COUNTABLE EXCHANGE AND FULL EXCHANGE RINGS

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Abstract. We show that a Dedekind-finite, semi-π-regular ring with a "nice" topology is an $\aleph_0$-exchange ring, and the same holds true for a strongly clean ring with a "nice" topology. We generalize the argument to show that a Dedekind-finite, semi-regular ring with a "nice" topology is a full exchange ring. Putting these results in the language of modules, we show that a cohopfian module with finite exchange has countable exchange, and all modules with Dedekind-finite, semi-regular endomorphism rings are full exchange modules. These results are generalized further.

1. Introduction

The exchange property was first studied in 1964 by Crawley and Jónsson [3], and is defined for modules as follows: A right k-module M_k has the $\aleph$-*exchange property* if whenever $A = M \oplus N = \bigoplus_{i\in I} A_i$, with $|I| \leqslant \aleph$, then there are submodules $A_i' \subseteq A_i$, with $A = M \oplus \left(\bigoplus_{i\in I} A_i'\right)$. If M has $\aleph$-exchange for all cardinals $\aleph$ then we say M has *full exchange*. If the same holds just for the finite cardinals, we say M has *finite exchange*. It is easy to show that 2-exchange is equivalent to finite exchange. An outstanding question in module theory is whether or not finite exchange implies full exchange.

It turns out that the finite exchange property is an endomorphism ring invariant; putting $E = \text{End}(M_k)$, then M_k has finite exchange if and only if E_E has finite exchange. A ring, R, such that R_R has finite exchange is called an *exchange ring*, following Warfield [14], and this turns out to be a left-right symmetric condition. Nicholson [9] calls a ring *suitable* if for each equation $x+y=1$ there are (orthogonal) idempotents $e \in Rx$ and $f \in Ry$ with $e+f=1$. Suitable rings and exchange rings turn out to be the same objects. It is easy to show that semi-π-regular rings[1] are suitable, and while this is a large class it does not exhaust all suitable rings. Any corner ring in a suitable ring is suitable, and any direct product of suitable rings is suitable.

Continuous modules, and hence (quasi-)injective modules, always claim the exchange property [6]. Further, quasi-continuous modules with finite exchange have full exchange [7], [12]. There are many other classes of modules for which finite exchange implies full exchange, including modules which are direct sums of indecomposables [15], and modules with abelian endomorphism rings [11]. It also turns out that square-free modules[2] with finite exchange have countable exchange [8].

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[1] $R/J(R)$ is π-regular, and idempotents lift modulo $J(R)$. An ring R is π-regular, if for each $x \in R$ there is some $n \in \mathbb{Z}_+$ so that x^n is von Neumann regular.

[2] No non-zero submodule is isomorphic to a square $X \oplus X$.

Any endomorphism ring, $E = \mathrm{End}(M)$, is endowed with a ring topology, called the *finite topology*, in which a basis of neighborhoods of zero is given by annihilators of finite subsets of M. One says that a collection $\{x_i\}_{i\in I}$ of endomorphisms in E is *summable*, if for each $m \in M$ the set $\{i \mid x_i(m) \neq 0\}$ is finite. One may then easily define $\sum_{i\in I} x_i$ as the map $m \mapsto \sum_{i\in I} x_i(m)$. Central to the study of exchange modules is the following proposition:

Proposition 1.1 ([15, Proposition 3])**.** *For any given cardinal, $\aleph$, the following are equivalent:*

(1) *M has the $\aleph$-exchange property.*
(2) *If we have*
$$A = M \oplus N = \bigoplus_{i\in I} A_i$$
with $A_i \cong M$ for all $i \in I$, $|I| \leqslant \aleph$, then there are submodules $A_i' \subseteq A_i$ such that
$$A = M \oplus \bigoplus_{i\in I} A_i'.$$
(3) *Given a summable family $\{x_i\}_{i\in I}$ of elements of E, with $\sum_{i\in I} x_i = 1$, and with $|I| \leqslant \aleph$, then there are orthogonal idempotents $e_i \in Ex_i$ with $\sum_{i\in I} e_i = 1$.*

Now, let R be a topological ring with a linear, Hausdorff topology. This means that there is a ring topology with a basis of neighborhoods of zero, say $\mathfrak{U}$, consisting of left ideals, with $\bigcap_{U\in\mathfrak{U}} U = (0)$. We say that the collection $\{x_i\}_{i\in I}$ of elements of R is *summable to* $r \in R$, written as $\sum_{i\in I} x_i = r$, if for every $U \in \mathfrak{U}$ there is a finite set $F' \subseteq I$ such that $\sum_{i\in F} x_i - r \in U$ for all finite sets $F \supseteq F'$. The finite topology on E is linear and Hausdorff, and this new notion of summability agrees with the one defined above. Following [8], we can now extract from Proposition 1.1, property (3), a ring-theoretic version of $\aleph$-exchange.

Definition 1.2. Let R be a ring with a linear, Hausdorff topology. We say that R is an *$\aleph$-exchange ring* if for each summable family $\{x_i\}_{i\in I}$ in R, with $|I| \leqslant \aleph$ and with $\sum_{i\in I} x_i = 1$, there are summable, orthogonal idempotents $\{e_i\}_{i\in I}$ with $e_i \in Rx_i$ and $\sum_{i\in I} e_i = 1$.[3] If this holds for all cardinalities $\aleph$, we say the ring is a *full exchange ring*.[4]

Notice that, by (1)$\Leftrightarrow$(3) in Proposition 1.1, a module has $\aleph$-exchange if and only if E (with the finite topology) is an $\aleph$-exchange ring. Also notice, in the definition above we require $\{e_i\}_{i\in I}$ to be a summable family. When trying to verify that a ring is an $\aleph$-exchange ring, we often need to assume some condition which forces families of this sort to be summable. The following is such a condition: We say a summable family $\{x_i\}_{i\in I}$ is *left multiple summable* if for every arbitrary family $\{r_i\}_{i\in I}$, the collection $\{r_ix_i\}_{i\in I}$ is also summable. We say that a topology is *left multiple summable* if all summable families are left multiple summable. Finally, we

[3]The definition we have given is not equivalent to the one given in [8]. We've added the word "orthogonal," and removed the word "complete." However, from a private correspondence with the first author of [8], it was made clear that the definition given here is the one they intended.

[4]One easily sees that the notion of a full exchange ring is not equivalent to the notion of an exchange ring, since the former requires a ring topology to define. However, to avoid any confusion we will always refer to exchange rings as suitable rings.

say that a topological ring, R, has a *nice topology* if the topology is linear, Hausdorff, and left multiple summable. One can easily show that a complete, linear, Hausdorff topology is nice. We work with nice ring topologies rather than complete, linear, Hausdorff topologies simply for sake of broader generality. However, nice topologies are very nearly complete as follows:

Changing the terminology of [8], we say that a family $\{x_i\}_{i\in I}$ is *S-Cauchy* if for each $U \in \mathfrak{U}$ there is a finite set, F_U, with $a_i \in U$ for all $i \notin F_U$. Note that the definition of S-Cauchy is weaker than the definition of Cauchy that is usually given (hence the change in terminology from [8]). For a general introduction to Cauchy families and filters in topological rings the reader is referred to [1]. It is well known that a summable family is always S-Cauchy, and the converse is true if the ring is complete [2, Chapter 3]. If we replace completeness with left multiple summability we still have the converse.

Lemma 1.3. *Let R be a topological ring with a linear, Hausdorff topology. Every S-Cauchy family is summable if and only if the ring topology on R is left multiple summable.*

Proof. The forward direction is easy since $\mathfrak{U}$ is made up of left ideals, so left multiples of S-Cauchy families are still S-Cauchy. For the other direction let $X = \{x_i\}_{i\in I}$ be an S-Cauchy family. For each $U \in \mathfrak{U}$ fix a finite set $F_U \subseteq I$ so that $x_i \in U$ for each $i \notin F_U$. Let I' be an copy of I, disjoint from I. For each subset $A \subseteq I$, let $A' \subseteq I'$ be the corresponding subset under the identification of I with I'. Let $T = \{x_i, -x_{i'}\}_{i\in I, i'\in I'}$, so T is the disjoint union of X and $-X$. Then we claim that T is summable and sums to 0. In fact, for each finite set $F \supseteq F_U \cup F'_U$ we see that $\sum_{i\in F\cap I} x_i + \sum_{i'\in F\cap I'} -x_{i'} \in U$. Thus T satisfies the definition of summability.

By left multiple summability, we can multiply the elements of $T \cap X$ by 1 and the elements of $T \cap (-X)$ by 0, and we still have a summable family. But this new family is just X (along with extra copies of 0), which proves the claim. □

In §2 we develop some machinery that allows us to work recursively with suitable rings. In §3 we describe a general construction which shows that certain classes of nice topological rings are $\aleph_0$-exchange rings. Then in §4 we generalize the construction to show full exchange, as long as two technical conditions are met. We do some specific examples in §6, but before that we push the argument through the Jacobson radical in §5. These results are reinterpreted in module theoretic language in §7. We close with some results on abelian rings and commutative rings in §8 and then some final remarks in §9, where we provide a counter-example to the converse of Theorem 3.1.

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2. Tools for Suitable Rings

Throughout this paper we let k be a ring, M_k be a right k-module, and put $E = \mathrm{End}(M_k)$, which acts on the left of M. If we have two modules N and N' we write $N \subseteq^{\oplus} N'$ to mean that N is a direct summand of N'. Also throughout, we let R be a ring, $U(R)$ the group of units, and $J(R)$ the Jacobson radical. Rings are associative with 1, and modules are unital.

In our study of $\aleph$-exchange rings, we first investigate the behavior of idempotents in suitable rings. To begin, we define a useful equivalence relation on idempotents, as in [11, §5].

Definition 2.1. Let $e, e' \in R$ be idempotents. We say that e and e' are *left associate* if $e'e = e'$ and $ee' = e$.[5] We write this relation as $e \sim_\ell e'$, and it is easy to check that this is an equivalence relation. One also has the dual notion of right associate idempotents, which we denote by $e \sim_r e'$.

Lemma 2.2. *Let $e, e' \in R$ be idempotents. The following are equivalent:*

(1) $e \sim_\ell e'$.
(2) $Re = Re'$.
(3) $e' = e + (1-e)re$ *for some* $r \in R$.
(4) $e' = ue$ *for some* $u \in U(R)$.
(5) $e' = ue$ *for some* $u \in U(R)$*, with* $u(1-e) = (1-e)$.
(6) $(1-e) \sim_r (1-e')$.

Proof. The equivalence of properties (1) through (4) and (6) is a simple exercise [La_2, Exercise 21.4]. Clearly (5) implies (4). Finally we show (3) implies (5). By hypothesis $e' = e + (1-e)re$ for some $r \in R$. Putting $u = 1 + (1-e)re$, we see that $e' = ue$, u is a unit with inverse $u^{-1} = 1 - (1-e)re$, and $u(1-e) = (1-e)$ □

As an aside, although we don't need any further properties of the unit constructed above it is also true that $u(1-e') = (1-e')$, $eu = e$, $e'u = e'$, and $(1-e)u^{-1} = 1 - e'$.

The next two lemmas give us computational tools we will use to work recursively with suitable rings. They were proven in [11, Lemmas 4 and 5], but for completeness we reproduce the proofs here.

Lemma 2.3. *Let R be a suitable ring, and let $x_1+x_2+x_3 = 1$ be an equation in R. Suppose that x_1 is an idempotent. Then there are pair-wise orthogonal idempotents $e_1 \in Rx_1$, $e_2 \in Rx_2$, and $e_3 \in Rx_3$, such that $e_1 + e_2 + e_3 = 1$ and $x_1 \sim_\ell e_1$.*

Proof. Let $f = 1 - x_1$, and multiply by f on the left and right of $x_1 + x_2 + x_3 = 1$ to obtain $fx_2f + fx_3f = f$. Since corner rings in suitable rings are suitable [9, Proposition 1.10], fRf is suitable. Hence, there are orthogonal idempotents $f_2 \in fRf(fx_2f)$ and $f_3 \in fRf(fx_3f)$ summing to f, the identity in fRf. Write $f_2 = fr_2fx_2f$ and $f_3 = fr_3fx_3f$ for some $r_2, r_3 \in R$.

Let $e_2 = f_2r_2fx_2 \in Rx_2$ and let $e_3 = f_3r_3fx_3 \in Rx_3$. By an easy calculation we see that e_2 and e_3 are orthogonal idempotents. Let $e_1 = 1 - e_2 - e_3$, so e_1 is orthogonal to e_2 and e_3, with $e_1 + e_2 + e_3 = 1$. Then

$$\begin{aligned} e_1x_1 &= (1-e_2-e_3)(1-f) = 1 - e_2 - e_3 - f + e_2f + e_3f \\ &= e_1 - f + f_2 + f_3 = e_1 - f + f = e_1, \end{aligned}$$

so $e_1 \in Rx_1$. Finally, since $fe_2 = e_2$ and $fe_3 = e_3$, we see $x_1e_1 = x_1(1-e_2-e_3) = x_1$. □

[5]In [11] we called such idempotents *left strongly isomorphic*, since this relation strengthens the notion of isomorphic idempotents. However, our new terminology seems more appropriate in light of Lemma 2.2, property (4).

Lemma 2.4. *Let $e, e' \in R$ be idempotents, with $e \sim_\ell e'$. Assume R has a linear, Hausdorff topology. Also assume that $e = \sum_{i\in I} g_i$ where $\{g_i\}_{i\in I}$ is a summable family of orthogonal idempotents. Then $\{e'g_i\}_{i\in I}$ is a summable family of orthogonal idempotents, summing to e', with $g_i \sim_\ell e'g_i$. Further, if $e' = ue$ then $e'g_i = ug_i$. Finally, if f is any idempotent orthogonal to e, then f is orthogonal to each g_i.*

Proof. Notice that $g_i e = g_i = eg_i$ and $ee' = e$. Therefore

$$(e'g_i)(e'g_j) = e'(g_i e)e'g_j = e'g_i(ee')g_j = e'g_i e g_j = e'g_i g_j = \delta_{i,j} e'g_i.$$

So they are orthogonal idempotents. Also $g_i(e'g_i) = (g_i e)(e'g_i) = g_i e g_i = g_i$ and clearly $(e'g_i)g_i = e'g_i$. Thus $g_i \sim_\ell e'g_i$. If $e' = ue$ then $e'g_i = ueg_i = ug_i$. The final statement is another easy calculation. □

It will turn out that we will be working with families of idempotents that are "almost" orthogonal, which we want to modify into truly orthogonal families. The following lemmas give us the mathematical framework to make this happen.

Lemma 2.5. *Let $\{e_i\}_{i\in I}$ be a summable family of idempotents in a ring, R, with a linear, Hausdorff topology, and assume I is well-ordered. Suppose that $e_i e_j \in J(R)$ whenever $i < j$, and that $\sum_{i\in I} e_i = u \in U(R)$. Then $\{u^{-1}e_i\}_{i\in I}$ is a summable family of orthogonal idempotents, summing to 1.*

Proof. Follows from [8, Lemma 8]. □

Lemma 2.6. *Let $\{e_i\}_{i\in I}$ be a summable family of idempotents in a ring, R, with a linear, Hausdorff topology, and assume I is well-ordered. Put $e = \sum_{i\in I} e_i$ and suppose that $e_i e_j = 0$ whenever $i < j$. If $e^n r = 0$, for some $r \in R$ and some $n \in \mathbb{Z}_+$, then we have $e_i r = 0$ for all $i \in I$. In particular, $er = 0$.*

Proof. We proceed by induction. Since $e_i e_j = 0$ for $i < j$, this implies $e_1 e = e_1$ (where 1 is the first element of I). Therefore $e_1 e^n = e_1$, and so $e_1 r = e_1 e^n r = 0$. This finishes the base case.

Now, suppose that $e_i r = 0$ for all $i < \beta$. Then $er = \left(\sum_{i\geqslant\beta} e_i\right) r$. Again since $e_i e_j = 0$ for $i < j$, we have

$$e^{n-1}\left(\sum_{i\geqslant\beta} e_i\right) = e^{n-2}\left(\sum_{i<\beta} e_i + \sum_{i\geqslant\beta} e_i\right)\left(\sum_{i\geqslant\beta} e_i\right)$$
$$= e^{n-2}\left(\sum_{i\geqslant\beta} e_i\right)^2 = \cdots = \left(\sum_{i\geqslant\beta} e_i\right)^n.$$

So,

$$0 = e_\beta e^n r = e_\beta e^{n-1}\left(\sum_{i\geqslant\beta} e_i\right) r = e_\beta \left(\sum_{i\geqslant\beta} e_i\right)^n r = e_\beta r.$$

This finishes the inductive step. It is now clear that $er = 0$ also. □

Lemma 2.7. *Let R be a suitable ring with a linear, Hausdorff topology. Then $J(R)$ is closed.*

Proof. This is [8, Lemma 11]. The lemma they prove is for endomorphism rings, but the argument already works in this more general situation. □

Lemma 2.8. *Let R be a suitable ring, and put $\overline{R} = R/J(R)$. If $\varepsilon \in \overline{R}\overline{x}$ is an idempotent, then there is an idempotent $e \in Rx$ with $\overline{e} = \varepsilon$.*

Proof. Follows easily from [8, Corollary 7]. □

3. $\aleph_0$-Exchange Rings

The motivation for our first result comes from a simple construction showing that 2-exchange is equivalent to finite exchange for modules, based upon ideas in [9]. Unfortunately, the method fails when trying to pass to countable exchange. In the course of the proof we construct an element of our ring which we need to be a unit. For large classes of suitable rings this element will be a unit, and so we can show that these rings are $\aleph_0$-exchange rings.

Theorem 3.1. *Let R be a suitable ring with a nice topology. If the condition $(*)$ (which is defined and boxed below) holds then R is an $\aleph_0$-exchange ring.*

Proof. Let $\{x_i\}_{i\in\mathbb{Z}_+}$ be a summable family of elements in R, with $\sum_{i=1}^{\infty} x_i = 1$. For notational ease, set $y_j = \sum_{i>j} x_i$. For each $j \in \mathbb{Z}_+$ we will construct elements $e_{i,j} \in Rx_i$ (for $i \leqslant j$), $f_j \in Ry_j$, and $v_j \in U(R)$ such that the following conditions hold:

(A) The family $\{e_{1,j}, e_{2,j}, \ldots, e_{j,j}, f_j\}$ consists of orthogonal idempotents that sum to 1.
(B) For all $i \leqslant j$, $v_j e_{i,i} = e_{i,j}$ and $v_j f_j = f_j$.

Set $v_1 = 1$. Since R is suitable, the equation $x_1 + y_1 = 1$ implies that there are orthogonal idempotents $e_{1,1} \in Rx_1$ and $f_1 \in Ry_1$ with $e_{1,1} + f_1 = 1$. It is easy to check that condition (A) holds for $j = 1$, and condition (B) holds trivially in this case. This finishes the base case. Suppose, by induction, we have fixed elements $e_{i,j} \in Rx_i$ (for all $i \leqslant j$), $f_j \in Ry_j$, and $v_j \in U(R)$ satisfying the conditions above, for each $j \leqslant n$. Writing $f_n = r_n y_n$ for some $r_n \in R$, we have

$$1 = e_{1,n} + \cdots + e_{n,n} + f_n = (e_{1,n} + \cdots + e_{n,n}) + r_n x_{n+1} + r_n y_{n+1}.$$

Lemma 2.3 allows us to pick pair-wise orthogonal idempotents

$$h_1 \in R(e_{1,n} + \cdots + e_{n,n}), \qquad h_2 \in Rr_n x_{n+1}, \qquad h_3 \in Rr_n y_{n+1}$$

with $h_1 + h_2 + h_3 = 1$ and $h_1 \sim_\ell \sum_{i=1}^{n} e_{i,n}$. By Lemma 2.2, property (5), there exists $u_{n+1} \in U(R)$ such that $u_{n+1}(e_{1,n} + \cdots + e_{n,n}) = h_1$ and $u_{n+1} f_n = f_n$. Putting $e_{i,n+1} = u_{n+1} e_{i,n} \in Rx_i$ (for $i \leqslant n$), $e_{n+1,n+1} = h_2 \in Rx_{n+1}$, and $f_{n+1} = h_3 \in Ry_{n+1}$, Lemma 2.4 shows that condition (A) above holds.

By Lemma 2.2, property (6), $(e_{n+1,n+1} + f_{n+1})$ is right associate to f_n, hence $f_n e_{n+1,n+1} = e_{n+1,n+1}$ and $f_n f_{n+1} = f_{n+1}$. Putting $v_{n+1} = u_{n+1} v_n$, and remembering $u_{n+1} f_n = f_n$, we calculate

$$\begin{aligned} v_{n+1} f_{n+1} &= (u_{n+1} v_n)(f_n f_{n+1}) = u_{n+1} v_n f_n f_{n+1} \\ &= u_{n+1} f_n f_{n+1} = f_n f_{n+1} = f_{n+1} \end{aligned}$$

and similarly $v_{n+1} e_{n+1,n+1} = e_{n+1,n+1}$. Finally, for $i < n+1$,

$$v_{n+1} e_{i,i} = u_{n+1} v_n e_{i,i} = u_{n+1} e_{i,n} = e_{i,n+1}.$$

Therefore, condition (B) holds. This finishes the inductive step.

So we have constructed elements $e_{i,j}$ (for $i \leqslant j$), f_j, and v_j satisfying the properties above, for all $j \in \mathbb{Z}_+$. Since $\{x_i\}_{i\in\mathbb{Z}_+}$ is summable, and the topology is left multiple summable, the family $\{e_{i,i}\}_{i\in\mathbb{Z}_+}$ is also summable. We put $\varphi = \sum_{i\in\mathbb{Z}_+} e_{i,i}$ and assume that the following condition holds:

$$\boxed{(*) \quad \varphi \text{ is a unit.}}$$

Now, for $i < j$ we have $e_{i,i}e_{j,j} = v_j^{-1}v_j e_{i,i}e_{j,j} = v_j^{-1}e_{i,j}e_{j,j} = 0 \in J(R)$. So, by Lemma 2.5, $\{\varphi^{-1}e_{i,i}\}_{i\in\mathbb{Z}_+}$ is a summable, orthogonal set of idempotents, summing to 1. Finally, $e_i = \varphi^{-1}e_{i,i} \in Rx_i$, so R satisfies the definition of an $\aleph_0$-exchange ring. □

There are quite a few properties that the element φ has, even if we don't assume $(*)$ holds. For example, it is a convergent limit of units. In fact, since $\lim_{n\to\infty} y_n = 0$, and the topology is linear, we have $\lim_{n\to\infty} f_n = 0$. Therefore,

$$\varphi = \sum_{i=1}^{\infty} e_{i,i} = \lim_{n\to\infty}\left(\sum_{i=1}^{n} e_{i,i} + f_n\right) = \lim_{n\to\infty} v_n^{-1}\left(\sum_{i=1}^{n} e_{i,n} + f_n\right) = \lim_{n\to\infty} v_n^{-1}.$$

A natural question to ask is what convergent limits of units look like in general. We claim that in any ring with a linear, Hausdorff topology, a convergent limit of units is always a left non-zero-divisor. To see this, let $w = \lim_{i\in I} w_i$ with each w_i a unit, and with I directed. Let $U \in \mathfrak{U}$ be an arbitrary, open (left ideal) neighborhood of 0. If $wr = 0$ then $\lim_{i\in I} w_i r = 0$ and so, in particular, for a large index N we have $w_N r \in U$. But U being a left ideal means $r = w_N^{-1}w_N r \in U$. Therefore $r \in \bigcap_{U\in\mathfrak{U}} U = (0)$. So $r = 0$.

Theorem 3.1 gives us the following chain of corollaries.

Corollary 3.2. *Let R be a suitable ring with a nice topology, and set $\overline{R} = R/J(R)$. If $\overline{R}_{\overline{R}}$ is cohopfian, then R is an $\aleph_0$-exchange ring.*

Proof. We have $\varphi = \lim_{n\to\infty} v_n^{-1}$ where $v_n \in U(R)$. By Theorem 3.1, it suffices to show that φ is a unit. An element $r \in R$ is a unit if and only if $\overline{r} \in \overline{R}$ is a unit. Therefore it suffices to show that $\overline{\varphi}$ is a unit. Further, by Lemma 2.7, we have $\overline{\varphi} = \lim_{n\to\infty} \overline{v}_n^{-1}$ in the quotient topology. Since $\overline{\varphi}$ is a convergent limit of units it is a left non-zero-divisor. By [4, Exercise 4.16], which is an exercise we leave to the reader, $\overline{R}_{\overline{R}}$ is cohopfian if and only if all left non-zero-divisors are units. Thus $\overline{\varphi}$ is a unit. □

Corollary 3.3. *Let R be a ring with a nice topology. If R is a Dedekind-finite, semi-π-regular ring then R is an $\aleph_0$-exchange ring.*

Proof. All semi-π-regular rings are suitable rings. So, from the previous corollary it suffices to show that $\overline{R}_{\overline{R}}$ is cohopfian.

Fix $x \in \overline{R}$ which is a left non-zero-divisor. Since $\overline{R}$ is π-regular, fix $n \geqslant 1$, and $y \in \overline{R}$, such that $x^n = x^n y x^n$. Then $x^n(\overline{1} - yx^n) = 0$. Since x is a left non-zero-divisor so is x^n. Therefore $\overline{1} = yx^n$, and so x is left-invertible. From the Dedekind-finiteness, which passes to $\overline{R}$, x is invertible. □

Corollary 3.4. *Let R be a ring with a nice topology. If R is a strongly π-regular ring then R is an $\aleph_0$-exchange ring.*

Proof. Strongly π-regular rings are always Dedekind-finite and π-regular. □

There is another nice property that φ exhibits, arising from the fact it is a sum of "almost" orthogonal idempotents. For this we need another lemma.

Lemma 3.5. *Let $\{e_i\}_{i\in I}$ be a summable family of idempotents in a ring, R, with a linear, Hausdorff topology, and assume I is well-ordered. Suppose that $e_ie_j = 0$ whenever $i < j$, and set $\varphi = \sum_{i\in I} e_i$. Let $K \subseteq R$ be a right ideal such that $(1-\varphi)K \supseteq K$. Then $\varphi K = (0)$.*

Proof. It suffices to show that $e_iK = (0)$ for each $i \in I$. We work by induction. Let $r \in K$. Then $r = (1-\varphi)r'$ for some $r' \in K$. Hence $e_1r = e_1(1-\varphi)r' = 0$, where 1 is the first element of I. Since $r \in K$ is arbitrary, $e_1K = (0)$. Suppose now, by induction, that $e_iK = (0)$ for all $i < \beta$. Then

$$e_\beta r = e_\beta(1-\varphi)r' = e_\beta r' - \sum_{i\geqslant\beta} e_\beta e_i r' = -\sum_{i>\beta} e_\beta e_i r' = 0.$$

Hence $e_\beta K = (0)$. This finishes our induction. □

Recall, an element $r \in R$ is *clean* when we can write $r = u + e$, where $u \in U(R)$ and $e^2 = e \in R$. It is *strongly clean* if we can choose u and e so that they commute: $ue = eu$. A ring is *clean* (respectively, *strongly clean*) when every element is clean (respectively, strongly clean). Nicholson proved that all clean rings are suitable [9, Proposition 1.8], and every strongly π-regular ring is strongly clean [10, Theorem 1]. Using these facts we can strengthen Corollary 3.4 to the following:

Corollary 3.6. *Let R be a ring with a nice topology. If $R/J(R)$ is strongly clean, and idempotents lift modulo $J(R)$, then R is an $\aleph_0$-exchange ring.*

Proof. First, R is a suitable ring by [9, Proposition 1.5].[6] So it suffices to show that $\varphi = \sum_{i=1}^{\infty} e_{i,i}$ (as constructed in Theorem 3.1) is a unit. Second, since φ is a unit if and only if $\overline{\varphi}$ is a unit, it suffices to show the latter. Third, by Lemma 2.7, we have $\overline{\varphi} = \sum_{i=1}^{\infty} \overline{e}_{i,i}$.

Now, since $\overline{\varphi}$ is strongly clean, we can write $\overline{\varphi} = u+e$ with $u \in U(\overline{R})$, $e^2 = e \in \overline{R}$, and $ue = eu$. We compute $\overline{(1-\varphi)}e = (\overline{1} - u - e)e = -ue$. Hence $\overline{(1-\varphi)}$ acts as the unit $-u$ on $e\overline{R}$. Further, since u and e commute, this means $\overline{(1-\varphi)}$ is an automorphism of $e\overline{R}$. By Lemma 3.5 (applied to $\overline{R}$), $\overline{\varphi}e\overline{R} = (0)$ and in particular $\overline{\varphi}e = 0$. But $\overline{\varphi}$ is a limit of units, and hence a left non-zero-divisor. This means $e = 0$, so $\overline{\varphi} = u + e = u$ is a unit. □

Any strongly clean ring (with a nice topology) satisfies the conditions of this corollary. On the other hand, by [13] there are rings which are strongly clean modulo their radicals, but not strongly clean themselves (although idempotents lift through the radical). Thus, the hypotheses of the lemma are not superfluous.

Although we stated Lemma 3.5 for a general index set (so that we can use the result in later sections) in this section we are most interested in the case when $I = \mathbb{Z}_+$. Supposing this equality holds, one can improve Lemma 3.5 in a straightforward manner to show that $\varphi K = 0$ where $K = \bigcap_{i=1}^{\infty}(1-\varphi)^iR$. In particular, if φ is a left non-zero-divisor we must have $K = 0$. This yields:

Proposition 3.7. *Let R be a suitable ring with a nice topology. If R is such that, for $x \in R$, $\bigcap_{i=1}^{\infty}(1-x)^iR = 0$ implies $x \in U(R)$, then R is an $\aleph_0$-exchange ring.*

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[6] In fact, R is clean.

4. Full Exchange Rings

When trying to push the proof of Theorem 3.1 up to full exchange one runs into problems when passing through limit ordinals. However, we can get around these roadblocks by assuming a few more conditions. Just as in the proof of Theorem 3.1, we will define these conditions in the body of the proof.

Theorem 4.1. *Let R be a suitable ring with a nice topology. If R satisfies conditions $(*_1)$ and $(*_2)$ (defined and boxed below) then R is a full exchange ring.*

Proof. Let $\{x_i\}_{i\in I}$ be a summable collection of elements of R, summing to 1, with I an indexing set of arbitrary cardinality. Without loss of generality, we may assume that I is a well-ordered set with first element 1 and last element κ. Put $y_j = \sum_{i>j} x_i$ and $y'_j = x_j + y_j = \sum_{i\geqslant j} x_i$.

For each $j \in I$ we will inductively construct elements $e_{i,j} \in Rx_i$ (for $i \leqslant j$), $f_j \in Ry_j$, and $v_j \in U(R)$ such that the following two conditions hold:

(A) The family $\{e_{1,j}, e_{2,j}, \ldots, e_{j,j}, f_j\}$ consists of summable, orthogonal idempotents, summing to 1.

(B) For each $i \leqslant j$, $v_j e_{i,i} = e_{i,j}$ and $v_j f_j = f_j$.

Put $v_1 = 1$. Since R is suitable the equation $x_1 + y_1 = 1$ implies that there are orthogonal idempotents $e_{1,1} \in Rx_1$ and $f_1 \in Ry_1$ with $e_{1,1} + f_1 = 1$. This provides the base step of our inductive definition. Now suppose (by transfinite induction) that for all $j < \alpha$ we have constructed elements $e_{i,j}$ (for all $i \leqslant j$), f_j, and v_j satisfying the conditions above. We have two cases.

Case 1. α is not a limit ordinal.

In this case we proceed exactly as in the proof of Theorem 3.1. Writing $f_{\alpha-1} = r_{\alpha-1}y_{\alpha-1}$ for some $r_{\alpha-1} \in R$, we have

$$1 = \sum_{i<\alpha} e_{i,\alpha-1} + f_{\alpha-1} = \sum_{i<\alpha} e_{i,\alpha-1} + r_{\alpha-1}x_\alpha + r_{\alpha-1}y_\alpha.$$

Lemma 2.3 allows us to pick orthogonal idempotents

$$h_1 \in R\left(\sum_{i<\alpha} e_{i,\alpha-1}\right), \qquad h_2 \in Rr_{\alpha-1}x_\alpha, \qquad h_3 \in Rr_{\alpha-1}y_\alpha$$

with $h_1 + h_2 + h_3 = 1$ and $h_1 \sim_\ell \sum_{i<\alpha} e_{i,\alpha-1}$. By Lemma 2.2, property (5), there exists $u_\alpha \in U(R)$ such that $h_1 = u_\alpha\left(\sum_{i<\alpha} e_{i,\alpha-1}\right)$ and $u_\alpha f_{\alpha-1} = f_{\alpha-1}$. Putting $e_{i,\alpha} = u_\alpha e_{i,\alpha-1} \in Rx_i$ (for $i < \alpha$), $e_{\alpha,\alpha} = h_2 \in Rx_\alpha$, and $f_\alpha = h_3 \in Ry_\alpha$, then Lemma 2.4 implies that these are orthogonal idempotents. Also clearly

$$\sum_{i\leqslant\alpha} e_{i,\alpha} + f_\alpha = 1.$$

Therefore, condition (A) holds when $j = \alpha$. Checking that condition (B) holds for $v_\alpha = u_\alpha v_{\alpha-1}$ is done exactly as before. This completes the inductive definition of the elements we need, when α is a successor ordinal.

Case 2. α is a limit ordinal.

This case is much harder. Set $\varphi = \sum_{i<\alpha} e_{i,i}$. We calculate that if $i < j < \alpha$, then $e_{i,i}e_{j,j} = v_j^{-1}v_je_{i,i}e_{j,j} = v_j^{-1}e_{i,j}e_{j,j} = 0$. So φ is a sum of "almost" orthogonal idempotents. We assume

$(*_1)$ There exists an idempotent p such that $\varphi' = \varphi + p$ is a unit and $\varphi p = 0$.

For notational ease put $v'_\alpha = (\varphi')^{-1}$. From our work above, and Lemma 2.6, we see that $e_{i,i}p = 0$ for all $i < \alpha$. This means that the decomposition $\varphi' = \sum_{i<\alpha} e_{i,i} + p$ satisfies the hypotheses of Lemma 2.5. This yields $\sum_{i<\alpha} v'_\alpha e_{i,i} + v'_\alpha p = 1$, where the summands are orthogonal idempotents. Put $e'_{i,\alpha} = v'_\alpha e_{i,i}$ for all $i < \alpha$, and $f'_\alpha = v'_\alpha p$. The following argument shows that $f'_\alpha = p$, and in particular $v'_\alpha f'_\alpha = f'_\alpha$, which we will need later. First note that $v'_\alpha p e'_{i,\alpha} = f'_\alpha e'_{i,\alpha} = 0$ and so $pe'_{i,\alpha} = 0$. We already saw $e_{i,i}p = 0$ and so $e'_{i,\alpha}p = 0$ for all $i < \alpha$. Hence $r = \sum_{i<\alpha} e'_{i,\alpha} + p$ is a sum of orthogonal idempotents, and hence an idempotent. Further, if $rs = 0$ for some $s \in R$, then orthogonality implies $e'_{i,\alpha}s = 0$ and $ps = 0$. Then $(\varphi + p)s = 0$ and so $s = 0$. But this means r is a left non-zero-divisor and an idempotent, so $r = 1$. Thus, $p = 1 - \sum_{i<\alpha} e'_{i,\alpha} = f'_\alpha$.

We also claim $f_j f'_\alpha = f'_\alpha$ for all $j < \alpha$. To see this we compute

$$e_{i,j} f'_\alpha = v_j e_{i,i} f'_\alpha = v_j v'^{-1}_\alpha v'_\alpha e_{i,i} f'_\alpha = v_j v'^{-1}_\alpha e'_{i,\alpha} f'_\alpha = 0$$

and so

$$f_j f'_\alpha = \left(1 - \sum_{i \leqslant j} e_{i,j}\right) f'_\alpha = f'_\alpha. \tag{1}$$

Notice that we put hash marks on the idempotents we constructed. This is because they are not quite the ones we set out to construct. We need a few more modifications. The first problem with the idempotents we constructed above is that f'_α is not a left multiple of y'_α. We can fix this problem by finding a new idempotent $f''_\alpha \in Ry'_\alpha$, which is right associate to f'_α. The construction is as follows:

For use shortly, we note

$$\lim_{i \to \alpha} y_i = y'_\alpha, \tag{2}$$

where by $\lim_{i\to\alpha} y_i$ we mean the limit in the ring topology on R. (More formally, if $\mathfrak{U}$ is the given basis of neighborhoods of zero, then for each $U \in \mathfrak{U}$ there is some index $j \in I$, $j < \alpha$, so that for each $i \in (j, \alpha)$, $y_i - y'_\alpha \in U$.) Also by construction, for $i < \alpha$ we have $f_i \in Ry_i$, and so we can fix elements $r_i \in R$ with $f_i = r_i y_i$. We claim that left multiplication by y'_α gives an isomorphism $f'_\alpha R \to y'_\alpha f'_\alpha R$. It suffices to show that if $y'_\alpha f'_\alpha r = 0$ then $f'_\alpha r = 0$. Using equations (1) and (2) above, we see

$$f'_\alpha r = \lim_{i\to\alpha} f_i f'_\alpha r = \lim_{i\to\alpha} r_i y_i f'_\alpha r = \lim_{i\to\alpha} r_i y'_\alpha f'_\alpha r = 0$$

as claimed. Let $r'_\alpha : y'_\alpha f'_\alpha R \to f'_\alpha R$ be the isomorphism which is the inverse to $y'_\alpha|_{f'_\alpha R}$. In our calculation above, we saw that $r'_\alpha = \lim_{i\to\alpha} r_i|_{y'_\alpha f'_\alpha R}$. Notice that, *a priori*, the map r'_α does not extend to an element in $R = \mathrm{End}(R_R)$, since this limit might not converge on all of R. However, if r'_α did extend to an element in R, that would be equivalent to:

$(*_2)$ There exists $r'_\alpha \in R$ such that $r'_\alpha y'_\alpha f'_\alpha = f'_\alpha$.

We assume $(*_2)$ holds.

Set $f''_\alpha = f'_\alpha r'_\alpha y'_\alpha$. We do the calculations to check that f''_α is right associate to f'_α and is an idempotent. First,

$$f''_\alpha f''_\alpha = f'_\alpha r'_\alpha y'_\alpha f'_\alpha r'_\alpha y'_\alpha = f'_\alpha (r'_\alpha y'_\alpha f'_\alpha) r'_\alpha y'_\alpha = f'_\alpha f'_\alpha r'_\alpha y'_\alpha = f''_\alpha.$$

Second, one easily sees $f'_\alpha f''_\alpha = f''_\alpha$. Finally,

$$f''_\alpha f'_\alpha = f'_\alpha r'_\alpha y'_\alpha f'_\alpha = f'_\alpha (r'_\alpha y'_\alpha f'_\alpha) = (f'_\alpha)^2 = f'_\alpha.$$

We have shown $f'_\alpha \sim_r f''_\alpha$. Therefore the equivalence of properties (1) and (6) in Lemma 2.2 implies $(1-f'_\alpha) \sim_\ell (1-f''_\alpha)$. So, again by Lemma 2.2, property (5), we pick some unit v''_α such that $v''_\alpha(1-f'_\alpha) = 1-f''_\alpha$ and $v''_\alpha f'_\alpha = f'_\alpha$. Set $e''_{i,\alpha} = v''_\alpha e'_{i,\alpha}$, for $i < \alpha$. We have $\sum_{i<\alpha} e''_{i,\alpha} + f''_\alpha = 1$, and $\{e''_{i,\alpha}\ (\forall\ i<\alpha), f''_\alpha\}$ is a summable family of orthogonal idempotents by Lemma 2.4.

With all the machinery we have built up, it is now an easy matter to construct $e_{i,\alpha}$ (for each $i \leqslant \alpha$), f_α, and v_α. To do so, notice we have the equation

$$1 = \sum_{i<\alpha} e''_{i,\alpha} + f''_\alpha = \sum_{i<\alpha} e''_{i,\alpha} + r'_\alpha x_\alpha + r'_\alpha y_\alpha.$$

Now we use exactly the same ideas as in Case 1 to construct the elements we need. Lemma 2.3 allows us to pick orthogonal idempotents

$$h_1 \in R\left(\sum_{i<\alpha} e''_{i,\alpha}\right), \qquad h_2 \in Rr'_\alpha x_\alpha, \qquad h_3 \in Rr'_\alpha y_\alpha$$

with $h_1+h_2+h_3 = 1$ and $h_1 \sim_\ell \sum_{i<\alpha} e''_{i,\alpha}$. By Lemma 2.2, property (5), there exists $u_\alpha \in U(R)$ such that $h_1 = u_\alpha\left(\sum_{i<\alpha} e''_{i,\alpha}\right)$ and $u_\alpha f''_\alpha = f''_\alpha$. Putting $e_{i,\alpha} = u_\alpha e''_{i,\alpha} \in Rx_i$ (for $i<\alpha$), $e_{\alpha,\alpha} = h_2 \in Rx_\alpha$, and $f_\alpha = h_3 \in Ry_\alpha$, then Lemma 2.4 implies that these are orthogonal idempotents. Also clearly

$$\sum_{i\leqslant\alpha} e_{i,\alpha} + f_\alpha = 1.$$

Therefore, condition (A) holds when $j=\alpha$.

We put $v_\alpha = u_\alpha v''_\alpha v'_\alpha$. It is clear that $v_\alpha e_{i,i} = e_{i,\alpha}$ for $i<\alpha$, so we just need to see that left multiplication by v_α acts as the identity on $e_{\alpha,\alpha}$ and f_α. First, remember $f'_\alpha = v'_\alpha f'_\alpha$. Second, we chose v''_α so that $v''_\alpha f'_\alpha = f'_\alpha$ holds. Third, u_α was chosen so that $u_\alpha f''_\alpha = f''_\alpha$. Finally, $e_{\alpha,\alpha}$ and f_α are both fixed by left multiplication by f''_α and f'_α since $(e_{\alpha,\alpha}+f_\alpha) \sim_r f''_\alpha \sim_r f'_\alpha$. Therefore,

$$\begin{aligned} v_\alpha f_\alpha &= (u_\alpha v''_\alpha v'_\alpha)(f'_\alpha f_\alpha) = u_\alpha(v''_\alpha v'_\alpha f'_\alpha) f_\alpha \\ &= u_\alpha f'_\alpha f_\alpha = u_\alpha f_\alpha = u_\alpha(f''_\alpha f_\alpha) = f''_\alpha f_\alpha = f_\alpha \end{aligned}$$

and similarly, $v_\alpha e_{\alpha,\alpha} = e_{\alpha,\alpha}$. This finishes Case 2.

By transfinite induction, we have constructed the elements we wanted for all $j \in I$. Recall that we well-ordered I so that it had a last element κ. Let $e_i = e_{i,\kappa}$ for each $i \leqslant \kappa$. Then $\{e_i\}_{i\in I}$ is a summable family of orthogonal idempotents, summing to $1 - f_\kappa = 1$ (since $f_\kappa \in Ry_\kappa = (0)$), with $e_i \in Rx_i$ for each $i \in I$. This completes the proof. □

5. Lifting through the Jacobson Radical

Mohamed and Müller have shown in [6] that if M is a module such that $E/J(E)$ is regular and abelian, with idempotents lifting modulo $J(E)$, then M has full exchange. In particular, they use this to establish that continuous modules have full exchange. Similarly, one way of further generalizing Theorem 4.1 is to try and lift the argument through the Jacobson radical.

Let R be a suitable ring with a nice topology, and set $\overline{R} = R/J(R)$. Using the same constructions and terminology as in Theorem 4.1, consider the following two conditions:

$(*'_1)$	There exists an idempotent $\rho \in \overline{R}$ such that $\overline{\varphi} + \rho$ is a unit and $\overline{\varphi}\rho = 0$.

and

$$\boxed{(*_2') \quad \text{There exists } \overline{r}'_\alpha \in \overline{R} \text{ such that } \overline{r}'_\alpha \overline{y}'_\alpha \overline{f}'_\alpha = \overline{f}'_\alpha.}$$

Theorem 5.1. *Let R be a suitable ring with a nice topology. Then $(*_1') \Longleftrightarrow (*_1)$. Also, if $(*_1')$ and $(*_2')$ both hold then R is a full exchange ring.*

Proof. We show $(*_1') \Longrightarrow (*_1)$, noting that the converse is trivial. Suppose $\rho \in \overline{R}$ is chosen so that $(*_1')$ holds. Since R is suitable, idempotents lift modulo $J(R)$. Hence, there is some idempotent $\widetilde{p} \in R$ such that $\overline{\widetilde{p}} = \rho$. Set $u = \varphi + \widetilde{p}$. Notice that u is a unit, since it is a unit modulo $J(R)$ by hypothesis. Lemma 2.5 tells us that $\sum_{i<\alpha} u^{-1}e_{i,i} + u^{-1}\widetilde{p} = 1$, where the summands are orthogonal idempotents. Setting $p = u^{-1}\widetilde{p}$, then we have $u^{-1}e_{i,i}p = 0$ for all $i < \alpha$, and hence $e_{i,i}p = 0$ for all $i < \alpha$. In particular, $\varphi p = 0$.

The same argument we used in Theorem 4.1 to show that $p = f'_\alpha$ (under the old use of p) shows that $\rho = \overline{u}^{-1}\rho$. Therefore, $\varphi' = \varphi + p$ modulo $J(R)$ equals $\overline{\varphi} + \rho$, and thus must be a unit.

We now show $(*_1') + (*_2') \Longrightarrow R$ is a full exchange ring. First note that $\overline{R}$ is a ring with a linear, Hausdorff topology, by Lemma 2.7. Let $\{x_i\}_{i\in I}$ be a summable family of R, summing to 1. Then, $\{\overline{x}_i\}_{i\in I}$ is a summable family summing to $\overline{1}$, and is left-multiple summable since $\{x_i\}_{i\in I}$ is. Therefore, the same argument as used in the proof of Theorem 4.1, now applied to $\overline{R}$, shows that we can find orthogonal idempotents $\varepsilon_i \in \overline{R}\overline{x_i}$ summing to $\overline{1}$. (In Theorem 4.1 we didn't need R to have a left multiple summable topology, only that $\{x_i\}_{i\in I}$ is a left multiple summable family.)

By Lemma 2.8, we can lift each ε_i to an idempotent $e'_i \in Rx_i$. These are still summable idempotents, summing to a unit (since modulo $J(R)$ they sum to $\overline{1}$). Letting $u = \sum_{i\in I} e'_i$ then Lemma 2.5 says that $\{u^{-1}e'_i\}_{i\in I}$ is a summable family of orthogonal idempotents summing to 1. Clearly, $e_i = u^{-1}e'_i \in Rx_i$, so we are done. □

6. Examples of Full Exchange Rings

We need to find rings that satisfy $(*_1)$ and $(*_2)$ (or their counterparts). If we assume R_R (respectively, $\overline{R}_{\overline{R}}$) has (C_2), then $(*_2)$ (respectively, $(*_2')$) holds, since we have $y'_\alpha f'_\alpha R \cong f'_\alpha R \subseteq^{\oplus} R_R$ (respectively, the same equation with bars everywhere). Not surprisingly, Dedekind-finite, semi-π-regular rings satisfy $(*_1')$. So we have:

Theorem 6.1. *Let R be a ring with a nice topology. If R is Dedekind-finite, semi-π-regular, and either R_R or $\overline{R}_{\overline{R}}$ has (C_2), then R is a full exchange ring.*

Proof. Clearly R is a suitable ring. Since either $(*_2)$ or $(*_2')$ holds, then by Theorems 4.1 and 5.1 it suffices to show $(*_1')$ holds.

We have $\overline{\varphi}$ is π-regular, and so $\varphi^n\psi\varphi^n - \varphi^n \in J(R)$ for some $\psi \in R$ and some $n \geqslant 1$. In particular, $\rho = \overline{1} - \overline{\psi}\overline{\varphi}^n$ is an idempotent. Now, $\overline{\varphi} = \sum_{i<\alpha} \overline{e}_{i,i}$, by Lemma 2.7, where $e_{i,i}e_{j,j} = 0$ for $i < j$, and $e_{i,i}$ is an idempotent. Since $\overline{\varphi}^n\rho = 0$, Lemma 2.6 tells us $\overline{\varphi}\rho = 0$, and $\overline{e}_{i,i}\rho = 0$ for all $i < \alpha$.

All we need is that $u = \overline{\varphi} + \rho$ is a unit. From the argument in Corollary 3.3, it suffices to show that u is a left non-zero-divisor. Suppose $us = 0$ for some $s \in \overline{R}$. If $\overline{\varphi}s = 0$ then

$$0 = us = \overline{\varphi}s + (\overline{1} - \overline{\psi}\overline{\varphi}^n)s = s.$$

So we may assume $\overline{\varphi}s \neq 0$, and hence there is a minimal index β so that $\overline{e}_{\beta,\beta}s \neq 0$. Then

$$0 = \overline{e}_{\beta,\beta}us = \overline{e}_{\beta,\beta}(\overline{\varphi} + \rho)s = \overline{e}_{\beta,\beta}s$$

a contradiction. Thus $s = 0$, whence u is a left non-zero-divisor as claimed. □

Corollary 6.2. *Let R be a ring with a nice topology. If R is Dedekind-finite and semi-regular, then R is a full exchange ring.*

Proof. In this case $\overline{R}_{\overline{R}}$ has (C_2) since $\overline{R}$ is regular. Thus Theorem 6.1 applies. □

Corollary 6.3. *Let R be a ring with a nice topology. If R is a unit regular ring, then R is a full exchange ring.*

Proof. Unit regular rings are always Dedekind-finite and regular. □

Theorem 6.4. *Let R be a ring with a nice topology. If $\overline{R}$ is strongly clean, idempotents lift modulo $J(R)$, and either R_R or $\overline{R}_{\overline{R}}$ has (C_2), then R is a full exchange ring.*

Proof. Just as in Theorem 6.1, it suffices to prove $(*_1')$. Write $\overline{\varphi} = u + e$ where $u \in U(\overline{R})$, $e^2 = e \in \overline{R}$, and $ue = eu$. We just take $\rho = e$. By the same argument in the second paragraph of the proof of Corollary 3.6, we obtain $\overline{\varphi}e = 0$. (We can't say $e = 0$ since φ might not be a left non-zero-divisor in this case.) Then

$$\begin{aligned}\overline{\varphi} + \rho &= \overline{\varphi}((\overline{1} - e) + e) + e = \overline{\varphi}(\overline{1} - e) + \overline{\varphi}e + e \\ &= (u + e)(\overline{1} - e) + 0 + e = u(\overline{1} - e) + e\end{aligned}$$

which is a unit with inverse $u^{-1}(\overline{1} - e) + e$, since u and e commute. This yields $(*_1')$. □

7. Exchange Modules

What do the previous results say concerning modules? We have the following unsettling asymmetry, motivated by [5, Proposition 8.11].

Lemma 7.1. *Let M_k be a module and $E = \mathrm{End}(M_k)$ as usual. If E_E is cohopfian, respectively has (C_2), then the same is true for M. The converses do not hold.*

Proof. First, suppose that E_E is cohopfian. Let $x \in E$ be an injective endomorphism on M. If $xr = 0$ for some $r \in E$ then $xr(m) = 0$ for all $m \in M$. But x being injective implies $r(m) = 0$ for all $m \in M$. Therefore $r = 0$. Since r was arbitrary, x is a left non-zero-divisor. The cohopfian condition on E_E then implies x is a unit. This shows that M is cohopfian.

Now suppose instead that E_E has (C_2). Consider the situation where $N' \subseteq M$ and $N' \cong N \subseteq^{\oplus} M$. Let $e \in E$ be an idempotent with $e(M) = N$, and let $\varphi : N \to N'$ be an isomorphism. Without loss of generality, we may assume $\varphi \in E$ by setting φ equal to 0 on $(1 - e)(M)$.

Consider the map, $eE \to \varphi eE$, given by left multiplication by φ. Clearly this is surjective. To show injectivity, suppose that $\varphi er = 0$ for some $r \in E$. Then $\varphi er(m) = 0$ for all $m \in M$. In particular, $\varphi(er(M)) = 0$. But $er(M) \subseteq e(M)$ and φ is injective on $e(M) = N$, therefore $er(M) = 0$. But then $er = 0$. This shows injectivity.

Thus φeE is isomorphic to eE, a direct summand of E_E. Therefore φeE is generated by an idempotent, say f. Clearly $f\varphi e = \varphi e$, and $f = \varphi ey$ for some $y \in E$.

So $f(M) = \varphi ey(M) \subseteq \varphi e(M) = N'$, and $f(M) \supseteq f(\varphi e(M)) = \varphi e(M) = N'$. Hence $N' = f(M)$ is a direct summand.

A single example will show that both converses do not hold. Let $k = \mathbb{Z}$ and let M be the Prüfer p-group, for any prime p. Then E is isomorphic to the ring of p-adic integers. M_k is cohopfian while E_E is not, by [5, Proposition 8.11]. Notice that the only idempotents in E are 0 and 1. Thus, the only direct summands in either M_k or E_E are the trivial ones. One easily sees that multiplication by p yields $pE \cong E_E$, but pE is not a summand. Therefore E_E does not have the (C_2) property. On the other hand, any submodule isomorphic to M must contain elements killed by multiplication by p, and hence must equal M. Thus, the only submodule of M isomorphic to M is M itself, and trivially the only submodule of M isomorphic to (0) is (0). Hence M has the (C_2) property. □

Due to this lemma, it would appear that one could not work with the weaker notion of a cohopfian module and hope to prove a result analogous to Corollary 3.2. However, in an endomorphism ring a limit of units is very special.

Theorem 7.2. *Let M be a cohopfian module with finite exchange. Then M has countable exchange.*

Proof. In the endomorphism ring, E, a convergent limit of units must be an injective endomorphism (since nothing in the limit process has a kernel). But then the cohopfian condition forces this endomorphism to be an isomorphism, or in other words a unit in E. Thus convergent limits of units are units. So M has countable exchange from Theorem 3.1, since $(*)$ holds (φ is a limit of units, and hence is a unit). □

While the cohopfian condition was sufficient to show that φ is a unit, it isn't necessary. In the general case, we have an embedding $\varphi : M \to M$. But, since $\varphi = \lim_{n\to\infty} v_n^{-1}$, this embedding is locally split by the units v_n. In particular, in Theorem 7.2 we could replace the cohopfian condition with the assumption that all locally split monomorphisms (split by units) from M to M are globally split by a unit.

We now ask if one can also tweak Theorems 6.1 and 6.4 so we are working with the weaker hypothesis that M has the (C_2) property. The answer is yes.

Theorem 7.3. *Let M be a finite exchange module with the (C_2) property, and also be such that $(*_1)$ holds for E in the finite topology. Then M has full exchange.*

Proof. Following Theorem 4.1, with $R = E$, the only thing we need to show is that $(*_2)$ holds.

Consider the map $f'_\alpha(M) \to y'_\alpha f'_\alpha(M)$, given by left-multiplication by y'_α. It is clearly surjective. For any $m \in M$ we have

$$f'_\alpha(m) = \lim_{i\to\alpha} f_i f'_\alpha(m) = \lim_{i\to\alpha} r_i y_i f'_\alpha(m) = \lim_{i\to\alpha} r_i y'_\alpha f'_\alpha(m),$$

and so the map above must also be injective. The (C_2) hypothesis now implies $y'_\alpha f'_\alpha(M) = g_\alpha(M)$ for some idempotent g_α.

Define r'_α by the rule $r'_\alpha|_{(1-g_\alpha)(M)} = 0$ and $r'_\alpha|_{g_\alpha(M)} = \lim_{i\to\alpha} r_i|_{y'_\alpha f'_\alpha(M)}$, and extend linearly to M. While it is true that $\lim_{i\to\alpha} r_i$ does not necessarily converge in general, it does converge on $y'_\alpha f'_\alpha(M) = g_\alpha(M)$ since we saw above that

$r'_\alpha y'_\alpha f'_\alpha(m) = \lim_{i\to\alpha} r_i y'_\alpha f'_\alpha(m) = f'_\alpha(m)$.[7] Since m is arbitrary, $r'_\alpha y'_\alpha f'_\alpha = f'_\alpha$. This gives $(*_2)$. □

Theorems 6.1 and 6.4, and Corollaries 3.3 through 6.3 immediately translate over to the endomorphism ring case. In particular, we have:

Corollary 7.4. *If M has a Dedekind-finite, semi-π-regular endomorphism ring then M has countable exchange. Similarly, if M has finite exchange and $E/J(E)$ is strongly clean, then M has countable exchange. In either of these cases, if we add the hypothesis that M has (C_2) then M has full exchange.*

Corollary 7.5. *If M has a Dedekind-finite, semi-regular endomorphism ring, then M has full exchange.*

Recall we have the isomorphism $y'_\alpha : f'_\alpha M \to y'_\alpha f'_\alpha M$. This induces a monomorphism $f'_\alpha M \to y'_\alpha f'_\alpha M \subseteq M$ which is locally split by the maps $f'_\alpha r_i$. But $(*_2)$ is equivalent to this monomorphism being globally split. So we could replace the (C_2) hypothesis in Theorem 7.3 and Corollary 7.4 with the weaker assumption that all locally split monomorphisms from a summand of M into M are globally split.

8. Abelian Rings and Commutative Rings

The methods developed in this paper are abstracted from the methods we used to prove that abelian modules[8] with finite exchange have full exchange in [11]. The proof given there relies on a module-theoretic construction that doesn't easily translate over to the language of topological rings. To demonstrate this difficulty, suppose R is a suitable ring with a nice topology. As in the proof of Theorem 4.1, we have the equations

$$1 = \sum_{i\leqslant\beta} e_{i,\beta} + f_\beta = \sum_{i\leqslant\beta} e_{i,\beta} + r_\beta\left(\sum_{i\in(\beta,\alpha)} x_\beta\right) + r_\beta y'_\alpha$$

for each $\beta < \alpha$. Hence, by suitability, we can find orthogonal idempotents $a_\beta \in R\left(\sum_{i\leqslant\beta} e_{i,\beta}\right)$, $b_\beta \in R\left(\sum_{i\in(\beta,\alpha)} x_\beta\right)$, and $c_\beta \in Ry'_\alpha$, with $a_\beta + b_\beta + c_\beta = 1$.

If we assume that R satisfies $(*_1)$ then we can define f'_α, just as before. Now notice that $a_\beta f'_\alpha = 0$ since $e_{i,\beta} f'_\alpha = v_\beta e_{i,i} f'_\alpha = 0$. Also, $\lim_{\beta\to\alpha}\sum_{i\in(\beta,\alpha)} x_\beta = 0$ and hence $\lim_{\beta\to\alpha} b_\beta = 0$. Thus

$$\lim_{\beta\to\alpha} c_\beta f'_\alpha = \lim_{\beta\to\alpha} a_\beta f'_\alpha + b_\beta f'_\alpha + c_\beta f'_\alpha = f'_\alpha$$

Writing $c_\beta = s_\beta y'_\alpha$, this says

$$\lim_{\beta\to\alpha} s_\beta y'_\alpha f'_\alpha = f'_\alpha. \tag{3}$$

Now suppose that R is abelian. We will show that we get $(*_1)$ for free. Idempotents commute and so $\varphi = \sum_{i<\alpha} e_{i,i}$ is a sum of orthogonal idempotents, and hence φ is an idempotent. We can put $p = 1 - \varphi$, and then $(*_1)$ holds.

[7] One should also check that r'_α is a well-defined homomorphism, which we leave to the reader.

[8] Recall that a ring is called *abelian* if all idempotents are central. A module is *abelian* if its endomorphism ring is an abelian ring.

What about $(*_2)$? After replacing s_β by $s_\beta y'_\alpha s_\beta$ if necessary, we have $c_\beta = s_\beta y'_\alpha = y'_\alpha s_\beta$. And hence equation (3) yields

$$f'_\alpha = \lim_{\beta\to\alpha} f'_\alpha s_\beta f'_\alpha y'_\alpha f'_\alpha = \lim_{\beta\to\alpha} f'_\alpha y'_\alpha f'_\alpha s_\beta f'_\alpha.$$

If we assume that $R = \mathrm{End}(M)$ is an endomorphism ring in the finite topology, then the equation $f'_\alpha = \lim_{\beta\to\alpha} f'_\alpha s_\beta f'_\alpha y'_\alpha f'_\alpha$ implies that $f_\alpha y'_\alpha f'_\alpha$ is an injective endomorphism on the $f'_\alpha(M)$. Similarly, the equation $f'_\alpha = \lim_{\beta\to\alpha} f'_\alpha y'_\alpha f'_\alpha s_\beta f'_\alpha$ implies $f'_\alpha y'_\alpha f'_\alpha$ is surjective on $f'_\alpha(M)$. Therefore $f'_\alpha y'_\alpha f'_\alpha \in U(f'_\alpha R f'_\alpha)$. We just let r'_α be the inverse (in the corner ring $f'_\alpha R f'_\alpha$) and then $(*_2)$ holds. This yields:

Proposition 8.1 ([11, Theorem 2])**.** *An abelian module with finite exchange has full exchange.*

More generally, let I be a directed set and let R be a ring with a linear, Hausdorff topology. Suppose we are given elements $s_i \in R$ for each $i \in I$. We can ask what hypotheses need to be placed on R so that any element $r \in R$ satisfying the equations

$$1 = \lim_{i\in I} rs_i = \lim_{i\in I} s_i r$$

must be a unit. We saw above that it sufficed to assume R is an endomorphism ring in the finite topology. Unfortunately even if we assume $R = \mathrm{End}(M)/J(\mathrm{End}(M))$, using the finite topology on $\mathrm{End}(M)$ and then the quotient topology on R, it doesn't appear that these equations imply $r \in U(R)$. This is one obstruction to showing that square-free modules with finite exchange have full exchange.

There is another avenue we can explore if we assume a little more than R being abelian.

Lemma 8.2. *Let R be a ring with a complete, linear, Hausdorff topology. If every open neighborhood of zero contains an open (two-sided) ideal neighborhood of zero, then convergent limits of units are units.*

Proof. Let $u = \lim_{i\in I} u_i$ be a limit in R, with I a directed set and $u_i \in U(R)$. Let U be an arbitrary, open (left ideal) neighborhood of 0, and let $V \subseteq U$ be an open two-sided ideal neighborhood of 0. There is some $k \in I$ so that for all $j \geqslant k$ we have $u_j - u_k \in V$. Left multiply by u_j^{-1} and right multiply by u_k^{-1} to obtain

$$u_j^{-1}(u_j - u_k)u_k^{-1} = u_k^{-1} - u_j^{-1} \in u_j^{-1} V u_k^{-1} = V \subseteq U.$$

Therefore, $\{u_i^{-1}\}_{i\in I}$ is a Cauchy system. Since the topology is complete, $v = \lim_{i\in I} u_i^{-1}$ exists. Then one verifies

$$uv = \lim_{i\in I} u_i \lim_{i\in I} u_i^{-1} = \lim_{i\in I} u_i u_i^{-1} = 1$$

and similarly $vu = 1$. Therefore, $u \in U(R)$ as claimed. □

With all this talk of convergent limits, one might expect to use Theorem 3.1 to show countable exchange. However, we can do better.

Proposition 8.3. *Let R be a suitable, commutative ring with a complete, linear, Hausdorff topology. Then R is a full exchange ring.*

Proof. First notice that for any idempotent $e \in R$ the corner ring eRe is a ring with a complete, linear, Hausdorff topology. Second, it is well known that commutative suitable rings are strongly clean. From the argument above we see that $(*_1)$ holds. Therefore, it suffices to show that $(*_2)$ holds.

Using the notation of Theorem 4.1, we have

$$y'_\alpha f'_\alpha = \lim_{i \to \alpha} y_i f'_\alpha$$

where $y_i f'_\alpha \in U(f'_\alpha R f'_\alpha)$ with inverse $f'_\alpha r_i$.[9] Thus $y'_\alpha f'_\alpha$ is a limit of units, and hence is a unit by Lemma 8.2, in the corner ring $U(f'_\alpha R f'_\alpha)$. But then we just set $r'_\alpha = (f'_\alpha y'_\alpha f'_\alpha)^{-1}$ (again in the corner ring) and we have $(*_2)$ holding. □

If a family $\{x_i\}_{i \in I}$ is left-multiple summable in R, then the same holds true of the family $\{\overline{x}_i\}_{i \in I}$ in $R/J(R)$. However, in general the completeness of a topology does not pass to quotient rings, and so we can't lift the previous proposition through the radical.

9. Final Remarks

In [11] we defined what we called *finitely complemented* modules. These are modules whose direct summands have only finitely many complement summands. We showed that a finitely complemented module with a regular endomorphism ring has full exchange. We claim (but do not give the proof here) that using the methods derived above, one can remove the condition that E is regular, and replace it with M having finite exchange and (C_2).

There is another class of modules we can apply these techniques to; namely, square-free modules. Suppose that M is a square-free module with finite exchange. Mohamed and Müller have shown that $E/J(E)$ is abelian [8, Lemmas 11 and 15], and hence strongly clean. If M has (C_2) then Corollary 7.4 shows full exchange for M. For square-free modules the (C_2) property is equivalent to cohopfianness. So what we have shown is that a cohopfian, square-free module with finite exchange has full exchange. (One could even improve this to: square-free modules with finite exchange, where all locally split monomorphisms from summands of M into M are globally split, are full exchange modules.)

As far as we know, the only major classes of modules where it is known that finite exchange implies countable exchange but not known if it further implies full exchange are cohopfian modules, finitely complemented modules, and modules where $E/J(E)$ is strongly clean; all of which follows easily from the ideas used in the proof of Theorem 3.1.

Unfortunately, the converse of Theorem 3.1 is not true. For example, let k be a field and let M_k be a countably infinite-dimensional vector space over k, say with basis $\{m_i\}_{i \in \mathbb{Z}_+}$. Let $R = E = \operatorname{End}(M_k)$ with the finite topology, which is a suitable ring. Define $x_i \in R$, for $i \in \mathbb{Z}_+$, as follows:

$$x_i(m_j) = \begin{cases} -m_i & \text{if } j = i - 1 \\ m_i + m_{i+1} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

[9]Here we are using the fact that $r_i y_i = f_i \sim_r f'_\alpha$, and that idempotents commute.

Notice that $\sum_{i=1}^{\infty} x_i = 1$. Set $y_i = \sum_{j>i} x_j$. Then one calculates easily

$$y_i(m_j) = \begin{cases} 0 & \text{if } j < i \\ -m_{i+1} & \text{if } j = i \\ m_j & \text{if } j > i. \end{cases}$$

Since x_1 is already an idempotent we can take $e_{1,1} = x_1$ and $f_1 = y_1$. We leave the calculations to the reader, and claim that by induction $f_i x_{i+1} f_i + f_i y_{i+1} f_i = f_i$ is already a sum of orthogonal idempotents, and so we may take $e_{i+1,i+1} = f_i x_{i+1} f_i x_{i+1}$ and $f_{i+1,i+1} = f_i y_{i+1} f_i y_{i+1}$. It turns out that

$$e_{i,i}(m_j) = \begin{cases} -m_i - m_{i+1} & \text{if } j = i-1 \\ m_i + m_{i+1} & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_{i,i}(m_j) = \begin{cases} 0 & \text{if } j < i \\ -m_{i+1} & \text{if } j = i \\ m_j & \text{if } j > i. \end{cases}$$

Setting $\varphi = \sum_{i=1}^{\infty} e_{i,i}$, we see that φ is not a unit since it isn't surjective: $m_1 \notin \mathrm{im}(\varphi)$. However, M has full exchange (being semisimple) and so E is a full exchange ring in the finite topology. Therefore, this shows that the converse of Theorem 3.1 is not true. Modifying the example slightly, we see that the methods of this paper will also not work for a direct sum of an infinite number of copies of a single non-zero module.

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