JACOBSON'S LEMMA FOR DRAZIN INVERSES

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Abstract. If a ring element $\alpha = 1 - ab$ is Drazin invertible, so is $\beta = 1 - ba$. We show that the Drazin inverse of β is expressible by a simple formula (in generalization of "Jacobson's Lemma") in terms of the Drazin inverse and the spectral idempotent of α. The commutation rules $\alpha\, a = a\, \beta$ and $b\, \alpha = \beta\, b$ are extended to the Drazin inverses, and the spectral idempotents of α and β are shown to be isomorphic. In a related direction, we also prove that Jacobson's Lemma holds for π-regular elements and unit π-regular elements in arbitrary rings.

1. Introduction

For any ring R (with identity), "Jacobson's Lemma" states that, if $\alpha = 1 - ab$ is a unit, so is $\beta = 1 - ba$; indeed,

$$\beta^{-1} = 1 + b\, \alpha^{-1} a. \tag{1.1}$$

Halmos observed in [11] that this can be "formally" checked by writing

$$\begin{aligned} (1 - ba)^{-1} &= 1 + ba + baba + bababa + \cdots \\ &= 1 + b\,(1 + ab + abab + \cdots)\, a \\ &= 1 + b\,(1 - ab)^{-1} a, \end{aligned}$$

and pointed out that this nice trick is "usually ascribed to Jacobson." Kaplansky, widely known for his gift with words, once remarked to one of us that, using this trick, you can always reproduce formula (1.1) without fail "even if you are thrown up on a desert island, with all your books and papers lost" [16, p. 4]. Resonating with Kaplansky's witty comment, some of our students and close associates have sometimes fondly referred to (1.1) as the "Desert Island Formula".

Over the years, it has been realized that Jacobson's Lemma has suitable analogues for various kinds of "generalized inverses". An element $r \in R$ is said to be (von Neumann) *regular* if $r = rxr$ for some $x \in R$ (called an *inner inverse* of r), and r is said to be *unit-regular* if it has a unit inner inverse. It is known that if $\alpha = 1 - ab$ is regular (resp. unit-regular), then so is $\beta = 1 - ba$, and the formula (1.1) continues to hold if "inverse" is replaced by "an inner inverse". See, for example, [5].

In [9], Drazin introduced a new notion of invertibility, which is now commonly known as *Drazin invertibility.* An element $r \in R$ is called *Drazin invertible* if there exists $s \in R$ such that $rs = sr$, $s = srs$, and $r^n = r^{n+1} s$ for some $n \geq 0$. If s exists, it is unique. It is called the *Drazin inverse* of r, and is denoted by r' (following Drazin). The smallest integer n for which the equation $r^n = r^{n+1} r'$ holds is called the *Drazin index* of r, and is denoted by $i\,(r)$. In case $i\,(r) \leq 1$, r is said to be *group invertible* (or strongly regular), and its Drazin inverse is called the *group inverse* of r. According to [9, Theorem 4], $r \in R$ is Drazin invertible iff r is *strongly π-regular* in the sense that $r^n \in r^{n+1} R \cap R\, r^{n+1}$ for some integer $n \geq 1$. In this case, we say that r has *associated idempotent* $r'r$ and *spectral idempotent* $1 - r'r$.

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In 2009, Patrício and Veloso da Costa [23] asked if the Drazin invertibility of $\alpha = 1 - ab$ implies that of $\beta = 1 - ba$. In [22] and [7], Patrício-Hartwig and Cvetković-Ilić-Harte answered this question affirmatively, showing in fact that α and β always have the same Drazin index. In their papers, however, no formula was given that expresses β' in terms of α'. Subsequently, such a formula was found by Castro-González, Mendes-Araújo, and Patrício; see Theorem 3.6 in their recent paper [4]. In [4], the Drazin inverse formula for β' was derived from a preliminary group inverse formula, so the proof was long and indirect. (See also [24] which deals with a more general notion of Drazin inverses.) In Theorem 2.1 of the present note, we offer a formally simpler formula (2.2) that computes β' directly — without going through (but of course subsuming) the group inverse case. While (2.2) does completely recapture the formula in Theorem 3.6 in [4], its proof is short, self-contained, and does not assume knowledge of any parts of [22], [7], [4], or [24]. It also contains more information on the associated idempotent of β as well as its "Azumaya realization" as a Drazin invertible element.

In §3, we use our techniques to derive commutation rules for the Drazin inverses α', β', and the spectral idempotents e, f of α and β, proving in particular the hitherto unknown fact that e and f are *isomorphic* idempotents. We then place our computations in the general context of "Jacobson pairs" (pairs of the form $(1-xy,\, 1-yx)$ over rings), and show that several pairs naturally associated with (α, β) are also Jacobson pairs. In Theorem 3.8, we further demonstrate the universal nature of Jacobson's Lemma by proving that it holds for *π-regular elements* and *unit π-regular elements* in arbitrary rings. While we show in §4 that Jacobson's Lemma does not hold for "clean elements" and "suitable elements" in the sense of Nicholson [18], this work paves the way to a further study of Jacobson's Lemma for other significant classes of ring elements, which we hope to present in the forthcoming work [17].

Throughout this paper, R denotes an arbitrary ring with $1 \in R$, and $\mathrm{U}(R)$ denotes the group of units in R. Other standard notations in ring theory follow those in [15].

2. Formula for Drazin Inverse

We begin this section by proving the theorem below, where new compact expressions are given for the Drazin inverse and associated idempotent of $1 - ba$ in terms of the Drazin inverse and spectral idempotent of $1 - ab$.

Theorem 2.1. *Suppose $\alpha := 1 - ab \in R$ is Drazin invertible with index k, Drazin inverse α', and spectral idempotent $e = 1 - \alpha'\alpha$. Then $\beta := 1 - ba$ is also Drazin invertible with index k, and has Drazin inverse*

$$\beta' = (1 - b\,ea)^n + b\,\alpha' a \quad \text{for any } n \geq k. \tag{2.2}$$

The associated idempotent for β is $(1 - b\,ea)^n$ for any $n \geq k$.

This means that, to get the new formula from (1.1), all we need to do is to interpret "inverse" as the Drazin inverse, and simply replace the "1" on the right-hand-side of (1.1) *by the associated idempotent of β.* In this way, (2.2) retains the simplicity and memorability of a "Desert Island Formula".

To prove Theorem 2.1, let $\gamma := 1 + b\,\alpha' a$ (an expression inspired by the formula (1.1)) and $\varepsilon := b\,ea$, recalling that $e = 1 - \alpha'\alpha = 1 - \alpha\,\alpha'$. Since $\beta\,b = b\,\alpha$, iteration shows that $\beta^i\,b = b\,\alpha^i$ for all $i \geq 0$. Similarly, we have $a\,\beta^i = \alpha^i\,a$ for all $i \geq 0$. These two important commutation rules will be used freely (without further mention) below.

Lemma 2.3. (A) $\beta^k \varepsilon = 0$, (B) $\varepsilon\,\gamma = \varepsilon$, (C) $\beta\,\gamma = \gamma\,\beta = 1 - \varepsilon$, (D) $\beta^{k+1}\gamma = \beta^k$.

Proof. Keeping in mind that $\alpha^k = \alpha^{k+1}\alpha'$, and $e\,\alpha' = (1-\alpha'\alpha)\,\alpha' = 0$, we have

$$\begin{aligned}
\beta^k\varepsilon &= \beta^k b\,(1-\alpha\,\alpha')\,a = b\,\alpha^k(1-\alpha\,\alpha')\,a = 0,\\
\varepsilon\,\gamma &= b\,ea\,(1+b\,\alpha'a) = b\,e\,\big(a+(1-\alpha)\,\alpha'a\big)\\
&= b\,e\,\big(1+(1-\alpha)\,\alpha'\big)\,a = b\,e\,(e+\alpha')\,a = \varepsilon,\\
\beta\,\gamma &= \beta\,(1+b\,\alpha'a) = (1-b\,a)+b\,\alpha\,\alpha'a\\
&= 1-b\,(1-\alpha\,\alpha')\,a = 1-\varepsilon, \text{ and}\\
\gamma\,\beta &= (1+b\,\alpha'a)\,\beta = (1-b\,a)+b\,\alpha'\alpha\,a\\
&= 1-b\,(1-\alpha'\alpha)\,a = 1-\varepsilon.
\end{aligned}$$

Finally, $\beta^{k+1}\gamma = \beta^k\,(\beta\,\gamma) = \beta^k\,(1-\varepsilon) = \beta^k$ by (A). (We'll show later that actually $\gamma \in \mathrm{U}(R)$; see the proof of Theorem 3.10(D).) □

Proof of Theorem 2.1. The point of (C) and (D) above is that they give an explicit "Azumaya realization" of β as a strongly π-regular element, in the sense of [2, Theorem 3]. With this, the rest of the proof is at hand. First, (C) and (D) imply that β is Drazin invertible with $i\,(\beta) \le k = i\,(\alpha)$, and hence equality must hold on account of symmetry. Next, the Drazin inverse of β can be computed via Drazin's Theorem 4 in [9]; namely, for any $n \ge k$, $\beta' = \beta^n\gamma^{n+1}$. Therefore, by (C) in Lemma 2.3,

$$\begin{aligned}
\beta' &= (\beta\,\gamma)^n\,\gamma = (1-\varepsilon)^n\,(1+\gamma-1)\\
&= (1-\varepsilon)^n + (1-\varepsilon)^n\,(\gamma-1) = (1-\varepsilon)^n+\gamma-1\\
&= (1-\varepsilon)^n + b\,\alpha'a \quad (\text{for any } n \ge k),
\end{aligned}$$

where, toward the end, we have used the fact that $\varepsilon\,(\gamma-1) = 0$ (by (B) in Lemma 2.3). Since the computation above is valid for any $n \ge k$, it follows that the powers $(1-\varepsilon)^n$ are all equal for $n \ge k$. Thus, by (C) again, the associated idempotent for β is

$$\beta\,\beta' = \beta^{k+1}\gamma^{k+1} = (\beta\,\gamma)^{k+1} = (1-\varepsilon)^{k+1} = (1-b\,ea)^k. \qquad \square$$

The next theorem gives two explicit expressions for the spectral idempotent f of β (in case α is Drazin invertible of index k). The first expression for f shows that our formula (2.2) above completely recaptures [4, Theorem 3.6] (which also appeared in [24, Corollary 2.4]. The second expression for f will provide the groundwork for some further generalizations of Theorem 2.1 in [17].

Theorem 2.4. *Keep the notations and assumptions in Theorem 2.1, and let $u := \alpha - e$. Then $u \in \mathrm{U}(R)$, and the spectral idempotent f for β is given by $b\,era = -b\,eu^{-1}a$, where $r = 1+\alpha+\cdots+\alpha^{k-1}$ (and $k = i\,(\alpha)$). In particular, $R\,eR = R\,fR$.*

Proof. For $\delta := 1-\varepsilon \in R$, Theorem 2.1 shows that

$$f = 1-\delta^k = (1-\delta)\,(1+\delta+\cdots+\delta^{k-1}) = b\,e\,a\,(1+\delta+\cdots+\delta^{k-1}). \tag{2.5}$$

Since e commutes with ab, we have $a\,\delta = a - ab\,ea = (1-eab)\,a$. Therefore,

$$e\,a\,\delta^i = e\,(1-eab)^i\,a = (e-eab)^i\,a = e\,\alpha^i a \quad (\text{for all } i \ge 0). \tag{2.6}$$

From (2.5) and (2.6), it follows that $f = b\,era$ where $r = 1+\alpha+\cdots+\alpha^{k-1}$. [Note: if $k = 0$ (that is, $\alpha \in \mathrm{U}(R)$), r is to be interpreted as 0.] In particular, $R\,fR \subseteq R\,eR$, so equality holds by symmetry.

For the rest of the theorem, recall that $u := \alpha - e \in \mathrm{U}(R)$ is a standard consequence of Fitting's Decomposition Theorem. To prove the second formula $f = -b\,eu^{-1}a$, it suffices to check that $er =$

$-eu^{-1}$. Since $ur = ru$, this amounts to $eur = -e$. Now

$$\begin{aligned} eur &= e\,(\alpha - 1 + 1 - e)\,r \\ &= e\,(\alpha - 1)\,(1 + \alpha + \cdots + \alpha^{k-1}) \\ &= e\,(\alpha^k - 1) = (e\,\alpha)^k - e. \end{aligned}$$

As $(e\,\alpha)^k = 0$ (again from Fitting's Decomposition Theorem), we get $eur = -e$. □

For $k = i\,(\alpha) \le 1$, we have the following special case of Theorems 2.1 and 2.4.

Corollary 2.7. *If $\alpha = 1 - ab \in R$ is group invertible with spectral idempotent e, then $\beta = 1 - b\,a$ is also group invertible, with group inverse $\beta' = 1 + b\,(\alpha' - e)\,a$ and spectral idempotent $f = b\,ea$. (By symmetry, we also have $e = afb$.)*

In the case where α is *nilpotent,* Theorem 2.1 can be refined a bit as follows.

Corollary 2.8. *Let $\alpha = 1 - ab \in R$ be nilpotent of index k. Then*

(A) *$\beta = 1 - b\,a$ is Drazin invertible with index k, and its associated idempotent is $\beta' = \beta^k = \beta^{k+1} = \cdots$. The spectral idempotent f of β is a full idempotent, in the sense that $R\,fR = R$.*
(B) *If b is not a right 0-divisor in R, then $\beta^k = 0$.*
(C) *If R is a Dedekind-finite ring, then β is similar to α. In this case, β is also nilpotent of index k.*

Proof. (A) follows from Theorem 2.1 since in the notations there, $\alpha' = 0$ and $e = 1$. Indeed, the equation $\beta^k = \beta^{k+1}$ can be directly seen as follows. From $\beta^k b = b\,\alpha^k = 0$, we have $\beta^{k+1} = \beta^k(1 - b\,a) = \beta^k$. If b is not a right 0-divisor, then $\beta^k b = 0 \Rightarrow \beta^k = 0$, proving (B). Finally, for (C), assume that R is Dedekind-finite. Since α is nilpotent, we have $ab = 1 - \alpha \in \mathrm{U}(R)$, and hence $b \in \mathrm{U}(R)$ too. Then $\beta = b\,\alpha\,b^{-1}$ is similar to α. In particular, β is also nilpotent of index k. □

Remark 2.9. In the case $k = 1$, we have $\alpha = 0$, so $ab = 1$. Then $(b\,a)^2 = b\,a$, and $(1 - b\,a)^2 = 1 - b\,a$ too (consistently with all conclusions of (A) in Corollary 2.8). In general, however, $b\,a$ may not be 1, so $\beta = 1 - b\,a$ may not be 0. This shows that the assumption on b cannot be dropped in (B) of Corollary 2.8.

3. Commutation Rules, and Jacobson Pairs

In this section, we present another main result in this paper relating the Drazin inverses and the spectral idempotents of α and β. These relations are "formal extensions" of the basic commutation rules $\beta\,b = b\,\alpha$ and $\alpha\,a = a\,\beta$. However, they have not been noticed in the literature before, and their proofs do require some work.

Theorem 3.1. *Suppose $\alpha = 1 - ab \in R$ is Drazin invertible, and let e, f be the spectral idempotents of α and $\beta := 1 - ba$. Then $\beta' b = b\,\alpha'$, $\beta'' b = b\,\alpha''$, and $f\,b = b\,e$. (By symmetry, we also have $\alpha' a = a\,\beta'$, $\alpha'' a = a\,\beta''$, and $e\,a = a\,f$.)*

Proof. Let $k = i\,(\alpha)$, and let $\gamma = 1 + b\,\alpha' a$ as in Lemma 2.3. We claim that the following two equations hold: (A) $\gamma\,b = b\,(e + \alpha')$, and (B) $\alpha' = \alpha^k\,(e + \alpha')^{k+1}$.

To prove (A), note that $\gamma\,b = (1 + b\,\alpha' a)\,b = b\,(1 + \alpha'(1 - \alpha)) = b\,(e + \alpha')$. (B) is a standard fact on any Drazin invertible element α with index k. It can be proved formally by expanding $(\alpha\,(e + \alpha'))^k = (\alpha\,e + (1 - e))^k$ into $\alpha^k e + (1 - e)$. More conceptually, it can be seen easily by thinking of α as a Fitting endomorphism on a right module M (over some ring). If $M = I \oplus K$ is the Fitting decomposition of α on M (so that $\alpha\,|\,I$ is an automorphism and $\alpha\,|\,K$ is nilpotent of index k), then $\alpha^k(e + \alpha')^{k+1}$ is zero on K and is the inverse of α on I, so it agrees precisely with α' on M.

Using (A), (B), and the earlier equation $\beta' = \beta^k \gamma^{k+1}$, we have now

$$\beta' b = \beta^k \gamma^{k+1} b = \beta^k b\,(e+\alpha')^{k+1} = b\,\alpha^k\,(e+\alpha')^{k+1} = b\,\alpha'. \tag{3.2}$$

This leads to $b\,e = b - b\,\alpha'\alpha = b - \beta' b\,\alpha = b - \beta'\beta\, b = f\,b$. Finally, to deal with the "double" Drazin inverses, recall from [9] that $\beta'' = \beta^2\,\beta'$. From this, we have

$$\beta''\,b = \beta^2 \beta' b = \beta^2\, b\,\alpha' = b\,\alpha^2 \alpha' = b\,\alpha''. \tag{3.3}$$

Note that there is no need to discuss further "higher-order" Drazin inverses, since $\alpha''' = \alpha'$ and a similar equation holds for β, as long as α is Drazin invertible. ☐

The referee pointed out that the theorem above can be seen as a consequence of a very general commutation rule proved by Drazin in his recent paper [10]. Indeed, for any $x, y \in R$, let us define the "intertwining set" of the ordered pair (x, y) to be

$$\mathrm{I}\,(x, y) := \{\, r \in R : x\,r = r\,y\}. \tag{3.4}$$

For any pair of Drazin invertible elements $x, y \in R$, Drazin's commutation rule in [10, Theorem 2.2] states that $\mathrm{I}\,(x, y) \subseteq \mathrm{I}\,(x', y')$. In the notations of Theorem 3.1, since we know that $a \in \mathrm{I}\,(\alpha, \beta)$ and $b \in \mathrm{I}\,(\beta, \alpha)$, Drazin's result gives immediately that $a \in \mathrm{I}\,(\alpha', \beta')$, $b \in \mathrm{I}\,(\beta', \alpha')$, and therefore the same holds for the double Drazin inverses (and spectral idempotents). While this gives a quicker and more conceptual proof of Theorem 3.1 (assuming Drazin's new result), we have chosen to preserve our original proof of the theorem above as it gives a good illustration of the Fitting decomposition technique, and it will also be applicable to the setting of the remark below.

Remark 3.5. In the recent paper [24], Jacobson's Lemma was studied for a generalized notion of Drazin invertibility introduced by Koliha and Patrício. In the setting of [14] (see also [13]), the role of nilpotent elements in one of the standard characterizations of Drazin inverses is replaced by that of *quasi-nilpotent* elements. (An element $q \in R$ is quasi-nilpotent if $1 - qx \in \mathrm{U}(R)$ for every $x \in R$ such that $qx = xq$.) If we interpret α' and β' as the generalized Drazin inverses of $\alpha = 1 - ab$ and $\beta = 1 - ba$, Theorem 2.3 and Corollary 2.4 in [24] give explicit formulas expressing β' and the spectral idempotent f of β in terms of α' and the spectral idempotent e of α. Assuming these two results, it is relatively routine to check that the formulas $\beta' b = b\,\alpha'$ and $f\,b = b\,e$ proved in Theorem 3.1 do remain valid in the setting of *generalized* Drazin inverses introduced in [14]. The details of these verifications are left to the reader.

To facilitate the further study and generalization of Jacobson's Lemma, it is now convenient to introduce the following terminology.

Definition 3.6. We say that $(\alpha, \beta) \in R^2$ is a *Jacobson pair* if there exist $a, b \in R$ such that $\alpha = 1 - ab$ and $\beta = 1 - ba$. In this case, of course (β, α) is also a Jacobson pair. The diagonal elements $(r, r) \in R^2$ are always Jacobson pairs. In the case of a commutative ring R, these are the *only* Jacobson pairs.

Example 3.7. Two idempotents $e, f \in R$ are said to be *isomorphic* (written $e \cong f$) if $eR \cong fR$ as right R-modules (or equivalently, $R\,e \cong R\,f$ as left R-modules). A criterion for $e \cong f$ is that there exist $x, y \in R$ such that $e = xy$ and $f = yx$: see [15, (21.20)]. From this, it follows that $e \cong f$ *iff* $(1 - e, 1 - f)$ *is a Jacobson pair.*

Recall that an element $x \in R$ is said to be *unit-regular* if $x = xux$ for some $u \in \mathrm{U}(R)$. Also, $y \in R$ is said to be π*-regular* (resp. *unit* π*-regular*) if y^n is regular (resp. unit-regular) for some integer $n \geq 1$.

Theorem 3.8. *If* $(\alpha, \beta) \in R^2$ *is a Jacobson pair, so is* (α^n, β^n) *for any integer* $n \geq 1$*. If* α *is* π*-regular (resp. unit* π*-regular), then so is* β.

Proof. Say $\alpha = 1-ab$ and $\beta = 1-ba$. Then by [4, Lemma 2.3], we have (for any $n \ge 1$) $\alpha^n = 1-(ra)\,b$ and $\beta^n = 1 - b\,(ra)$, where $r := 1+\alpha+\cdots+\alpha^{n-1} \in R$. Thus, (α^n, β^n) is a Jacobson pair. If α is π-regular, then α^n is regular for some $n \ge 1$. By what we have said at the beginning of the paper, this implies that β^n is also regular, so β is π-regular. If α is unit π-regular, the same argument works, by using Chen's result [5, Lemma 2.1] that the unit-regularity of α^n implies that of β^n (as long as (α^n, β^n) is a Jacobson pair). □

Corollary 3.9. *Let $\alpha \in \mathrm{U}(R)$ and $\beta \in R$. If (α, β) is a Jacobson pair, so is (α^n, β^n) for any $n \in \mathbb{Z}$.*

Proof. By the "original" Jacobson's Lemma, $\beta \in \mathrm{U}(R)$ too, so the notation (α^n, β^n) makes sense for any $n \in \mathbb{Z}$. In view of Theorem 3.8, it suffices to prove that $(\alpha^{-1}, \beta^{-1})$ is a Jacobson pair. Again, write $\alpha = 1-ab$, and $\beta = 1-ba$. By Jacobson's Lemma, $\beta^{-1} = 1+b\,(\alpha^{-1}a)$, and $\alpha^{-1} = 1+(a\,\beta^{-1})\,b$. But $\alpha\,a = a\,\beta$ implies that $\alpha^{-1}a = a\,\beta^{-1}$, so $(\alpha^{-1}, \beta^{-1})$ is indeed a Jacobson pair. □

Theorem 3.10. *Let $(\alpha, \beta) \in R^2$ be a Jacobson pair such that α (and hence also β) is Drazin invertible. Let e and f be the spectral idempotents of α and β. Then*

(A) *e and f are also the spectral idempotents of α' and β', and we have $e \cong f$.*
(B) *The associated idempotents $1-e$, $1-f$ for α and β form a Jacobson pair.*
(C) *(α', β') is also a Jacobson pair, with group invertible entries.*
(D) *$e+\alpha'$, $f+\beta' \in \mathrm{U}(R)$, with inverses $e+\alpha''$ and $f+\beta''$. For any $n > 0$,*

$$\big(e+(\alpha')^n,\ f+(\beta')^n\big) \quad \text{and} \quad \big(e+(\alpha'')^n,\ f+(\beta'')^n\big)$$

are Jacobson pairs, with unit entries.

Proof. (A) In view of $\alpha'' = \alpha^2\alpha'$, α' has spectral idempotent

$$1-\alpha''\alpha' = 1-(\alpha^2\alpha')\,\alpha' = 1-\alpha\,(\alpha'\alpha\alpha') = 1-\alpha\,\alpha' = e.$$

Similarly, β' has spectral idempotent f. To see that $e \cong f$, write $\alpha = 1-ab$ and $\beta = 1-ba$. By Theorem 2.4, we have $f = b\,era$ and (by symmetry) $e = afsb$, where $r = 1+\alpha+\cdots+\alpha^{k-1}$ and $s = 1+\beta+\cdots+\beta^{k-1}$, with $k = i\,(\alpha) = i\,(\beta)$. From $\alpha\,a = a\,\beta$, we have $ra = as$, so the relation $ea = af$ in Theorem 3.1 gives

$$f = b\,e\,(ra) = b\,e\,(as) = b\,(afs) \cong (afs)\,b = e.$$

(B) By Example 3.7, $e \cong f$ implies that $(1-e, 1-f)$ is a Jacobson pair. To show the synergy of ideas, here is another proof. From Theorem 2.1, we know that $1-e = (1-b\,ea)^k$, and similarly, $1-f = (1-afb)^k$. Since $ea = af$, $\big(1-b\,(ea),\ 1-(af)\,b\big)$ is a Jacobson pair. By Theorem 3.8, so is $\big((1-b\,ea)^k, (1-afb)^k\big) = (1-e, 1-f)$. (This, in turn, implies that $e \cong f$.)

(C) By Theorems 2.1 and 2.4, we have

$$\beta' = 1-b\,(er-\alpha')\,a, \quad \text{and by symmetry,} \quad \alpha' = 1-a\,(fs-\beta')\,b. \tag{3.11}$$

Using again the relations $ra = as$, $\alpha'a = a\,\beta'$, and $ea = a\,f$, we may define

$$a_1 := (er-\alpha')\,a = e\,(as) - a\,\beta' = a\,(fs-\beta'). \tag{3.12}$$

Then, by (3.11), $(\alpha', \beta') = (1-b\,a_1, 1-a_1b)$ is a Jacobson pair (and it is well known that α', β' are group invertible, as long as they exist).

(D) To prove (D), we go back to the element $\gamma = 1+b\,\alpha'a$ used in the proofs of both Lemma 2.3 and Theorem 3.1. Using the rule $\alpha'a = a\,\beta'$ in Theorem 3.1, we can rewrite

$$\gamma = 1+b\,\alpha'a = 1+b\,a\,\beta' = 1+(1-\beta)\,\beta' = f+\beta'. \tag{3.13}$$

(As the reader might have noticed, this new expression for γ was partly inspired by the earlier relation $\gamma\,b = b\,(e+\alpha')$ in the proof of Theorem 3.1.) Now $\gamma = 1+b\,(\alpha'a)$ forms a Jacobson pair with

$$1+(\alpha'a)\,b = 1+\alpha'(1-\alpha) = e+\alpha'.$$

Therefore, $(e+\alpha',\, f+\beta')$ is a Jacobson pair. A straightforward calculation shows that $e+\alpha'$ and $f+\beta'$ are *units* in R, with inverses given respectively by

$$e+(1-e)\,\alpha = e+\alpha'' \quad \text{and} \quad f+(1-f)\,\beta = f+\beta''. \tag{3.14}$$

Applying Corollary 3.9, we can then get other Jacobson pairs by forming the powers $\big((e+\alpha')^n,\,(f+\beta')^n\big)$ $(n\in\mathbb{Z})$. For $n>0$, we get $\big(e+(\alpha')^n,\, f+(\beta')^n\big)$ (since $e\alpha'=\alpha' e=0$), and for $n=-m<0$, we get $\big(e+(\alpha'')^m,\, f+(\beta'')^m\big)$, so all of these powers are Jacobson pairs, with unit entries. (Note that the latter case can also be deduced from the former case by using part (A) above.) ☐

Remark 3.15. Of course, direct applications of Theorem 3.1 also lead to other Jacobson pairs arising from (α,β). Some examples are, say,

$$(1-a\,\alpha,\,1-a\,\beta),\quad (1-a\,\alpha',\,1-a\,\beta'),\quad \text{and} \quad (1-ae,\,1-af).$$

However, these Jacobson pairs are somewhat less interesting, since their entries may no longer be Drazin invertible (when α itself is).

Remark 3.16. In general, in part (B) of Theorem 3.10, *the associated idempotents* $1-e$ *and* $1-f$ *of* α *and* β *need not be isomorphic.* For instance, if $a,b\in R$ are such that $ab=1\neq ba$, then $\alpha=1-ab=0$ has associated idempotent 0 (and spectral idempotent $e=1$). If $\beta=1-ba$ has an *isomorphic* associated idempotent, then the latter is 0 too, and hence $f=1$. But Theorem 2.4 implies that $f=b\,ea=b\,a\neq 1$, a contradiction. If R happens to be an IC ring (a ring with internal cancellation in the sense of [12]), then of course Theorem 3.10(A) would imply that the associated idempotents of α and β are isomorphic (and hence similar). This is the case, for instance, when R is a unit-regular ring. More deeply, it is also the case when R is a strongly π-regular ring (that is, when all elements in R are Drazin invertible), since Ara has proved in [1] that such a ring R has stable range one, which (in view of the work in [12]) is equivalent to R being an IC ring.

4. Clean Elements and Suitable Elements

The idea behind Jacobson's Lemma suggests that, if $\mathcal{C}_R$ is a class of elements in a ring R (defined, say for all rings, by some specific ring-theoretic property), it is of interest to investigate whether $\alpha\in\mathcal{C}_R \Rightarrow \beta\in\mathcal{C}_R$ for every Jacobson pair $(\alpha,\beta)\in R^2$. If this is the case, let us say that $\mathcal{C}_R$ *satisfies Jacobson's Lemma* (or Jacobson's Lemma holds for $\mathcal{C}_R$). We have seen in the previous sections that this is indeed the case if $\mathcal{C}_R$ is, say, the set of units, regular or unit-regular elements, π-regular or strongly π-regular elements, or Drazin invertible elements of a fixed Drazin index n in R. Another case in which Jacobson's Lemma is satisfied (by all rings) is where we define

$$\mathcal{C}_R=\{\,\alpha\in R:\ (1-\alpha)^n=(1-\alpha)^{n+1} \ \text{for some}\ n\geq 1\,\}. \tag{4.1}$$

Here, if $1-ab\in\mathcal{C}_R$, then $(ab)^n=(ab)^{n+1}$ for some $n\geq 1$, so left-multiplying by b and right-multiplying by a gives $(ba)^{n+1}=(ba)^{n+2}$. This shows that $1-ba\in\mathcal{C}_R$.

On the other hand, there are families $\{\mathcal{C}_R\}$ such that Jacobson's Lemma is satisfied only for some *specific* classes of rings R. In such cases, the Lemma would "define" these classes of rings. The two simplest examples are as follows. If $\mathcal{C}_R:=\{0\}$ for all R, then $\mathcal{C}_R$ satisfies Jacobson's Lemma iff R is Dedekind-finite, and if $\mathcal{C}_R:=\{1\}$ for all R, then $\mathcal{C}_R$ satisfies Jacobson's Lemma iff R is *reversible* $(ab=0\Rightarrow ba=0)$.

Note that there do exist some families $\{\mathcal{C}_R\}$ which are in spirit close to those named in the first paragraph of this section, but for which Jacobson's Lemma fails to hold. Each of the four families $\mathcal{C}_R$ below has the "complement property" that $\alpha\in\mathcal{C}_R\Rightarrow 1-\alpha\in\mathcal{C}_R$, so to say that Jacobson's Lemma fails for $\mathcal{C}_R$ (for a given ring R) is tantamount to saying that there exist $a,b\in R$ such that $ab\in\mathcal{C}_R$ but $ba\notin\mathcal{C}_R$.

The first example is where $\mathcal{C}_R$ is the set of all idempotents in R. For this family, Jacobson's Lemma fails for the matrix ring $R = \mathbb{M}_2(S)$ over any nonzero ring S. Indeed, for $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}$ in R (with $s \neq 0$), $AB = 0$ is an idempotent, but $BA = B$ is *not* an idempotent (since $B^2 = 0 \neq B$).

For more examples, let us recall the terminology of *suitable elements, clean and uniquely clean elements* introduced by Nicholson in [18], [20] and Nicholson-Zhou in [21]. Let $\operatorname{suit}(R)$ be the set of all suitable elements in R; that is, elements $\alpha \in R$ for which there exists an idempotent $e \in R\alpha$ such that $1 - e \in R(1-\alpha)$. (By [19], "left suitable" is equivalent to "right suitable", so we can simply refer to such α as being *suitable.*) Next, let $\operatorname{cn}(R)$ (resp. $\operatorname{ucn}(R)$) be the set of clean (resp. uniquely clean) elements of R; that is, elements $\alpha \in R$ that are expressible (resp. expressible uniquely) in the form $e + u$ where e is an idempotent and u is a unit in R.[1] An equation $\alpha = e + u$ of this form is called a *clean decomposition* of α. It is easy to check that all three sets introduced above have the complement property. Somewhat surprisingly, we have the following "negative" result.

Proposition 4.2. *In general, Jacobson's Lemma fails for* $\operatorname{suit}(R)$*,* $\operatorname{cn}(R)$*, and* $\operatorname{ucn}(R)$*.*

Proof. We shall prove the failure of Jacobson's Lemma (for all three sets) for the test ring $R = \mathbb{M}_2(\mathbb{Z})$. For $\operatorname{ucn}(R)$, this is easy. For $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in R, we have $AB = 0$, which is easily seen to be in $\operatorname{ucn}(R)$. However, the following distinct clean decompositions show that $BA \notin \operatorname{ucn}(R)$:

$$BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.3}$$

For the other two sets, recall Nicholson's important observation (in [18]) that $\operatorname{cn}(R) \subseteq \operatorname{suit}(R)$ (for any ring R). To show that $\operatorname{cn}(R)$ and $\operatorname{suit}(R)$ do not satisfy Jacobson's Lemma for our test ring $R = \mathbb{M}_2(\mathbb{Z})$, it suffices to name two matrices $C, D \in R$ such that $CD \in \operatorname{cn}(R)$, but $DC \notin \operatorname{suit}(R)$. Let $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $D = \begin{pmatrix} n & 1 \\ 0 & 0 \end{pmatrix}$ in R, where $n \in \mathbb{Z} \setminus \{-1, 0, 1, 2\}$. Then $CD \in \operatorname{cn}(R)$ since

$$CD = \begin{pmatrix} n & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ n-1 & 1 \end{pmatrix} + \begin{pmatrix} n & 1 \\ 1-n & -1 \end{pmatrix} \in R, \tag{4.4}$$

where the RHS is a clean decomposition (that is, the first matrix is idempotent, and the second is invertible). *We finish by showing that* $P := DC = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} \notin \operatorname{suit}(R)$. Indeed, if otherwise, there would exist an idempotent matrix $E \in R \cdot P$ such that $I_2 - E \in R \cdot (I_2 - P)$. Then E has the form $\begin{pmatrix} nx & 0 \\ ny & 0 \end{pmatrix}$, and $E = E^2$ implies that $n^2x^2 = nx$ and $n^2xy = ny$, so $x = y = 0$; that is, $E = 0$. But then

$$I_2 - E = I_2 \in R \cdot (I_2 - P) = R \cdot \begin{pmatrix} 1-n & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.5}$$

which is impossible in $R = \mathbb{M}_2(\mathbb{Z})$ since $n \notin \{0, 2\} \Rightarrow |1-n| \neq 1$. (Note that, if $n \in \{-1, 0, 1, 2\}$ instead, the matrix $P = DC$ would have been in $\operatorname{cn}(R)$.) ☐

What about the implication "$ab \in \mathcal{C}_R \Rightarrow ba \in \mathcal{C}_R$" for π-regular and strongly π-regular elements? If we take $\mathcal{C}_R$ to be the subset R^π of π-regular elements in R, we see easily that the complement property does not always hold. By Theorem 3.8, Jacobson's Lemma *does hold* for R^π. However, this no longer guarantees that $AB \in R^\pi \Rightarrow BA \in R^\pi$. For instance, taking the idempotents $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

[1] It may be said that $\operatorname{cn}(R)$ is the "additive version" of the set of unit-regular elements in R, since the latter are elements of the form eu where e is an idempotent and u is a unit.

and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ in the ring $R = \mathbb{M}_2(\mathbb{Z})$, $AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is regular (with unit inner inverse $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$) and hence π-regular. But $C := BA = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is *not* π-regular, since $C^n = C^n\,(x_{ij})\,C^n \in R$ for some $n \geq 1$ would have led to an impossible equation $2^n = 2^{2n} x_{11}$ in $\mathbb{Z}$.

As for the set of Drazin invertible elements in R (which is often denoted by R^{D} in the literature), the classical "Cline's Formula" in [6] (see also [3]) does guarantee that $ab \in R^{\mathrm{D}} \Rightarrow ba \in R^{\mathrm{D}}$ for all a, b in any ring R. However, this in itself does not imply Jacobson's Lemma for R^{D} (or vice versa), since the set R^{D} does not have the complement property in general. [For instance, in the ring $\mathbb{Z}$, $r = -1$ is (invertible and hence) Drazin invertible, but $1 - r = 2$ is not.] Indeed, the proof of Jacobson's Lemma for R^{D} for a general ring (as given in Theorem 2.1) is considerably harder than the classical proof for Cline's Formula in [6] or in [3].

While Proposition 4.2 might have been a bit of a disappointment, it turns out that there is a good "reason" behind the failure of Jacobson's Lemma for $\mathrm{cn}\,(R)$ and $\mathrm{ucn}\,(R)$; namely, the lack of a "commuting property" in the clean decompositions $\alpha = e + u$ used in the definitions of these sets. In a forthcoming work [17], we will show that, as soon as a commuting property is imposed on the intervening decompositions, a strong form of Jacobson's Lemma will hold for several classes of ring elements of interest to current researchers in ring theory, including the strongly clean elements of Nicholson [20], the strongly nil-clean elements of Diesl [8], and the quasipolar elements of Zhuang, Chen, and Cui [24], etc.

References

1. Pere Ara, *Strongly π-regular rings have stable range one*, Proc. Amer. Math. Soc. **124** (1996), no. 11, 3293–3298. MR 1343679 (97a:16024)
2. Gorô Azumaya, *Strongly π-regular rings*, J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1954), 34–39. MR 0067864 (16,788e)
3. Adi Ben-Israel and Thomas N. E. Greville, *Generalized Inverses*, second ed., CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 15, Springer-Verlag, New York, 2003, Theory and applications. MR 1987382 (2004b:15008)
4. N. Castro-González, C. Mendes-Araújo, and Pedro Patrício, *Generalized inverses of a sum in rings*, Bull. Aust. Math. Soc. **82** (2010), no. 1, 156–164. MR 2669152 (2011f:15011)
5. Huanyin Chen, *Unit-regularity and stable range one*, Bull. Korean Math. Soc. **47** (2010), no. 3, 653–661. MR 2666385 (2011f:16023)
6. R. E. Cline, *An application of representations for the generalized inverse of a matrix*, Tech. Summary Rep. **592** (1965).
7. Dragana Cvetkovic-Ilic and Robin Harte, *On Jacobson's lemma and Drazin invertibility*, Appl. Math. Lett. **23** (2010), no. 4, 417–420. MR 2594854 (2011g:16067)
8. Alexander James Diesl, *Classes of strongly clean rings*, ProQuest LLC, Ann Arbor, MI, 2006, Thesis (Ph.D.)–University of California, Berkeley. MR 2709132
9. M. P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly **65** (1958), 506–514. MR 0098762 (20 #5217)
10. ———, *Commuting properties of generalized inverses*, Lin. Multilin. Alg. (to appear), 7 p.
11. P. R. Halmos, *Does mathematics have elements?*, Math. Intelligencer **3** (1980/81), no. 4, 147–153. MR 642132 (83e:00004)
12. Dinesh Khurana and T. Y. Lam, *Rings with internal cancellation*, J. Algebra **284** (2005), no. 1, 203–235. MR 2115012 (2006d:16008)
13. J. J. Koliha, *A generalized Drazin inverse*, Glasgow Math. J. **38** (1996), no. 3, 367–381. MR 1417366 (98b:46065)
14. J. J. Koliha and Pedro Patrício, *Elements of rings with equal spectral idempotents*, J. Aust. Math. Soc. **72** (2002), no. 1, 137–152. MR 1868712 (2003b:16037)
15. T. Y. Lam, *A First Course in Noncommutative Rings*, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001. MR 1838439 (2002c:16001)
16. ———, *Exercises in Classical Ring Theory*, second ed., Problem Books in Mathematics, Springer-Verlag, New York, 2003. MR 2003255 (2004g:16001)
17. T. Y. Lam and Pace P. Nielsen, *Jacobson pairs and Bott-Duffin decompositions*, (in preparation).

18. W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278. MR 0439876 (55 #12757)
19. ______, *On exchange rings*, Comm. Algebra **25** (1997), no. 6, 1917–1918. MR 1446139
20. ______, *Strongly clean rings and Fitting's lemma*, Comm. Algebra **27** (1999), no. 8, 3583–3592. MR 1699586 (2000d:16046)
21. W. K. Nicholson and Y. Zhou, *Rings in which elements are uniquely the sum of an idempotent and a unit*, Glasg. Math. J. **46** (2004), no. 2, 227–236. MR 2062606 (2005d:16052)
22. Pedro Patrício and R. E. Hartwig, *The link between regularity and strong-pi-regularity*, J. Aust. Math. Soc. **89** (2010), no. 1, 17–22. MR 2727061 (2011k:16031)
23. Pedro Patrício and António Veloso da Costa, *On the Drazin index of regular elements*, Cent. Eur. J. Math. **7** (2009), no. 2, 200–205. MR 2506960 (2010e:16019)
24. Guifen Zhuang, Jianlong Chen, and Jian Cui, *Jacobson's lemma for the generalized Drazin inverse*, Linear Algebra Appl. **436** (2012), no. 3, 742–746. MR 2854904 (2012k:15006)

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