cap R\,(1-a)$. Letting $A = \text{im}\,(e)$ and $B = \ker\,(e)$ (with $M = A \oplus B$), we see from Lemma 4.1 that $K' \subseteq A \subseteq I$ and $K \subseteq B \subseteq I'$, as desired. For the last part of the theorem, assume that M_k has the finite exchange property. In this case, Warfield's results in [Wa] imply that $R = \text{End}_k\,(M)$ is an exchange ring (or equivalently, a suitable ring). Thus, the first part of the theorem applies to every $a \in R$. $\square$

Remark 4.6. (1) The conclusion of the above theorem in the case of a *strongly clean* endomorphism $a \in R$ is not new. Indeed, assume that $a = f + u$ where $f \in \text{idem}\,(R)$ and $u \in R$ is a unit such that $fu = uf$. Combining the proofs of Theorem 3.6(1) and Theorem 4.5, we know that, for $e = 1 - f$, $A := \text{im}\,(e)$ and $B := \ker\,(e)$ satisfy $M = A \oplus B$, with $K' \subseteq A \subseteq I$ and $K \subseteq B \subseteq I'$ (in the notations of (4.2)). On the other hand, going back to the idempotent f, we have $A = \ker\,(f)$ and $B = \text{im}\,(f)$, so the equation $M = A \oplus B$ is precisely the "$ABAB$-decomposition" of a with respect to the strongly clean representation $a = f + u$, in the sense of [CK] (after [Ni$_3$]). The inclusion relations $K' \subseteq A \subseteq I$ and $K \subseteq B \subseteq I'$ could have been checked directly without invoking Theorem 3.6(1) or Theorem 4.5.

(2) As we have mentioned in the Introduction, a special case of a being strongly clean is where $a \in R$ is *strongly π-regular.* The latter is well known to be equivalent to $a : M \to M$ being a *Fitting endomorphism* of M; that is, there exists an integer $n \geq 1$ such that $M = A \oplus B$ where $A = \mathrm{im}\,(a^n)$ and $B = \ker\,(a^n)$. Letting f be the projection of M onto B with kernel A, we can show easily that $af = fa$, and $a - f \in R$ is a unit. (This is why a is strongly clean in R.) Therefore, we see from (1) above that $K' \subseteq A \subseteq I$ and $K \subseteq B \subseteq I'$. Again, these inclusion relations could have been checked directly using just the definitions $A = \mathrm{im}\,(a^n)$ and $B = \ker\,(a^n)$ (with $M = A \oplus B$).

In view of the above remarks, we see that, in some sense, Theorem 4.5 (holding under the assumption that $(1 - a^2)^2 \in \mathrm{suit}\,(R)$) can be thought of as a partial analogue of the $ABAB$-decomposition in [Ni$_3$, CK], which is in turn a partial analogue of the classical Fitting decomposition in [Fi].

Finally, we note that the main conclusion of Theorem 4.5 applies to any linear operator a on a vector space M over a division ring k, since every vector space has the full exchange property. However, the conclusion of that theorem does not seem to be known even in this linear algebra setting, in case M_k is infinite-dimensional. Here, the operator $a : M \to M$ may not be strongly clean in R. Nevertheless, R is a regular ring, which makes it a rather special kind of an exchange ring.

Acknowledgements

This work was partially supported by a grant from the Simons Foundation (#315828 to Pace Nielsen). We thank the anonymous referee for providing several comments which improved the article.

References

[AF] F. W. Anderson and K. R. Fuller: *Rings and Categories of Modules.* Second Edition, Graduate Texts in Math., Vol. **13**, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[AG] P. Ara, K. Goodearl, K. C. O'Meara, and E. Pardo: *Separative cancellation for projective modules over exchange rings.* Israel J. Math. **105** (1998), 105–137.

[CK] V. Camillo, D. Khurana, T. Y. Lam, W. K. Nicholson, and Y. Zhou: *Continuous modules are clean.* J. Algebra **304** (2006), 94–111.

[Fi] H. Fitting: *Die theorie der Automorphismenringen Abelscher Gruppen und ihr Analogon bei nicht kommutativen Gruppen.* Math. Ann. **107** (1933), 514–542.

[Fu] L. Fuchs: *On quasi-injective modules.* Ann. Scuola Norm. Sup. Pisa **23** (1969), 541–546.

[Go] K. R. Goodearl: *Von Neumann Regular Rings.* Krieger Publishing Company, Malabar, Florida, 1991.

[GW] K. R. Goodearl and R. B. Warfield: *Algebras over zero-dimensional rings.* Math. Ann. **223** (1976), 157–168.

[KG] D. Khurana and R. N. Gupta: *Lifting idempotents and projective covers.* Kyungpook Math. J. **41** (2001), 217–227.

[KLN] D. Khurana, T. Y. Lam, and P. P. Nielsen: *Exchange elements in rings, and the equation* $XA - BX = I$. Trans. A.M.S., to appear.

[La] T. Y. Lam: *A First Course in Noncommutative Rings.* Second Edition, Graduate Texts in Math., Vol. **131**, Springer-Verlag, Berlin-Heidelberg-New York, 2001.

[vN_1] J. von Neumann: *Lectures in Continuous Geometry.* Planographed notes, Institute for Advanced Study, Edwards Brothers, Ann Arbor, 1937.

[vN_2] J. von Neumann: *Continuous Geometry.* Edited by I. Halperin, Princeton Univ. Press, Princeton, N.J., 1960.

[Ni_1] W. K. Nicholson: *Lifting idempotents and exchange rings.* Trans. A.M.S. **229** (1977), 269–278.

[Ni_2] W. K. Nicholson: *On exchange rings.* Comm. Alg. **25** (1997), 1917–1918.

[Ni_3] W. K. Nicholson: *Strongly clean rings and Fitting's lemma.* Comm. Algebra **27** (1999), 3583–3592.

[NZ] W. K. Nicholson and Y. Zhou: *Strong lifting.* J. Alg. **285** (2005), 795–818.

[Wa] R. B. Warfield: *Exchange rings and decompositions of modules.* Math. Ann. **199** (1972), 31–36.

2010 *AMS Subject Classification*: 16D40, 16D50, 16E50, 16U99.
Keywords: Two-sided properties, idempotents, exchange elements, suitable elements, exchange rings, suitable rings, endomorphism rings.

Department of Mathematics
Panjab University
Chandigarh 160 014, India
dkhurana@pu.ac.in

Department of Mathematics
University of California
Berkeley, CA 94720
lam@math.berkeley.edu

Department of Mathematics
Brigham Young University
Provo, UT 84602
pace@math.byu.edu