ABELIAN EXCHANGE MODULES

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Abstract. Let M_k be a right k-module with endomorphism ring $E = End(M_k)$. We prove that if E is an Abelian exchange ring then M_k has the full exchange property. We also give an extension of this result in the case E is regular.

1. Introduction

In 1964, Crawley and Jónsson introduced a module-theoretic property which they called exchange. The definition is as follows:

Definition 1. A module M_k is said to have the $\aleph$-*exchange property* (for some cardinal $\aleph$) if, whenever we have

$$A = M \oplus N = \bigoplus_{i \in I} A_i$$

with $|I| \leqslant \aleph$, then there are submodules $A'_i \subseteq A_i$ such that

$$A = M \oplus \bigoplus_{i \in I} A'_i.$$

If a module, M, has $\aleph$-exchange for all cardinals $\aleph$, we say that M has *(full) exchange*. A module is said to have *finite exchange* when it has n-exchange for all $n \in \mathbb{Z}_+$. A straightforward argument shows that finite exchange is equivalent to 2-exchange. Further, if a module is finitely generated then finite exchange implies full exchange. It is easy to show that $\aleph$-exchange is preserved in finite direct sums, and in direct summands, of $\aleph$-exchange modules.

We call a ring, R, an *exchange ring* if R_R has the finite (hence full) exchange property. Warfield (1972) proved that this is a left-right symmetric notion, and that a module has finite exchange if and only if its endomorphism ring is an exchange ring. A few years later Nicholson (1977) studied the exchange property by studying what he called *suitable* rings. We say a ring, R, is suitable if for each $a \in R$ there is some idempotent $e \in Ra$ with $1 - e \in R(1 - a)$. Nicholson showed that this is a left-right symmetric notion, and that a ring is suitable if and only if it is an exchange ring.

Using these results it is easy to see that all π-regular rings, clean rings (every element is the sum of a unit and an idempotent), semiperfect rings, and semiregular rings ($R/J(R)$ is regular, and idempotents lift modulo $J(R)$) are exchange rings. In particular, all local rings and regular rings have exchange.

A thorny unanswered question is: for all modules, does finite exchange imply full exchange? The answer is yes for modules that are (arbitrary) direct sums of indecomposible modules by (Zimmermann-Huisgen and Zimmermann, 1984) and for quasi-continuous modules by (Oshiro and Rizvi, 1996) and (Mohamed and Müller, 1995). Yu proved that an Abelian module with finite exchange has *countable* exchange (Yu, 1994), and recently this was extended to *full* exchange in (Dhonpongsa and Tansee, 2003). There seems to be an error in the proof, which we describe in §4. But before we get to that, in §§2 − 3 we set up some machinery to give an independent proof, which avoids the complication in §4, thus recovering their claim. In §5 we study a class of modules, which we call finitely complemented modules, that includes the Abelian modules. Finally, we show that finitely complemented modules whose endomorphism rings are regular have full exchange.

2. Some Conventions

Throughout this paper we let k and R be rings, M_k be a right k-module, and $E = End(M_k)$. If we have two modules M and N we write $N \subseteq^{\oplus} M$ to mean that N is a direct summand of M. Given a well-ordered set, I, we may refer to its elements as ordinals.

Definition 2. We call a family $(x_i)_{i\in I}$ of endomorphisms in E *summable* if, for each $m \in M$, $x_i(m) = 0$ for all but finitely many $i \in I$. We write $\sum_{i\in I} x_i$ for the endomorphism that has the obvious action $m \mapsto \sum_{i\in I} x_i(m)$.

With this concept in hand we are ready to state a key proposition.

Proposition 1. *The following are equivalent:*

(1) M has the $\aleph$-exchange property.

(2) If we have

$$A = M \oplus N = \bigoplus_{i\in I} A_i$$

with $A_i \cong M$ for all $i \in I$, and $|I| \leqslant \aleph$, then there are submodules $A'_i \subseteq A_i$ such that

$$A = M \oplus \bigoplus_{i\in I} A'_i.$$

(3) Given a summable family $(x_i)_{i\in I}$ of elements of E, with $|I| \leqslant \aleph$, and with $\sum_{i\in I} x_i = 1$, there are orthogonal idempotents $e_i \in Ex_i$ with $\sum_{i\in I} e_i = 1$.

Proof. This is proven by Zimmermann-Huisgen and Zimmermann (1984, Proposition 3). □

3. Abelian Modules

A ring is *Abelian* if all its idempotents are central. A module is *Abelian* if its endomorphism ring is an Abelian ring. We have the following alternate description of this property.

Lemma 1. *A module M_k is Abelian if and only if all direct summands of M are fully invariant.*

Proof. (⇒): Fix some $N \subseteq^{\oplus} M$. Let $p \in E = End(M_k)$ be some splitting of this inclusion, and let $f \in E$ be an arbitrary element. Fix some element $n \in N$. Then $f(n) = f(p(n)) = p(f(n)) \in N$ since p is an idempotent (and hence central). Thus, N is fully invariant. Since N was arbitrary, all direct summands are fully invariant.

(⇐): Now, suppose that all direct summands are fully invariant. Let $e \in E$ be an idempotent, and let $f \in E$ be arbitrary. Fix $m \in M$. Since $M = e(M) \oplus (1-e)(M)$, we can write $m = m_1 + m_2$ with $m_1 \in eM$ and $m_2 \in (1-e)M$. By hypothesis, $f(m_1) \in eM$ and $f(m_2) \in (1-e)M$. So we calculate

$$ef(m) = ef(m_1) + ef(m_2) = f(m_1) = f(e(m_1) + e(m_2)) = fe(m).$$

So $ef = fe$, since m was arbitrary. Thus M is Abelian. □

We will eventually attempt to take advantage of condition (3) of Proposition 1. To do so, we need the following lemma, which basically follows the proof of (Nicholson, 1977, Prop. 1.11), but for completeness we include the argument here.

Lemma 2. *Let $e_1+x_2+x_3 = 1$ be an equation in an Abelian, suitable ring, R, with e_1 an idempotent. Then there are orthogonal idempotents $e_2 \in Rx_2$ and $e_3 \in Rx_3$ such that $e_1 + e_2 + e_3 = 1$.*

Proof. Let $f = 1 - e_1$. Then multiplying the equation $e_1 + x_2 + x_3 = 1$ on the left and right by f yields

$$fx_2f + fx_3f = f$$

since $fe_1f = 0$ and since f is an idempotent. Keep in mind that f is central. Now, since fRf is also a suitable ring (Nicholson, 1977, Prop. 1.10), there are orthogonal idempotents $e_2 \in fRf(fx_2f) = Rfx_2 \subseteq Rx_2$ and $e_3 \in fRf(fx_3f) \subseteq Rx_3$ with $e_2 + e_3 = f = 1 - e_1$. □

Notice that e_1 is also orthogonal to e_2 and e_3. Now we are ready to prove:

Theorem 2. *If M has the finite exchange property and E is Abelian then M has the full exchange property.*

Proof. Let $(x_i)_{i\in I}$ be an arbitrary, summable family of elements of E with $\sum_{i\in I} x_i = 1$. By Proposition 1 we see that it suffices to find orthogonal idempotents $e_i \in Ex_i$ (which, then, are summable) with $\sum_{i\in I} e_i = 1$. We may assume that I is well-ordered with first element 1 and last element κ.

Since M has the finite exchange property, E is suitable. So, there are orthogonal idempotents $e_1 \in Ex_1$ and $f_1 \in E\left(\sum_{j>1} x_j\right)$ with $e_1 + f_1 = 1$. Suppose, by transfinite induction, that for each $\beta < \alpha$ we have constructed elements $e_\beta \in Ex_\beta$ and $f_\beta \in E\left(\sum_{j>\beta} x_j\right)$, such that

$$\sum_{j\leqslant\beta} e_j + f_\beta = 1$$

and the elements in the collection $\{e_j\ (j \leqslant \beta), f_\beta\}$ are pairwise orthogonal idempotents. Notice, this implies that $\{e_j\}_{j<\alpha}$ is a summable orthogonal set of idempotents, and $\sum_{j<\alpha} e_j$ is also an idempotent.

We ask the question: is $\left(1 - \sum_{j<\alpha} e_j\right) \in E\left(\sum_{j\geqslant\alpha} x_j\right)$? If α has a predecessor in I then this follows from our inductive assumption. So, assume that α is a limit ordinal. We will show that $y = \sum_{j\geqslant\alpha} x_j$ is an automorphism on $\left(1 - \sum_{j<\alpha} e_j\right) M$, and this will answer our question in the affirmative, since then there is some $y' \in E$ so that

$$\left(1 - \sum_{j<\alpha} e_j\right) = y'y\left(1 - \sum_{j<\alpha} e_j\right) = y'\left(1 - \sum_{j<\alpha} e_j\right) y \in Ey.$$

Let $m \in \left(1 - \sum_{j<\alpha} e_j\right) M$. Since $(x_i)_{i\in I}$ is summable and α is a limit ordinal there is some $\beta < \alpha$ such that, for each $\gamma \in [\beta, \alpha)$, we have $x_\gamma(m) = 0$. Write $f_\beta = z\left(\sum_{j>\beta} x_j\right)$ for some $z \in E$, and set $e = \sum_{j\leqslant\beta} e_j$. We then have a sum

$$1 = \sum_{j\leqslant\beta} e_j + f_\beta = e + z\left(\sum_{\beta<j<\alpha} x_j\right) + zy.$$

Applying Lemma 2, we obtain orthogonal idempotents

$$g_1 \in E\left(z \sum_{\beta<j<\alpha} x_j\right), \qquad g_2 \in E(zy)$$

(that are also orthogonal to e) with $e + g_1 + g_2 = 1$. We note that since $m \in \left(1 - \sum_{j<\alpha} e_j\right) M$ we have $e(m) = 0$, and from the choice of β we have $g_1(m) = 0$. Therefore $g_2(m) = m$. Write $g_2 = wy$, for some $w \in E$. Replacing w by wyw if necessary, we may further assume that yw is an idempotent.

Remembering that idempotents commute, we calculate

$$yw = ywyw = (wy)yw = wy(yw) = wywy = wy = g_2.$$

Consider the element $m' = w(m)$. Since E is Abelian, direct summands of M are fully invariant. So

$$m \in \left(1 - \sum_{j<\alpha} e_j\right) M \implies m' = w(m) \in \left(1 - \sum_{j<\alpha} e_j\right) M.$$

Further, $y(m') = yw(m) = g_2(m) = m$, and so y is surjective on the summand $\left(1 - \sum_{j<\alpha} e_j\right) M$. Also, if $y(m) = 0$ then $m = g_2(m) = wy(m) = 0$, so y is injective.

We have now shown $\left(1 - \sum_{j<\alpha} e_j\right) \in E\left(\sum_{j\geqslant\alpha} x_j\right)$. So there is some $r \in E$ such that

$$r\left(\sum_{j\geqslant\alpha} x_j\right) = \left(1 - \sum_{j<\alpha} e_j\right).$$

Then

$$1 = \left(\sum_{j<\alpha} e_j\right) + \left(1 - \sum_{j<\alpha} e_j\right) = \left(\sum_{j<\alpha} e_j\right) + r x_\alpha + r\left(\sum_{j>\alpha} x_j\right)$$

where $\sum_{j>\alpha} x_j = 0$ if $\alpha = \kappa$. Applying Lemma 2 once more, we obtain orthogonal idempotents

$$e_\alpha \in Erx_\alpha \subseteq Ex_\alpha, \qquad f_\alpha \in Er\left(\sum_{j>\alpha} x_j\right) \subseteq E\left(\sum_{j>\alpha} x_j\right)$$

(also both orthogonal to $\sum_{j<\alpha} e_j$) with

$$\left(\sum_{j<\alpha} e_j\right) + e_\alpha + f_\alpha = 1.$$

Notice that if an idempotent of E is orthogonal to $\sum_{j<\alpha} e_j$ then it is orthogonal to each e_j, $j < \alpha$. This finishes the inductive definition of the orthogonal idempotents e_i, for $i \in I$. It is clear that $\sum_{i\in I} e_i = 1$ with $e_i \in Ex_i$. □

Corollary 1. *A module whose endomorphism ring is strongly regular has the full exchange property.*

4. Cardinality Arguments

In their attempt to prove that Abelian modules have $\aleph$-exchange (for arbitrary $\aleph$), Dhompongsa and Tansee use a set theoretic construction, found in (Dhompongsa and Tansee, 2003, p. 865), that seems to break down if $\aleph > \aleph_1$. We describe this problem below.

Let I be a well-ordered set. We assume that $|I| > \aleph_1$, so I has an element which is preceded by uncountably many other elements of I. Put $\alpha \in I$ to be the least such element. For each $i \in I$ which is not a limit ordinal, set $i^- = i - 1$. Suppose, by way of contradiction, that for every limit ordinal $i \in I$ we can pick $i^- \in I$ such that: (1) $i^- < i$ and (2) $\sup\{j < i : \inf\{\ell^- : i > \ell > j\} = j\} = i$. (Note: Our condition (2) follows easily from (Dhompongsa and Tansee, 2003, Eq. 3.1), since $\inf\{\ell^- : \ell > j\} \leqslant (j + 1)^- = j$.)

Now, put $X = \{j < \alpha : \inf\{\ell^- : \alpha > \ell > j\} = j\}$. By hypothesis, $\sup(X) = \alpha$, and also $\alpha \notin X$. For each $x \in X$ put $Y_x = \{i \in I : i < x\}$, and put $Y = \{i \in I : i < \alpha\}$. From our hypothesis on α, we see that Y_x has only countably many elements, for each x. For any $\beta < \alpha$, since $\lim_{x\in X} x = \alpha$ we have $\beta < x_\beta$ for some $x_\beta \in X$. Therefore $\beta \in Y_{x_\beta}$, and so $Y = \bigcup_{x\in X} Y_x$. But Y is uncountable (from the definition of α), and hence cannot be a countable union of countable sets. Therefore X must be uncountable.

Let $x_1 < x_2 < x_3 < \dots$ be the first countably many elements of X, and let $\gamma = \lim_{i\in\mathbb{Z}_+} x_i$. We have $\gamma < \alpha$ since X is uncountable. Also, notice that γ is a limit ordinal. Hence $\gamma^- < \gamma$ implies $\gamma^- < x_N$ for some N. But then $\inf\{\ell^- : \alpha > \ell > x_N\} \leqslant \gamma^- < x_N$, contrary to the fact that $x_N \in X$.

This contradiction shows that one cannot choose i^- for each $i \in I$ satisfying the conditions above, when $|I| > \aleph_1$. But Dhompongsa and Tansee use this construction to show that Abelian modules have $|I|$-exchange. Thus, it appears that their proof breaks down if $|I|$ is large.

5. Some Generalizations

In §3 we saw that given an equation $\sum_{i\in I} x_i = 1$ in E, where E is an Abelian exchange ring, then one can pick orthogonal idempotents $e_i \in Ex_i$, one at a time, that sum to 1. This method is too straightforward to work in much generality. However, one can modify the process by introducing some type of finiteness condition on the module in question.

Definition 3. Consider a module M_k. If, for each $A \subseteq^\oplus M$, there are only finitely many submodules $B_1, \dots, B_n \subseteq M$ with $A \oplus B_1 = A \oplus B_2 = \cdots = A \oplus B_n = M$, then we say that M is *finitely complemented*.

Another way to phrase this property is that every direct summand of M has only finitely many complements in M. In an Abelian module, every summand has only one complement, so Abelian modules are finitely complemented. We need to put this property in the language of idempotents, so as to utilize Proposition 1. The correct notion is as follows:

Definition 4. Let e and e' be idempotents in a ring R. We say that e and e' are *left strongly isomorphic* when $e'e = e'$ and $ee' = e$. We write this as $e \sim e'$.

It is clear (by dualizing the definition) what is meant by saying idempotents are right strongly isomorphic. Notice that if idempotents are left (or right) strongly isomorphic then they are isomorphic. Also, idempotents are both left and right strongly isomorphic if and only if they are equal. A quick calculation shows that the relation of being left strongly isomorphic is an equivalence relation.

We have other characterizations of left strongly isomorphic idempotents:

Lemma 3. *Let e and e' be idempotents in a ring R. The following are equivalent:*

(1) e and e' are left strongly isomorphic.
(2) $Re = Re'$.
(3) $e' = e + (1-e)re$ for some $r \in R$.
(4) $e' = ue$ for some $u \in U(R)$.
(5) $(1-e)R = (1-e')R$.

Furthermore, if $R = End(M_k)$ for some module M_k, then the following properties are also equivalent to the ones above:

(6) $\ker(e) = \ker(e')$.
(7) $(1-e)M = (1-e')M$.

Proof. The equivalence of properties (1)-(5) is (Lam, 2001, Exercise 21.4). (6) $\Leftrightarrow$ (7) is trivial, and (1) $\Leftrightarrow$ (6) is an easy calculation. $\square$

Corollary 2. *M is finitely complemented if and only if every idempotent in E is left strongly isomorphic to only finitely many other idempotents.*

Proof. Clear from Lemma 3, using (1) $\Leftrightarrow$ (7). $\square$

Since we are working without the Abelian assumption we need a modified version of Lemma 2.

Lemma 4. *Let $x_1+x_2+x_3 = 1$ be an equation in a suitable ring, R, where x_1 is an idempotent. Then there are pair-wise orthogonal idempotents $e_1 \in Rx_1$, $e_2 \in Rx_2$, and $e_3 \in Rx_3$, such that $x_1e_2 = x_1e_3 = 0$, $e_1 + e_2 + e_3 = 1$, and $x_1 \sim e_1$.*

Proof. Let $f = 1-x_1$, and multiply by f on the left and right of $x_1+x_2+x_3 = 1$ to obtain $fx_2f + fx_3f = f$. Since fRf is suitable, there are orthogonal idempotents $f_2 \in fRf(fx_2f)$ and $f_3 \in fRf(fx_3f)$, with $f_2 + f_3 = f$. Write $f_2 = fr_2fx_2f$ and $f_3 = fr_3fx_3f$ for some $r_2, r_3 \in R$.

Let $e_2 = f_2r_2fx_2 \in Rx_2$ and let $e_3 = f_3r_3fx_3 \in Rx_3$. Notice that $e_2 = fe_2$ and $e_3 = fe_3$. Hence $x_1e_2 = x_1e_3 = 0$ as we wanted. Further, e_2 and e_3 are orthogonal idempotents (easy calculation). Let $e_1 = 1 - e_2 - e_3$, so e_1 is orthogonal to e_2 and e_3, and we also obtain $e_1 + e_2 + e_3 = 1$.

We calculate

$$\begin{aligned} e_1x_1 = (1-e_2-e_3)(1-f) &= 1 - f - e_2 - e_3 + e_2f + e_3f \\ &= e_1 - f + f_2 + f_3 = e_1 - f + f = e_1. \end{aligned}$$

Therefore $e_1 \in Rx_1$. Finally, $x_1e_1 = x_1(1-e_2-e_3) = x_1$. $\square$

Lemma 5. *Let R, x_1, x_2, x_3, e_1, e_2, and e_3 be as in the previous lemma. Also suppose that $x_1 = \sum_{i\in I} g_i$, where $(g_i)_{i\in I}$ is a summable collection of orthogonal idempotents. Then $(e_1 g_i)_{i\in I}$ is a summable collection of orthogonal idempotents with $e_1 g_i \sim g_i$ for each $i \in I$. Also, if an idempotent is orthogonal to e_1 it is orthogonal to $e_1 g_i$ for each $i \in I$.*

Proof. By orthogonality, we have the equations $g_i x_1 = g_i = x_1 g_i$. Also $x_1 e_1 = x_1$. Therefore

$$e_1 g_i e_1 g_j = e_1(g_i x_1)e_1 g_j = e_1 g_i (x_1 e_1) g_j = e_1 g_i x_1 g_j = e_1 g_i g_j = \delta_{i,j} e_1 g_i$$

so they are orthogonal idempotents. Also $g_i(e_1 g_i) = (g_i x_1)(e_1 g_i) = g_i x_1 g_i = g_i$ and $(e_1 g_i) g_i = e_1 g_i$, so $e_1 g_i \sim g_i$. The last part is another easy calculation. □

With all of this machinery we are ready to prove:

Theorem 3. *Suppose that M is a finitely complemented module with finite exchange. Then M has countable exchange.*

Proof. Let $(x_i)_{i\in\mathbb{Z}_+}$ be a summable collection of elements of E with $\sum_{i=1}^{\infty} x_i = 1$. As before, it suffices to *lift* this equation to one involving orthogonal idempotents. To ease notation, let $y_i = \sum_{j>i} x_j$, for each i. Since E is suitable there are orthogonal idempotents $e_{1,1} \in Ex_1$ and $f_1 \in Ey_1$ with $e_{1,1} + f_1 = 1$.

Suppose, by induction, we have fixed elements $e_{i,j} \in Ex_i$ and $f_j \in Ey_j$ for each $j \leqslant n$, and for each $i \leqslant j$, such that the following conditions hold: (i) $e_{1,j} + e_{2,j} + \cdots + e_{j,j} + f_j = 1$ for each $j \leqslant n$, (ii) the family $\{e_{1,j}, e_{2,j}, \ldots, e_{j,j}, f_j\}$ consists of orthogonal idempotents for each $j \leqslant n$, and (iii) $e_{i,j} \sim e_{i,\ell}$ for all $j, \ell \leqslant n$ $(j, \ell \geqslant i)$.

Write $f_n = r y_n$. We have

$$1 = e_{1,n} + \cdots + e_{n,n} + f_n = (e_{1,n} + \cdots + e_{n,n}) + r x_{n+1} + r y_{n+1}.$$

Lemma 4 allows us to pick orthogonal idempotents

$$z_1 \in E(e_{1,n} + \cdots + e_{n,n}), \qquad z_2 \in Erx_{n+1}, \qquad z_3 \in Ery_{n+1}$$

with $z_1 + z_2 + z_3 = 1$. Then let $f_{n+1} = z_3 \in Ey_{n+1}$, $e_{n+1,n+1} = z_2 \in Ex_{n+1}$, and $e_{i,n+1} = z_1 e_{i,n} \in Ex_i$ for each $i < n+1$. Clearly,

$$e_{1,n+1} + \cdots + e_{n+1,n+1} + f_{n+1} = 1.$$

Lemma 5 implies that $\{e_{1,n+1}, \ldots, e_{n+1,n+1}, f_{n+1}\}$ is a family of orthogonal idempotents. Also by Lemma 5, $e_{i,n} \sim e_{i,n+1}$. Therefore, by transitivity, $e_{i,j} \sim e_{i,\ell}$ for all $j, \ell \leqslant n+1$ $(j, \ell \geqslant i)$. Thus conditions (i), (ii), and (iii) above hold for the case $n+1$.

By induction, we have constructed idempotents $e_{i,j} \in Ex_i$, $f_j \in Ey_j$ for each $j \in \mathbb{Z}_+$ and each $i \leqslant j$, such that conditions (i), (ii), and (iii) above hold for all j.

Since M is finitely complemented, each idempotent is left strongly isomorphic to only finitely many other idempotents. But $e_{1,j} \sim e_{1,\ell}$ for any j, ℓ. Therefore, there is an infinite sequence $\alpha^1 : \alpha^1_1 < \alpha^1_2 < \alpha^1_3 < \ldots$ of positive integers, with $e_{1,\alpha^1_j} = e_{1,\alpha^1_\ell}$ for all $j, \ell \in \mathbb{Z}_+$.

Then, by the same reasoning, there is a subsequence $\alpha^2 : \alpha^2_1 < \alpha^2_2 < \ldots$ of the sequence α^1, such that $e_{2,\alpha^2_j} = e_{2,\alpha^2_\ell}$ for all $j, \ell \in \mathbb{Z}_+$. We write $\alpha^1 \supseteq \alpha^2$ to denote that α^2 is a subsequence of α^1.

Repeating this idea, we obtain a chain of sequences $\alpha^1 \supseteq \alpha^2 \supseteq \dots$ with $e_{n,\alpha^n_j} = e_{n,\alpha^n_\ell}$ for all $j, \ell \in \mathbb{Z}_+$. Put $e_i = e_{i,\alpha^i_1} \in Ex_i$ for each $i \in \mathbb{Z}_+$. Then

$$e_i e_{i'} = e_{i,\alpha_1^{\max\{i,i'\}}} e_{i',\alpha_1^{\max\{i,i'\}}} = \delta_{i,i'} e_i$$

where δ is the Kronecker delta function. Thus, these are orthogonal idempotents.

All that now remains is to show $\sum_{i=1}^\infty e_i = 1$. Fix some $m \in M$. Since $(x_i)_{i\in\mathbb{Z}_+}$ is a summable family, there is some $N \in \mathbb{Z}_+$ such that $x_i(m) = 0$ for all $i \geqslant N$. In particular $f_i(m) = 0$, $e_{i,j}(m) = 0$, and $e_i(m) = 0$ for all $i \geqslant N$. We calculate

$$\begin{aligned}\left(\sum_{i=1}^\infty e_i\right)(m) = \sum_{i=1}^N e_i(m) = \sum_{i=1}^N e_{i,\alpha_1^N}(m) \\ = \sum_{i=1}^{\alpha_1^N} e_{i,\alpha_1^N}(m) + f_{\alpha_1^N}(m) = \mathrm{id}(m) = m\end{aligned}$$

(where we implicitly used $\alpha_1^N \geqslant N$). Therefore $\sum_{i=1}^\infty e_i = 1$. □

Example. Let $M_\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}_{(p)}$, where p is a prime number. Then

$$E = End_\mathbb{Z}(M) \cong \begin{pmatrix} End_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z}) & Hom_\mathbb{Z}(\mathbb{Z}_{(p)}, \mathbb{Z}/p\mathbb{Z}) \\ Hom_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}_{(p)}) & End_\mathbb{Z}(\mathbb{Z}_{(p)}) \end{pmatrix}.$$

Since $\mathbb{Z}/p\mathbb{Z}$ is a torsion module, but $\mathbb{Z}_{(p)}$ is torsion-free, we have $Hom_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}_{(p)}) = 0$. Therefore,

$$E \cong \begin{pmatrix} \mathbb{Z}/p\mathbb{Z} & Hom_\mathbb{Z}(\mathbb{Z}_{(p)}, \mathbb{Z}/p\mathbb{Z}) \\ 0 & \mathbb{Z}_{(p)} \end{pmatrix}.$$

We write $\varphi_n \in Hom_\mathbb{Z}(\mathbb{Z}_{(p)}, \mathbb{Z}/p\mathbb{Z})$ for the endomorphism that sends 1 to $n \pmod p$ (for $n \in [0, p-1]$). Notice these are well-defined homomorphisms, and exhaust all elements of $Hom_\mathbb{Z}(\mathbb{Z}_{(p)}, \mathbb{Z}/p\mathbb{Z})$, since $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z}$.

Write $e = \begin{pmatrix} \alpha & \varphi \\ 0 & \beta \end{pmatrix}$ for an idempotent $e \in E$. An easy calculation (using $e^2 = e$) yields $\alpha^2 = \alpha$ and $\beta^2 = \beta$. Therefore $\alpha = 0$ or 1, and $\beta = 0$ or 1. (There are other relations necessary for e to be an idempotent but we don't need to consider them.) Since there are only finitely many choices of φ, this shows that there are only finitely many idempotents in E.

Next, notice that E is not Abelian since, for example, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ doesn't commute with $\begin{pmatrix} 0 & \varphi_1 \\ 0 & 1 \end{pmatrix}$. (These are actually left strongly isomorphic idempotents.)

We see that M is an exchange module since it is a finite direct sum of two exchange modules. However, M is not that interesting since it is noetherian and a finite direct sum of (strongly) indecomposable modules. We can fix both of these problems as follows:

Set $E' = \prod_{q \neq p,\ q \text{ prime}} \mathbb{Z}/q\mathbb{Z}$, and note that E' is an exchange ring. A straightforward calculation shows $End_\mathbb{Z}(E') \cong E'$, which is commutative, and hence $E'_\mathbb{Z}$ is an Abelian module with finite exchange. Therefore $E'_\mathbb{Z}$ is a full exchange module.

Setting $M'_\mathbb{Z} = M_\mathbb{Z} \oplus E'_\mathbb{Z}$, we have M' is a full exchange module. There are no non-trivial $\mathbb{Z}$-homomorphisms from M to E', since there are no non-trivial $\mathbb{Z}$-homomorphisms from M to $\mathbb{Z}/q\mathbb{Z}$ for $p \neq q$. Conversely, there are no non-trivial $\mathbb{Z}$-homomorphisms from E' to M, since E' is p-divisible. Thus, $End_\mathbb{Z}(M') \cong E \times E'$, which is a non-Abelian ring with each idempotent left strongly isomorphic to only finitely many other idempotents. Therefore, M' gives us another example of a non-Abelian, finitely complemented module with full exchange. We leave it as an exercise for the reader to show that M' is not a direct sum of indecomposables.

A natural question one might ask is whether the proof of Theorem 3 can be generalized to show full exchange. If one forces the endomorphism ring to be regular (i.e. the kernel and image of each endomorphism is a direct summand) then the answer is yes. Hence, we have the following:

Theorem 4. *Suppose that M is a finitely complemented module with a regular endomorphism ring E. Then M has full exchange.*

Proof. As usual, we start with a summable family $(x_i)_{i\in I}$, with $\sum_{i\in I} x_i = 1$. As before, well-order I so that it has first element 1 and a last element κ. Set $y_i = \sum_{j>i} x_j$ for each $i \in I$.

Since E is a regular ring it is suitable, so there are (orthogonal) idempotents $e_{1,1} \in Ex_1$ and $f_1 \in Ey_1$ with $e_{1,1} + f_1 = 1$.

Suppose, by transfinite-induction, that for some $\alpha \in I$ we have fixed elements $e_{i,j} \in Ex_i$ and $f_j \in Ey_j$ for each $j < \alpha$, and for each $i \leqslant j$, such that the following conditions hold: (i) $\sum_{i\leqslant j} e_{i,j} + f_j = 1$, for each $j < \alpha$, (ii) the family $\{e_{i,j}(i \leqslant j), f_j\}$ consists of orthogonal idempotents, for each $j < \alpha$, and (iii) $e_{i,j} \sim e_{i,\ell}$, for each $j, \ell < \alpha$ $(j, \ell \geqslant i)$.

If α is not a limit ordinal then we construct $e_{i,\alpha}$ (for $i \leqslant \alpha$) and f_α satisfying conditions (i), (ii), and (iii), just as in the previous theorem. So, assume that α is a limit ordinal.

We need to set up a few notions. By "a sequence J in I", we mean that J is a subset of I, ordered via the ordering in I. Suppose that I contains an upper bound for the elements of J. Then when we say "α is the limit of the sequence J in I" we mean that α is the smallest element of I that is also an upper bound for J.

We know, from the finitely complemented property and the fact that α is a limit cardinal, that there is an infinite sequence in I, say J_1, such that $e_{1,j} = e_{1,j'}$ for any two elements $j, j' \in J_1$, and the limit of the sequence J_1 in I is α. Put $e'_{1,\alpha} = e_{1,j}$ (for any $j \in J_1$).

Now, inductively pick $e'_{i,\alpha}$ for $i < \alpha$ such that:

$(\star)$ for each finite set $F \subset [1, i] \subseteq I$ with $i \in F$,
there exists a sequence J_F whose limit in I is α,
with $e'_{\ell,\alpha} = e_{\ell,j}$ for each $\ell \in F$ and $j \in J_F$.

We need to see that this picking process actually works! So, suppose that for some $\beta \in I, \beta > 1$, we have fixed each $e'_{i,\alpha}$ for $i < \beta$ satisfying the condition above. Assume, by way of contradiction, that $e'_{\beta,\alpha}$ cannot be chosen as above. Then, for each $e_{\beta,\gamma}$ $(\beta \leqslant \gamma < \alpha)$ there is some finite set (which we fix and call F_γ) with $F_\gamma \subset [1, \beta]$, $\beta \in F_\gamma$, such that for any sequence J_γ whose limit in I is α, then there is some $\ell \in F_\gamma$ and some $j \in J_\gamma$ with $e'_{\ell,\alpha} \neq e_{\ell,j}$ if $\ell \neq \beta$, or with $e_{\beta,\gamma} \neq e_{\beta,j}$ if $\ell = \beta$. (To put it simply, we have negated the condition $(\star)$ above in the hopes of reaching a contradiction.)

Since $e_{\beta,\beta} \sim e_{\beta,j}$ for each $j \in [\beta, \alpha)$ and since M is finitely complemented, there are only finitely many distinct idempotents in the family $(e_{\beta,j})_{j\in[\beta,\alpha)}$. So, fix $\gamma_1,\ldots,\gamma_n$ such that the set $\{e_{\beta,\gamma_1},\ldots,e_{\beta,\gamma_n}\}$ contains each of these distinct idempotents. Let $F = F_{\gamma_1}\cup F_{\gamma_2}\cup\cdots\cup F_{\gamma_n}$, set $F' = F-\{\beta\}$, and set $m = \max(F') < \beta$. From the choice of $e'_{m,\alpha}$, there is some sequence $J_{F'}$ whose limit in I is α, with $e'_{\ell,\alpha} = e_{\ell,j}$ for each $\ell\in F'$ and $j\in J_{F'}$. But then there is a subsequence of $J_{F'}$, call it $(J_{F'})'$, converging to α, and $e_{\beta,j} = e_{\beta,j'}$ for each $j,j'\in (J_{F'})'$ (this comes from the finitely complemented assumption, and the fact that $J_{F'}$ is infinite). Then for each $j\in (J_{F'})'$, $e_{\beta,j} = e_{\beta,\gamma_{n'}}$ for some fixed $n'\in[1,n]$, contradicting the definition of $F_{\gamma_{n'}}$ (see the previous paragraph).

We have thus defined $e'_{i,\alpha}$ for each $i<\alpha$. Further, from condition $(\star)$ above, it is clear that they are orthogonal idempotents. If we can show that

$$1-\sum_{i<\alpha} e'_{i,\alpha} \in E\left(\sum_{i\geqslant\alpha} x_i\right) \tag{1}$$

then we have $\sum_{i<\alpha} e'_{i,\alpha} + r x_\alpha + r y_\alpha = 1$ for some $r\in E$. In this case, we can proceed just as in the case when α was not a limit ordinal. Thus, it suffices to show equation (1) is true.

We set $y'_\alpha = \sum_{i\geqslant\alpha} x_i = y_\alpha + x_\alpha$. We also fix some $r_i\in E$, for each $i<\alpha$, such that $f_i = r_i y_i$. We define a map $s\in \mathrm{Hom}_k(y'_\alpha(M), M)$ as follows. Fix $m\in M$. Set $F = \{i\in I : x_i(m)\neq 0\}$, and notice that this is a finite set. (If $m=0$ then $F=\emptyset$.) Let $F' = \{i\in F : i<\alpha\}$ and set $\eta = \max(F')$. Then, from the definition of $e'_{\eta,\alpha}$, there is some index $\mu\in I$, $\eta<\mu<\alpha$, such that $e_{\ell,\mu} = e'_{\ell,\alpha}$ for each $\ell\in F$. We calculate

$$\begin{aligned} m = \sum_{i\leqslant\mu} e_{i,\mu}(m) + f_\mu(m) &= \sum_{i\in F'} e'_{i,\alpha}(m) + r_\mu y_\mu(m) \\ &= \sum_{i<\alpha} e'_{i,\alpha}(m) + r_\mu y'_\alpha(m). \end{aligned} \tag{2}$$

We define s so that $s(y'_\alpha(m)) = r_\mu y'_\alpha(m)$.

First, we need to show that this definition is independent of our choice of μ. If we had chosen a different $\mu'\in I$ with the same properties as μ, then calculating as above and using equation (2), we have

$$r_{\mu'} y'_\alpha(m) = m - \sum_{i<\alpha} e'_{i,\alpha}(m) = r_\mu y'_\alpha(m).$$

So, the definition is independent of our choice of μ. Second, we need to show that if $y'_\alpha(m) = y'_\alpha(m')$, then s acts the same on both sides. This is done as above, enlarging F to the set $\{i\in I : x_i(m)\neq 0 \text{ or } x_i(m')\neq 0\}$. Finally, we need $s\in\mathrm{Hom}(y'_\alpha(M),M)$. This follows easily using the same techniques as above.

Thus, by construction, we have

$$1-\sum_{i<\alpha} e'_{i,\alpha} = s y'_\alpha.$$

The next question we need to ask is whether we can extend s to some $r\in E$. This is where we need the fact that E is regular. Then $\mathrm{im}(y'_\alpha)$ is a direct summand, so such an extension can indeed be made. We have thus verified inclusion (1) above.

By induction, we have now defined $e_{i,j}$ and f_j (for $j\in I$ and $i\leqslant j$) satisfying properties (i), (ii), and (iii). (Note: $f_\kappa = 0$.) Set $e_i = e_{i,\kappa}$ for each $i\in I$. Then, by construction, $(e_i)_{i\in I}$ is a summable family of orthogonal idempotents summing to 1, with $e_i \in Ex_i$. This finishes the proof. $\square$

6. Final Remarks

Our assumption in Theorem 4 that E be a regular ring could be replaced with M being quasi-injective, and the same proof would work. However, this gives no new information since it is known that all quasi-injective modules have full exchange. Also notice that we did not use the fact that kernels of endomorphisms are direct summands. It would be nice to weaken the condition that E be a regular ring to E being an exchange ring. However, the ideas of Theorem 2 are not sufficient, since we make use of the fact that summands in Abelian modules are fully invariant. This does not hold true in finitely complemented modules.

Acknowledgements

My thanks to my advisor T.Y. Lam, for introducing me to this subject and carefully reading manuscripts and giving suggestions. Also a warm thanks to the referee, whose comments led to a substantial improvement in the manuscript.

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