UNIQUELY CLEAN ELEMENTS IN RINGS

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Abstract. It is well known that every uniquely clean ring is strongly clean. In this paper, we investigate the question of when this result holds element-wise. We first construct an example showing that uniquely clean elements need not be strongly clean. However, in case every corner ring is clean the uniquely clean elements are strongly clean. Further, we classify the set of uniquely clean elements for various classes of rings, including semiperfect rings, unit-regular rings, and endomorphism rings of continuous modules.

Introduction

An element a in a ring R is called a *clean element* if a is a sum of a unit and an idempotent in R. We call $a = e + u$, with $e \in \mathrm{idem}(R)$ and $u \in \mathrm{U}(R)$, a *clean decomposition* for a. A ring is clean if every element has a clean decomposition. Clean rings were first studied by Nicholson [10] in connection with exchange rings and lifting of idempotents. Subsequently, they have been shown to have connections with strongly π-regular elements and Fitting decompositions [11], Boolean rings [12], and continuous modules [1]. Unit-regular rings are always clean [2, 3].

Following the literature, an element $a \in R$ is *strongly clean* if it has a clean decomposition $a = e + u$ in which $eu = ue$. An element is *uniquely clean* if it has exactly one clean decomposition. Strongly clean rings and uniquely clean rings are defined similarly. It turns out that uniquely clean rings are always strongly clean [12, Theorem 20]. This paper is dedicated to the question of when this result holds element-wise.

We will employ the following notations. We let $\mathrm{ucn}(R)$ denote the set of uniquely clean elements and $\mathrm{scn}(R)$ is the set of strongly clean elements. Other notations are standard: $\mathrm{U}(R)$ is the group of units, $\mathrm{idem}(R)$ is the set of idempotents, $Z(R)$ is the center of R, and $\mathrm{rad}(R)$ is the Jacobson radical of R.

1. Uniquely clean elements are not strongly clean

In practice, it is difficult to find uniquely clean elements which are not strongly clean. In the remaining sections of this paper we will see many reasons for this phenomenon. First, however, we provide an example showing that the uniquely clean elements are not always strongly clean.

Example 1.1. There exists a ring R such that $\mathrm{ucn}(R) \nsubseteq \mathrm{scn}(R)$.

Construction and proof. In $R = \mathbb{M}_2(\mathbb{Z})$, we claim that the clean matrix

$$A = \begin{pmatrix} 8 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 8 & 3 \\ -3 & -1 \end{pmatrix} \tag{1.2}$$

is uniquely clean. Consider any clean decomposition $A = E + U$ in R. Following the approach in [5], writing $E = \left(\begin{smallmatrix} p & q \\ x & y \end{smallmatrix}\right)$ we have $\det(E) = py - qx = 0$, $3x - 8y = \pm 1$, and $y \equiv 1 \pmod{x}$. If $3x - 8y = -1$, then $x = 5 + 8t$ and $y = 2 + 3t$ for some $t \in \mathbb{Z}$. But by inspection, $y - 1 = 1 + 3t = k(5 + 8t)$ has no

2010 *Mathematics Subject Classification.* Primary 16U99, Secondary 16D50, 16E50.
Key words and phrases. clean ring, strongly clean elements, uniquely clean elements.

integer solution (t,k). If $3x-8y=1$, then $x=3+8t$ and $y=1+3t$ for some $t\in\mathbb{Z}$, in which case (again by inspection) $y-1=3t=k(3+8t)$ has a unique integer solution $(t,k)=(0,0)$. This means that we must have $(x,y)=(3,1)$. But then $E^2=E$ implies $p=q=0$, so $E=\left(\begin{smallmatrix}0&0\\3&1\end{smallmatrix}\right)$ is uniquely determined. This proves that $A\in\mathrm{ucn}\,(R)$. Obviously, $A\notin\mathrm{scn}\,(R)$, since the decomposition (1.2) above is not a strongly clean decomposition. Using the same method, we can check (for instance) that replacing 8 above by 10 or 11 also leads to uniquely clean matrices which are not strongly clean. On the other hand, replacing 8 by 5 (for instance) leads to a matrix with exactly three clean decompositions, none of which is strongly clean; namely

$$\begin{pmatrix}5&3\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\2&1\end{pmatrix}+\begin{pmatrix}5&3\\-2&-1\end{pmatrix}=\begin{pmatrix}2&1\\-2&-1\end{pmatrix}+\begin{pmatrix}3&2\\2&1\end{pmatrix}=\begin{pmatrix}3&2\\-3&-2\end{pmatrix}+\begin{pmatrix}2&1\\3&2\end{pmatrix}.\qquad\square$$

2. Properties of uniquely clean decompositions

In this section we record some conditions which imply that in a uniquely clean decomposition the idempotent is central.

Lemma 2.1. *Let $a\in\mathrm{ucn}\,(R)$, with a (unique) clean decomposition $a=e+u$. If $x\in R$ satisfies both*
(i) $(eu^{-1}-u^{-1}e)x(1-e)\in\mathrm{rad}\,(R)$ *and*
(ii) $(1-e)x(eu^{-1}-u^{-1}e)\in\mathrm{rad}\,(R)$,
then $ex=xe$.

Proof. Let x satisfy (i) and (ii). We have

$$a=e+u=[e+ex(1-e)]+[u-ex(1-e)],\tag{2.2}$$

and note that $e+ex(1-e)\in\mathrm{idem}\,(R)$. We want to prove $u-ex(1-e)\in\mathrm{U}(R)$, which will show that the second decomposition in (2.2) is clean. From (i) we have $j:=ueu^{-1}x(1-e)-ex(1-e)\in\mathrm{rad}\,(R)$, and thus

$$u-ex(1-e)=u-ueu^{-1}x(1-e)+j=u[1-e(u^{-1}x)(1-e)]+j\in\mathrm{U}(R),$$

where $1-e(u^{-1}x)(1-e)\in\mathrm{U}(R)$ since $e(u^{-1}x)(1-e)$ squares to zero. As $a\in\mathrm{ucn}\,(R)$ but (2.2) gives two clean decompositions, those clean decompositions are equal. As the two clean decompositions in (2.2) must be the same, this implies that $ex(1-e)=0$, or in other words $ex=exe$. Using (ii), we can similarly show that $xe=exe$. Hence $ex=exe=xe$. $\square$

We now record various consequences of Lemma 2.1.

Corollary 2.3. *Keeping the notations above, the conclusion $ex=xe$ can be drawn in any of the following cases:*
(i) $eu-ue\in\mathrm{rad}\,(R)$,
(ii) $x\in\mathrm{rad}\,(R)$, *or*
(iii) *all square-zero elements are in* $\mathrm{rad}\,(R)$.

Proof. (i) If $eu-ue\in\mathrm{rad}\,(R)$, then $eu^{-1}-u^{-1}e\in\mathrm{rad}\,(R)$. (ii) This case is clear since $\mathrm{rad}\,(R)$ is an ideal. (iii) Recall the expression $a=e+u=[e+ex(1-e)]+[u-ex(1-e)]$. Note that $(ex(1-e))^2=0$, so $ex(1-e)\in\mathrm{rad}\,(R)$ by assumption. This shows that $u-ex(1-e)\in\mathrm{U}(R)$. We now finish as in Lemma 2.1. $\square$

Recall that a ring is *abelian* if $\mathrm{idem}\,(R)\subseteq Z(R)$. Following [6] one calls a ring *unit-central* if $\mathrm{U}(R)\subseteq Z(R)$. Notice that condition (i) in the previous corollary holds if a ring is abelian or unit-central. On the other hand, condition (iii) holds for any ring whose nilpotent elements form an ideal.

Lemma 2.4. *Assume that idempotents lift modulo an ideal $I\subseteq\mathrm{rad}\,(R)$. If $a\in\mathrm{ucn}\,(R)$, then $\overline{a}\in\mathrm{ucn}\,(R/I)$. The converse holds if idempotents lift uniquely modulo I.*

Proof. Let $a = e + u$ be the unique clean decomposition in R. Any clean decomposition of $\overline{a}$ in R/I can be written as $\overline{a} = \overline{f} + \overline{v}$ where $f \in \text{idem}(R)$ (since we are assuming idempotents lift modulo I) and $v \in \text{U}(R)$. Then $a = e + u = f + (v + j)$ for some $j \in I$. As these are clean decompositions for $a \in \text{ucn}(R)$, we have $e = f$, and hence $\overline{e} = \overline{f}$. This verifies that $\overline{a} \in \text{ucn}(R/I)$.

Conversely, assume $\overline{a} \in \text{ucn}(R/I)$. Consider two clean decompositions $a = e_1 + u_1 = e_2 + u_2$. As $\overline{a} = \overline{e}_1 + \overline{u}_1 = \overline{e}_2 + \overline{u}_2$ are clean decompositions, we must have $\overline{e}_1 = \overline{e}_2$. But idempotents lift uniquely, hence $e_1 = e_2$ and so a is uniquely clean. □

Example 2.5. The assumption on idempotents lifting in Lemma 2.4 is not superfluous.

Proof and construction. Letting $R = \{\frac{m}{n} \in \mathbb{Q} : \gcd(6, n) = 1\}$, then $\text{rad}(R) = 6R$. As R has only trivial idempotents it is easy to see that $2 \in \text{ucn}(R)$. But in $R/\text{rad}(R)$, $\overline{2} = \overline{1} + \overline{1} = \overline{3} + \overline{5}$ are two distinct clean decompositions. So $\overline{2} \notin \text{ucn}(R/\text{rad}(R))$. □

Corollary 2.6. *Let $a \in \text{ucn}(R)$, with clean decomposition $a = e + u$. If either*

(i) *the element a is strongly clean, or*

(ii) *idempotents lift modulo* $\text{rad}(R)$*, and* $\overline{a} \in \text{scn}(R/\text{rad}(R))$*,*

then $e \in Z(R)$.

Proof. (i) Because of the uniqueness of the clean decomposition for a, and as a is strongly clean by assumption, we must have $eu = ue$. Applying Corollary 2.3(i), we obtain $ex = xe$ for all $x \in R$, and so $e \in Z(R)$.

(ii) From the assumptions, Lemma 2.4 implies $\overline{a} \in \text{ucn}(R/\text{rad}(R))$ with unique clean decomposition $\overline{a} = \overline{e} + \overline{u}$. Since we also have $\overline{a} \in \text{scn}(R/\text{rad}(R))$,this yields $\overline{eu} = \overline{ue}$. It follows that $ex = xe$ for all $x \in R$, by Corollary 2.3(i). □

Theorem 2.7. *Let $a \in \text{ucn}(R)$ with clean decomposition $a = e + u$. Assume one of the following conditions:*

(i) *Idempotents lift modulo* $\text{rad}(R)$ *and* $R/\text{rad}(R)$ *is strongly clean* (*e.g., R is strongly clean*)*.*

(ii) *The ring R is right quasi-duo* (*i.e. every maximal right ideal is a two-sided ideal*)*.*

Then $e \in Z(R)$ and hence $\text{ucn}(R) \subseteq \text{scn}(R)$*.*

Proof. (i) This follows from Corollary 2.6.

(ii) We show $R/\text{rad}(R)$ is abelian, and the result then follows by applying Corollary 2.3(i). To see that $R/\text{rad}(R)$ is abelian, let $x + \text{rad}(R) \in \text{idem}(R/\text{rad}(R))$ and let I be a maximal right ideal of R. By hypothesis I is an ideal, so R/I is a division ring and hence $x + I \in R/I$ is central. As $R/\text{rad}(R)$ is a subdirect product of $\{R/I : I \text{ is a maximal ideal of } R\}$, this implies that $x + \text{rad}(R) \in R/\text{rad}(R)$ is central. □

The following corollary recaptures some known results (e.g. see [8] and [13]).

Corollary 2.8. *Given $f \in \text{idem}(R)$, we have $f \in \text{ucn}(R)$ if and only if $f \in Z(R)$.*

Proof. (⇐): This was originally noted by Nicholson and Zhou [12].

(⇒): This direction follows by applying Corollary 2.6(i) to $f = (1-f)+(2f-1)$, which is a strongly clean decomposition. □

Although we will get a more general result shortly, we record the following easy consequence of Theorem 2.7.

Proposition 2.9. *Let R be a semiperfect ring. If $a \in \text{ucn}(R)$ with clean decomposition $a = e+u$, then $e \in Z(R)$. In particular,* $\text{ucn}(R) \subseteq \text{scn}(R)$*.*

Proof. For a semiperfect ring R, idempotents lift modulo $\text{rad}(R)$. Also $R/\text{rad}(R)$ is semisimple and hence strongly clean. Now apply Theorem 2.7(i). □

3. Uniquely clean elements and central idempotents

We saw in the previous section that uniquely clean decompositions often involve a central idempotent. In this section we collect some further results in this direction. We begin with the following lemma which is an easy consequence of the main theorem of [4]. For more information on Morita contexts readers are directed to [7].

Lemma 3.1. *Let R and S be clean rings. Let ${}_RM_S$ be an R-S-bimodule, and let ${}_SN_R$ be an S-R-bimodule. Assume $T := \left(\begin{smallmatrix} R & M \\ N & S \end{smallmatrix}\right)$ is a Morita context. Any element $\alpha := \left(\begin{smallmatrix} r & x \\ y & s \end{smallmatrix}\right) \in T$ has a clean decomposition whose idempotent part has a diagonal Peirce decomposition $\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$.*

If α is uniquely clean in T, then r and s are uniquely clean in R and S, respectively. Moreover, if $r = e + u$ and $s = f + v$ are their clean decompositions in R and S, then

$$\begin{pmatrix} r & x \\ y & s \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} u & x \\ y & v \end{pmatrix}$$

is the clean decomposition of α in T.

Proof. Let $r = e+u$ and $s = f+v$ be clean decompositions in R and S respectively. Then $s-yu^{-1}x \in S$ and $r - xv^{-1}y \in R$. As R and S are clean, there exist decompositions $s - yu^{-1}x = g + w$ and $r - xv^{-1}y = h + z$, where $g^2 = g \in S$, $w \in \mathrm{U}(S)$, $h^2 = h \in R$, and $z \in \mathrm{U}(R)$. According to the proof of [4, Lemma 1]

$$\begin{aligned} \begin{pmatrix} r & x \\ y & s \end{pmatrix} &= \begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix} + \begin{pmatrix} u & x \\ y & w + yu^{-1}x \end{pmatrix} \\ &= \begin{pmatrix} h & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} z + xv^{-1}y & x \\ y & v \end{pmatrix} \end{aligned}$$

are clean decompositions in T. This proves the first half of the lemma.

For the second half, assume $\left(\begin{smallmatrix} r & x \\ y & s \end{smallmatrix}\right)$ is uniquely clean in T. The two clean decompositions we obtained above must be the same, and so it immediately follows that $h = e$ and $g = f$. Therefore, we have a clean decomposition

$$\begin{pmatrix} r & x \\ y & s \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} u & x \\ y & v \end{pmatrix}.$$

This equality holds for any clean decompositions $r = e + u$ and $s = f + v$, and thus r and s must both be uniquely clean. □

Remark 3.2. Let M_R be a module such that $M = X \oplus Y$. Let $\alpha \in \mathrm{End}(M_R)$ be the projection to X with kernel Y. If f is uniquely clean in $\mathrm{End}(M_R)$ and if $\mathrm{End}(X_R)$ and $\mathrm{End}(Y_R)$ are clean, then $\alpha f \alpha$ is uniquely clean in $\mathrm{End}(X_R)$. Indeed, we have

$$\mathrm{End}(M_R) = \begin{pmatrix} \mathrm{End}(X_R) & \mathrm{Hom}_R(Y, X) \\ \mathrm{Hom}_R(X, Y) & \mathrm{End}(Y_R) \end{pmatrix} \text{ and } f = \begin{pmatrix} \alpha f \alpha & * \\ * & * \end{pmatrix}.$$

Thus, the claim follows by Lemma 3.1. In particular, if X is f-invariant, then $f|_X : X \to X$ is uniquely clean.

Theorem 3.3. *Let R be a ring such that every corner of R is clean. If $a = e + u \in \mathrm{ucn}\,(R)$, then $e \in Z(R)$. In particular, $\mathrm{ucn}\,(R) \subseteq \mathrm{scn}\,(R)$.*

Proof. By symmetry it suffices to show $eR(1 - e) = 0$. Fix $x \in R$ and consider $f := e + ex(1 - e)$, an idempotent for which $ef = f$ and $fe = e$. Let's compute the Peirce decomposition of e with respect to the idempotent f: The fRf-component is $fef = ef = f$, and the $fR(1 - f)$-component is $fe(1 - f) = fe - fef = e - f$. Since $e = f + (e - f)$, the other two components must be 0 and so e has the "matricial" Peirce decomposition $\left(\begin{smallmatrix} f & e-f \\ 0 & 0 \end{smallmatrix}\right)$ with respect to f. But by Lemma 3.1, a has a clean decomposition whose idempotent part is of the form $\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$. Invoking now the uniqueness of the clean decomposition of $a \in \mathrm{ucn}\,(R)$ we get $e - f = 0$. In other words $ex(1-e) = 0$, which is what we needed. ☐

Definition 3.4. For a ring R, let $\mathrm{ucn}_0(R) := \{e + j \,:\, e^2 = e \in Z(R), j \in \mathrm{rad}\,(R)\}$.

Lemma 3.5. *For any ring R, we have* $\mathrm{ucn}_0(R) \subseteq \mathrm{ucn}\,(R) \cap \mathrm{scn}\,(R)$.

Proof. Set $a = e + j$ where $e^2 = e \in Z(R)$ and $j \in \mathrm{rad}\,(R)$. The clean decomposition $a = (1-e) + [(2e-1)+j]$ is strongly clean. If $a = f + v$ is an arbitrary clean decomposition, then $e = f + (v - j) = (1-e) + (2e-1)$ are two clean decompositions. As $e \in \mathrm{ucn}\,(R)$ by Corollary 2.8, $f = 1 - e$. This shows $a \in \mathrm{ucn}\,(R)$. ☐

The containment in this lemma is proper, in general. For example, the element $2 \in \mathbb{Z}$ is uniquely clean and strongly clean, but does not belong to $\mathrm{ucn}_0(\mathbb{Z})$.

Proposition 3.6. *Assume that idempotents lift modulo the Jacobson radical. If* $\mathrm{ucn}_0(R/\mathrm{rad}\,(R)) = \mathrm{ucn}\,(R/\mathrm{rad}\,(R))$*, then* $\mathrm{ucn}_0(R) = \mathrm{ucn}\,(R)$*. The converse holds if idempotents lift uniquely.*

Proof. Letting $a \in \mathrm{ucn}\,(R)$, by Lemma 2.4 we know $\overline{a} \in \mathrm{ucn}\,(R/\mathrm{rad}\,(R))$, and so $\overline{a} \in \mathrm{ucn}_0(R/\mathrm{rad}\,(R))$ by hypothesis. Hence $\overline{a}$ is a central idempotent of $R/\mathrm{rad}\,(R)$. As idempotents lift, there exists $e^2 = e \in R$ such that $j := a - e \in \mathrm{rad}\,(R)$. If $e = f + v$ is a clean decomposition for e, then $a = f + (v + j)$ is a clean decomposition. As $a = (1-e) + [(2e-1)+j]$ is a clean decomposition, $a \in \mathrm{ucn}\,(R)$ implies $f = 1 - e$. So $e = (1-e) + (2e-1)$ is the unique clean decomposition of e in R. This shows $e \in \mathrm{ucn}\,(R)$. By Corollary 2.8 we have $e \in Z(R)$, and thus $a = e + j \in \mathrm{ucn}_0(R)$.

The converse holds, under the extra assumption, by an application of Lemma 2.4. ☐

4. Endomorphism rings of continuous modules

We have seen that the uniquely clean property is intricately connected to the centrality of idempotents. In this section we will prove that there is also a connection to two-sided invertibility. For ease of notation, if $a = e + u$ is a clean decomposition we say that a is *e-clean*, or *clean with respect to e*. If this is the only clean decomposition for a, we say that a is *e-uniquely clean*.

Theorem 4.1. *If $a \in \mathrm{ucn}\,(R)$ is left (or right) invertible in R, then $a \in \mathrm{U}(R)$.*

Proof. Assume $ab = 1$, $ba \neq 1$ for some $b \in R$. Also assume $a = h + u$ where $h^2 = h \in R$, and $u \in \mathrm{U}(R)$ say with inverse $v \in R$. It suffices to show that a is not uniquely clean.

With respect to the orthogonal idempotents $e = 1 - b^2a^2$ and $f = b^2a^2$, the Peirce decomposition for a is

$$a = \begin{pmatrix} eae & eaf \\ 0 & faf \end{pmatrix}.$$

Notice that $faf = b^2a^3b^2a^2 = b^2aa^2 = b^2ha^2 + b^2ua^2$. We see that $fb^2 = b^2$ and $a^2f = a^2$. Thus, the element b^2ua^2 is a unit in fRf, with inverse b^2va^2. The element b^2ha^2 is an idempotent in fRf.

On the other hand, if $eae \in eRe$ has a clean decomposition, say $eae = g + w$, $g^2 = g \in eRe$ and $w \in \mathrm{U}(eRe)$, we then have

$$a = \begin{pmatrix} eae & eaf \\ 0 & faf \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & b^2ha^2 \end{pmatrix} + \begin{pmatrix} w & eaf \\ 0 & b^2ua^2 \end{pmatrix}.$$

This is a clean decomposition. The last summand is a unit because it is upper-triangular and the diagonal components are units in the corner rings. The penultimate summand is clearly an idempotent.

Thus, it suffices to show that eae has more than one clean decomposition in eRe. Notice that $e = 1-b^2a^2$ decomposes into a sum of orthogonal idempotents $e_1 = 1-ba$ and $e_2 = ba-b^2a^2 = b(1-ba)a$. With respect to these two idempotents we have the Peirce decomposition

$$eae = \begin{pmatrix} 0 & e_1ae_2 \\ 0 & 0 \end{pmatrix}.$$

Thus eae is clean with respect to $e = e_1 + e_2$. (As eae is nilpotent, $-e + eae$ is a unit.) It is also clean with respect to $e' = e_2 - e_2be_1$. A quick computation proves $e'^2 = e'$. Also $eae - (e_2 - e_2be_1) = -e_2 + (1-ba)a + b(1-ba)$ is a unit with inverse $e_1 + (1-ba)a + b(1-ba)$. Visually, we have

$$\begin{pmatrix} 0 & e_1ae_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -e_2be_1 & e_2 \end{pmatrix} + \begin{pmatrix} 0 & e_1ae_2 \\ e_2be_1 & -e_2 \end{pmatrix}.$$

(The last matrix acts essentially like the invertible matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, which is where the motivation for the construction came from.) □

The *clean index* of an element is the number of clean decompositions it has. The previous theorem implies the following nice corollary.

Corollary 4.2. *If a is a left (or right) invertible element which is not a unit, then it has clean index 0 or ∞.*

Proof. Suppose that a has clean index $n \geq 1$. By the proof of Theorem 4.1, and using the same notations there, the element faf has n clean decompositions, and eae has at least two clean decompositions. Putting these together, we have at least $2n$ clean decompositions for a, and so $2n = n$. This is only possible when $n = \infty$. □

The following lemma is useful when working with the corner rings relative to a central idempotent.

Lemma 4.3. *Let e be a central idempotent of R and assume $a \in R$ is e-uniquely clean. Then ea is 1_{eRe}-uniquely clean in eRe and $(1-e)a$ is 0-uniquely clean in $(1-e)R(1-e)$.*

Proof. This is a straightforward application of Lemma 3.1. □

With all of the machinery we have developed we are now ready to classify the uniquely clean elements for a large collection of rings.

Theorem 4.4. *Let R be a (von Neumann) regular ring for which every corner ring is clean. Then $\operatorname{ucn}(R) = \operatorname{idem}(R) \cap Z(R)$.*

Proof. One inclusion follows from Corollary 2.8. For the other inclusion, we first show that 0 is the only 1-uniquely clean element in R. Suppose $a \in R$ is 1-uniquely clean. We denote the right annihilator of a in R by $\mathbf{r}_R(a)$. We have to show that $\mathbf{r}_R(a) = R$. In any case, since R is regular we have $R = \mathbf{r}_R(a) \oplus L$ for some right ideal L (which we want to show is 0). There exists an idempotent $e \in R$ such that

$$L = eR \text{ and } \mathbf{r}_R(a) = (1-e)R.$$

As $a(1-e) = 0$, we have $a = eae + (1-e)ae$ and so $a' = eae$ is 1_{eRe}-uniquely clean in $S = eRe$ (by Lemma 3.1).

Notice that if $\mathbf{r}_S(a') = 0$, then (as S is a regular ring) we know a' is left invertible in S. So by Theorem 4.1, a' is invertible and is thus 0-clean also. This contradicts the fact that a' is uniquely clean, unless $e = 0$ in which case we are done. So we can reduce to the case where we have $e \neq 0$ and $\mathbf{r}_S(a') \neq 0$. Fixing $0 \neq x \in \mathbf{r}_S(a') \subseteq eRe$ we note that $ax \in (1-e)R$ and so

$$xR \cap axR \subseteq eR \cap (1-e)R = 0.$$

Let $X = xR \oplus axR$. As R is a regular ring, all finitely generated right ideals are direct summands, and in particular X is a summand of R_R. Furthermore, since $a = ae$ we have $a^2x = aeaex = aa'x = 0$. This proves that X is a-invariant. By Lemma 3.1 (and the remark that follows), $a|_X \in \mathrm{End}(X_R)$ is uniquely clean. Notice that we have two clean decompositions

$$a|_X = 1_X + (a|_X - 1_X) \text{ and } a|_X = g + (a|_X - g)$$

where $g \in \mathrm{End}(X_R)$ is defined as $g(x) = g(ax) = x$. (Note that g is a well-defined R-homomorphism, for if $axr = 0$ then $xr \in \mathbf{r}_S(a') \cap \mathbf{r}_R(a) \subseteq eR \cap (1-e)R = (0)$.) Thus $g = 1_X$ and so $x = g(ax) = ax$. This shows that $x = a^2x = 0$, a contradiction. We have thus proven that 0 is the only 1-uniquely clean element in R. It also follows that 1 is the only 0-uniquely clean element in R.

Now suppose $x \in R$ is uniquely clean. By Theorem 3.3, $x = e + u$ for some central idempotent e. So by Lemma 4.3, ex is 1_{eRe}-uniquely clean in eRe and $(1-e)x$ is 0-uniquely clean in $(1-e)R(1-e)$. As eRe and $(1-e)R(1-e)$ are regular rings in which every corner ring is clean, in view of above discussion this implies that $ex = 0$ and $(1-e)x = 1-e$. Thus $x = ex + (1-e)x = 1-e$ is a central idempotent in R. □

With this theorem in hand, we are able to study the uniquely clean elements in continuous and quasi-continuous modules. For basic facts and definitions regarding continuous and quasi-continuous modules see [9] and [7]. We will freely use results about such modules from those sources.

Theorem 4.5. *If M_R is a continuous module, then* $\mathrm{ucn}\,(\mathrm{End}(M_R)) = \mathrm{ucn}_0(\mathrm{End}(M_R))$.

Proof. Let $S = \mathrm{End}(M_R)$. Since $S/\mathrm{rad}\,(S)$ is right continuous, regular, and idempotents of $S/\mathrm{rad}\,(S)$ lift to idempotents of S (see [9, Chapter 3]), by Proposition 3.6 we can assume that S is right continuous and regular. As every corner of S is right continuous, and hence clean by [1, Main Theorem], we have $\mathrm{ucn}\,(S) = \mathrm{ucn}_0(S)$ by Theorem 4.4. □

Corollary 4.6. *Let M_R be a quasi-continuous module with the finite exchange property. Then every uniquely clean element of* $\mathrm{End}(M_R)$ *is the sum of a unit and a central idempotent.*

Proof. Let $S = \mathrm{End}(M_R)$ and $\Delta = \{f \in S : \ker(f) \leq_e M\}$. As S is an exchange ring, idempotents lift modulo $\mathrm{rad}\,(S)$ and modulo Δ. Furthermore $\Delta \subseteq \mathrm{rad}\,(S)$ and $S/\Delta = S_1 \times S_2$ where S_1 is right self-injective and S_2 is reduced clean. (For more details on these facts, see [9] and also the proof of [1, Theorem 4.3].) As S_2 is abelian and clean, it is strongly clean. We also know that $\mathrm{ucn}\,(S_1) \subseteq \mathrm{scn}\,(S_1)$ by the previous theorem. Therefore, we have $\mathrm{ucn}\,(S/\Delta) \subseteq \mathrm{scn}\,(S/\Delta)$. If $a \in \mathrm{ucn}\,(S)$, then $a + \Delta \in \mathrm{ucn}\,(S/\Delta)$ by Lemma 2.4, so it is strongly clean in S/Δ. As $\Delta \subseteq \mathrm{rad}\,(S)$, $a + \mathrm{rad}\,(S)$ is strongly clean in $S/\mathrm{rad}\,(S)$. By Corollary 2.6(ii), the element a is the sum of a unit and a central idempotent. □

Example 4.7. The equality $\mathrm{ucn}\,(\mathrm{End}(M_R)) = \mathrm{ucn}_0(\mathrm{End}(M_R))$ does not hold for quasi-continuous modules in general.

Construction and proof. Let $R = \mathbb{Z}$, and consider $R_R = \mathbb{Z}_{\mathbb{Z}}$ which is a quasi-continuous modules. Notice that $\mathrm{ucn}_0(R)$ is the set of central idempotents. The element $2 \in \mathbb{Z}$ is uniquely clean but it is not an idempotent. □

Theorem 4.8. *Assume that idempotents lift modulo* $\mathrm{rad}\,(R)$ *and that* $R/\mathrm{rad}\,(R)$ *is a unit-regular ring. Then* $\mathrm{ucn}\,(R) = \mathrm{ucn}_0(R)$.

Proof. By Theorem 4.4, $\mathrm{ucn}_0(R/\mathrm{rad}\,(R)) = \mathrm{ucn}\,(R/\mathrm{rad}\,(R))$. Now apply Proposition 3.6. □

Corollary 4.9. *If R is unit-regular, then* $\mathrm{ucn}\,(R) = \mathrm{idem}\,(R) \cap Z(R)$.

The next corollary directly strengthens Proposition 2.9.

Corollary 4.10. *If R is a semiperfect ring, then* $\mathrm{ucn}\,(R) = \mathrm{ucn}_0(R)$.

We conclude the paper by raising the following questions.

Questions 4.11.

(1) Let R be a clean ring. Is it true that $\mathrm{ucn}\,(R) \subseteq \mathrm{scn}\,(R)$?
(2) Let R be a regular ring. Is it true that each $a \in \mathrm{ucn}\,(R)$ is a central idempotent?
(3) If M_R is a quasi-continuous exchange module, do we have $\mathrm{ucn}\,(\mathrm{End}(M_R)) = \mathrm{ucn}_0(\mathrm{End}(M_R))$? Equivalently, if R is a reduced clean ring, do we have $\mathrm{ucn}\,(R) = \mathrm{ucn}_0(R)$? Note that for a reduced clean ring every corner ring is still clean and all idempotents are central.
(4) Can one classify the rings R for which the equality $\mathrm{ucn}\,(R) \cap \mathrm{scn}\,(R) = \mathrm{ucn}_0(R)$ holds?

Acknowledgments

This work was partially supported by a grant from the Simons Foundation (#315828 to Pace Nielsen). It was also supported by a Discovery Grant from NSERC of Canada (#194196 to Yiqiang Zhou).

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