SQUARE-FREE MODULES WITH THE EXCHANGE PROPERTY

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Abstract. We prove that a square-free module with finite exchange has full exchange. More generally, if R is an exchange ring with $R/J(R)$ Abelian, and R is endowed with a left linear, Hausdorff, Σ-complete topology, then R is a full exchange ring. This provides an overarching framework for capturing many other results in the literature, such as the fact that quasi-continuous modules with finite exchange have full exchange. We further show that square-free modules with exchange satisfy an infinite version of the (C_3) property.

Overview

We begin in Section 1 with a dense introduction to exchange rings and the exchange property. Those unfamiliar with the terminology and results contained therein are encouraged to read [12], [24], and [11] (in that order). Section 2 introduces square-free modules, which are the main objects of study in this paper. Readers unfamiliar with the material in that section are referred to [8]. In Section 3 we provide a few simplifying lemmas, some of which are used to power the inductive constructions used in the proof of the main theorem, given in Section 4. The main theorem reads: square-free modules with finite exchange have full exchange. Finally, we give an interesting observation about such modules in Section 5. Throughout the paper, rings are associative with 1, and modules are unital. All modules are right Λ-modules, where Λ is a ring, unless otherwise specified. Endomorphisms are written on the left of module elements.

1. Introduction to Exchange Modules and Rings

Introduced by Crawley and Jónsson [3] in 1964, the exchange property has since become an important concept of study both in terms of direct sum decomposition theory (including the Krull-Schmidt-Azumaya theorem) [4] and in terms of lifting idempotents modulo ideals [12]. Let Λ be an associative ring with 1. A right Λ-module M_Λ has the $\aleph$-*exchange property*, for some cardinal $\aleph \geqslant 2$, if whenever there are right Λ-module decompositions $A = M \oplus N = \bigoplus_{i\in\aleph} A_i$ then there exist submodules $A_i' \subseteq A_i$, for each $i \in \aleph$, so that $A = M \oplus \left(\bigoplus_{i\in\aleph} A_i'\right)$. If M_Λ has the $\aleph$-exchange property for all cardinals $\aleph$ (respectively, all finite cardinals), then we say M_Λ has the *full exchange property* (respectively, the *finite exchange property*). Injective modules, or more generally continuous modules, always enjoy the full exchange property [7]. The $\aleph$-exchange property passes to summands and finite direct sums of modules with $\aleph$-exchange.

Following [22], a ring R is called an *exchange ring* if R_R has the finite exchange property. This is a left-right symmetric property, and M_Λ has finite exchange if and only if $\mathrm{End}(M_\Lambda)$ is an exchange ring. Given a ring R and a one-sided ideal I, one says that *idempotents lift modulo* I if given $x \in R$ with $x^2 - x \in I$ there exists some idempotent $e^2 = e \in R$ with $e - x \in I$. Idempotents *lift strongly* if, in addition, one can always choose $e \in xRx$. It is known that if idempotents lift modulo an ideal contained in the Jacobson radical $J(R)$, then they lift strongly [11, Lemma 6]. The seminal paper [12] of Nicholson characterized exchange rings as exactly those rings for which idempotents lift (strongly)

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modulo all left and right ideals (see also [13]). Furthermore, R is an exchange ring if and only if $R/J(R)$ is an exchange ring and idempotents lift (strongly) modulo $J(R)$. Regular rings (or, more generally, semi-π-regular rings) and clean rings (which include local rings) are all exchange rings. A C^*-algebra is an exchange ring, possibly without 1, if and only if it has real rank zero [1]. A criterion for when Leavitt path algebras are exchange rings can be given in terms of an intrinsic property of the underlying graph [2]. Exchange rings, and the exchange property, are intimately connected with the notions of cancellation and substitution in module theory [6]. For a thorough overview on exchange rings, including an extensive reference list, see [20].

If $\aleph' < \aleph$ and M has $\aleph$-exchange then M has $\aleph'$-exchange. It is natural to ask in what situations the reverse implication is true. It is known that 2-exchange always implies finite exchange. The question of whether finite exchange implies full exchange remains open, although there are numerous partial results. For example, if M is a projective module with countable exchange then M has full exchange, and regular projective modules (in the sense of Ware [21]) always have full exchange [18], [19]. If M is a direct sum of indecomposable modules and M has finite exchange then M has full exchange [24].

Given a module M_Λ set $E = \mathrm{End}(M_\Lambda)$. One says that a collection of elements $\{x_i\}_{i\in I}$ of E are *summable* if for each $m \in M$ the set $F_m = \{i \in I \mid x_i(m) \neq 0\}$ is finite. In that case, $\sum_{i\in I} x_i$ is a well-defined element of E, where $\left(\sum_{i\in I} x_i\right)(m) = \sum_{i\in F_m} x_i(m)$. One can simplify the criteria for checking the $\aleph$-exchange property as follows:

Proposition 1 ([24, Proposition 3]). *Given a module M and a cardinal $\aleph \geqslant 2$ the following are equivalent:*

(1) *The module M has the $\aleph$-exchange property.*
(2) *If we have*
$$A = M \oplus N = \bigoplus_{i\in I} A_i$$
with $A_i \cong M$ for all $i \in I$, and $|I| \leqslant \aleph$, then there are submodules $A_i' \subseteq A_i$ such that
$$A = M \oplus \bigoplus_{i\in I} A_i'.$$
(3) *Given a summable family $\{x_i\}_{i\in I}$ of elements of $E = \mathrm{End}(M_\Lambda)$, with $\sum_{i\in I} x_i = 1$, and with $|I| \leqslant \aleph$, then there are orthogonal idempotents $e_i \in Ex_i$ with $\sum_{i\in I} e_i = 1$.*

We want to provide a context in which item (3) of the previous proposition makes sense without reference to a module. To do so we must introduce some topological conditions which allow us to define summability in a (topological) ring. Following [11] and [16], one says that R has a left linear Hausdorff topology if there is a ring topology with a basis of neighborhoods of zero, $\mathfrak{U}$, consisting of left ideals, satisfying $\bigcap_{U\in\mathfrak{U}} U = (0)$. We say that a collection $\{x_i\}_{i\in I}$ of elements of R is *summable to* $r \in R$, written as $\sum_{i\in I} x_i = r$, if for every $U \in \mathfrak{U}$ there is a finite set $F' \subseteq I$ (depending on U) such that $\left(\sum_{i\in F} x_i\right) - r \in U$ for all finite sets $F \supseteq F'$.

One can describe a topology on $E = \mathrm{End}(M_\Lambda)$, called the *finite topology*, by taking a basis of neighborhoods of zero to be the annihilators of finite subsets of M. The finite topology is left linear and Hausdorff. Furthermore, both notions of summability agree in this topology. Given a ring R with a left linear Hausdorff topology, it is called an *$\aleph$-exchange ring* if for each summable family $\{x_i\}_{i\in I}$ in R, with $|I| \leqslant \aleph$ and with $\sum_{i\in I} x_i = 1$, then there are summable, orthogonal idempotents $\{e_i\}_{i\in I}$ with $e_i \in Rx_i$ and $\sum_{i\in I} e_i = 1$. If this holds for all cardinals $\aleph$, the ring is called a *full (topological) exchange ring* (not to be confused with "exchange ring"). A module has $\aleph$-exchange if and only if the endomorphism ring is an $\aleph$-exchange ring in the finite topology, by item (3) in the previous proposition. We have thus accomplished our goal of introducing a module-free context for summability.

In the definition of an $\aleph$-exchange ring, one needs the set $\{e_i\}_{i\in I}$ to be summable. For endomorphism rings, in the finite topology, this condition is automatically satisfied. In fact, if we fix $m \in M$ then $x_i(m) = 0$ for all but finitely many $i \in I$ and the same is true for any family $\{r_i x_i\}_{i\in I}$ consisting of left multiples of the x_i. Now, instead suppose we have a family $\{x_i\}_{i\in I}$ of elements in a ring R with a left linear Hausdorff topology. Following [15], we say this family is Σ-*Cauchy* if for each $U \in \mathfrak{U}$ (where $\mathfrak{U}$ is a basis of neighborhoods of zero, consisting of left ideals), there is a finite set $F_U \subseteq I$ so that $x_i \in U$ when $i \notin F_U$. Summable families are always Σ-Cauchy, and if the converse is true we say that R is Σ-complete. This is equivalent, by [16, Lemma 1.3], to the statement that if $\{x_i\}_{i\in I}$ is a summable family then every left-multiple of this family is also summable. In other words, any family of the form $\{r_i x_i\}_{i\in I}$, where $r_i \in R$ are arbitrary elements, is summable. Note that an endomorphism ring in the finite topology is Σ-complete, and in fact complete.

2. Square-free Modules

All of the terminology and results in this section are standard, and can be found in either [8] or [11]. A ring is said to be *Abelian* if all idempotents are central, and a module is *Abelian* when its endomorphism ring is an Abelian ring. A module is called *square-free* if there is no nonzero submodule isomorphic to a square $X \oplus X$.

Given a module M, the set $\Delta = \{f \in \operatorname{End}(M) \mid \ker(f) \subseteq_e M\}$ of endomorphisms with essential kernels is an ideal of $E = \operatorname{End}(M)$. Furthermore, if M has finite exchange then $\Delta \subseteq J(E)$, and both Δ and $J(E)$ are closed ideals in the finite topology [11, Lemma 11]. More generally, the Jacobson radical is closed in any exchange ring R with a left linear Hausdorff topology. If M is a square-free module then $\operatorname{End}(M)/\Delta$ is Abelian; in fact $\operatorname{End}(M)/\Delta$ is Abelian if and only if "complements are essentially unique" [11, Lemma 15].

A module M has (C_1) if every submodule is essential in a summand:

$$\forall\ N \subseteq M,\ \exists\ P \subseteq^{\oplus} M \text{ such that } N \subseteq_e P.$$

A module M has (C_2) if every submodule isomorphic to a summand is a summand:

$$N \subseteq M,\ N \cong P \subseteq^{\oplus} M \Longrightarrow N \subseteq^{\oplus} M.$$

A module M has (C_3) if the sum of two non-intersecting summands is a summand:

$$A, B \subseteq^{\oplus} M \text{ and } A \cap B = (0) \Longrightarrow A \oplus B \subseteq^{\oplus} M.$$

The property (C_2) implies property (C_3). A module is *continuous* (respectively *quasi-continuous*) if it satisfies (C_1) and (C_2) (respectively (C_1) and (C_3)). The (quasi-)continuous property is a direct generalization of injectivity. Quasi-continuous modules always decompose into a direct sum of a continuous module and a square-free module.

As stated in the introduction, continuous modules all have the full exchange property [7]. The same is not true for quasi-continuous modules, as demonstrated by $\mathbb{Z}_{\mathbb{Z}}$, since $\mathbb{Z}$ is not an exchange ring. However, nonsingular quasi-continuous modules with finite exchange were shown to have full exchange in [9], and shortly thereafter the same result without the nonsingularity hypothesis was demonstrated in [17] and [10]. Each of those proofs hinge on reducing to the case that M is a square-free quasi-continuous module. On the other hand, Yu [23] proved that Abelian modules with finite exchange have countable exchange, and this was generalized in [11] to square-free modules. In [14] the present author proves that Abelian modules with finite exchange have full exchange, but the method of proof relies on the fact that certain endomorphisms are injective and surjective. This prevents the techniques from working in the more general context of topological rings. In the present paper, a new set of techniques is developed, which are used to show that square-free modules with finite exchange have full exchange, thus generalizing and subsuming the results cited in this paragraph. The methods are general enough that they can be expressed in terms of the topological language introduced at the end of the previous section.

3. The Machinery

While we could make most of the results below apply to a module M_Λ with $\mathrm{End}(M_\Lambda)/\Delta$ an Abelian ring, it is easier to work directly with an Abelian ring and then later lift all of the results through the Jacobson radical. The first few lemmas tell us how to do such lifting.

Lemma 2. *If R is an exchange ring with a left linear Hausdorff topology then $J(R)$ is a closed ideal, $\overline{R} = R/J(R)$ is an exchange ring, and the quotient topology is a left linear Hausdorff topology. If $\{x_i\}_{i\in I}$ is summable to $r \in R$, then $\{\overline{x}_i\}_{i\in I}$ sums to $\overline{r} \in \overline{R}$.*

Proof. The proof of [11, Lemma 11], modified to the case where R is a topological ring as in [15, Lemma 3.2], demonstrates that $J(R)$ is a closed ideal. The rest of the claims are easily verified. □

Lemma 3. *Let R be an exchange ring and let $N \subseteq J(R)$ be an ideal. If R/N is an Abelian ring then so is $R/J(R)$.*

Proof. Let π denote an idempotent in $R/J(R)$, and let $r \in R$. We can lift π to an idempotent $p \in R$. Then $pr - rp \in N \subseteq J(R)$ since idempotents are central modulo N. But $\pi = p + J(R)$ and r is arbitrary, hence π commutes with all elements in $R/J(R)$. □

Proposition 4. *Suppose that $\{e_i\}_{i\in I}$ is a summable family of idempotents in an exchange ring R with a left linear Hausdorff topology. If $e_ie_j \in J(R)$ whenever $i \neq j$, and $u = \sum_{i\in I} e_i \in U(R)$, then $\{u^{-1}e_i\}_{i\in I}$ is a summable family of orthogonal idempotents which sum to 1.*

Proof. This is a weakening of [11, Lemma 8]. □

The next lemma is the pushing step in the main inductive construction. It allows one to pick off summands, one at a time.

Lemma 5 (cf. [14, Lemma 2])**.** *Let R be an Abelian exchange ring and suppose $e_1 + x_2 + x_3 = 1$ with e_1 an idempotent. There exist idempotents $e_2 \in Rx_2$ and $e_3 \in Rx_3$ so that $e_1 + e_2 + e_3 = 1$ and the e_i are pairwise orthogonal.*

Proof. Set $f = 1 - e_1$. Multiplying the original equation on the left and right by f, we obtain $fx_2f + fx_3f = f$. Corner rings of exchange rings are exchange rings, so fRf is an exchange ring. Hence there exist elements $r_2, r_3 \in R$ so that fr_2fx_2f and fr_3fx_3f are orthogonal idempotents adding to f. Letting $e_2 = fr_2x_2$ and $e_3 = fr_3x_3$, and noting that f is central, finishes the lemma. □

Fix an element $x \in E = \mathrm{End}(M_\Lambda)$ for some module M_Λ. Imagine that for each $m \in M$ there is an idempotent $e \in Ex$, depending on m, with $e(m) = m$. This is a *local* condition which we want to demonstrate, under certain circumstances, is *global*. In other words, we wish to prove that $1 \in Ex$. Notice that the local condition implies that x is injective, and it suffices to show that x is surjective; which is the method employed in [14], in the context where M is an Abelian module. The following proposition allows us to approach the problem from another angle. In essence we use the topology to glue all of the locally defined idempotents into one global idempotent, which then will equal 1.

Proposition 6. *Let R be an Abelian exchange ring with a left linear Hausdorff topology. Let I be an ordinal with a final element $\alpha \in I$, and further assume that α is a limit ordinal. Let $\{x_i\}_{i\leqslant\alpha}$ be a summable family of elements of R, with all left multiple families still summable. If for each $\beta < \alpha$ there exist a set of pairwise orthogonal, summable idempotents $\{e_{i,\beta}\}_{i\geqslant\beta}$ with $e_{i,\beta} \in Rx_i$ and satisfying $\sum_{i\geqslant\beta} e_{i,\beta} = 1$ then $1 \in Rx_\alpha$.*

Proof. The idea is to piece all of the $e_{\alpha,\beta}$ together to get 1, by using an inductive construction and taking the largest part of $e_{\alpha,\beta}$ not already used.

For each $i \in I$ we will construct idempotents $e'_{i,j} \in Rx_jx_i$ for $j < i$, so that for each $\beta \in I$ we have $\sum_{i \geqslant \beta}\left(e_{i,0} + \sum_{j<\beta} e'_{i,j}\right) = 1$, and all of the summands on the left-hand side are pairwise orthogonal idempotents. We will work by induction on β, constructing the needed idempotents as we go. Before we begin the induction argument, let's verify that the sum in question even makes sense. Fixing $\beta \in I$, and assuming there are elements $e'_{i,j} \in Rx_jx_i$ (for all $i \geqslant \beta$, and $j < i$) write $e'_{i,j} = r_{i,j}x_i$ with $r_{i,j} \in Rx_j$. The set $\{r_{i,j}\}_{j<i}$ is summable by hypothesis, so $e_{i,0} + \sum_{j<\beta} e'_{i,j} = e_{i,0} + \left(\sum_{j<\beta} r_j\right) x_i \in Rx_i$. Applying the summability hypothesis once more, we see that the sum $\sum_{i\geqslant\beta}\left(e_{i,0} + \sum_{j<\beta} e'_{i,j}\right)$ similarly consists of summable elements.

We now proceed with the inductive construction. When $\beta = 0$ the claim is trivially true by hypothesis on the set $\{e_{i,0}\}_{i\in I}$, and there are no new idempotents to construct. Assume, by way of induction on β, that the claim is true for all $\gamma < \beta$ and all necessary idempotents have been constructed in those cases.

Case 1: Suppose β is a successor ordinal. Write its predecessor as $\beta - 1$. We have, by inductive hypothesis, that

$$\sum_{i\geqslant\beta-1}\left(e_{i,0} + \sum_{j<\beta-1} e'_{i,j}\right) = 1$$

is a sum of orthogonal idempotents. Set $p = e_{\beta-1,0} + \sum_{j<\beta-1} e'_{\beta-1,j}$, which is an idempotent living in $Rx_{\beta-1}$. Recall that we have the sum $\sum_{i\geqslant\beta} e_{i,\beta} = 1$. (This is where we use the fact that α is a limit ordinal and the last element of I, hence $\beta < \alpha$ in the case we are considering.) Left multiply by p to obtain $\sum_{i\geqslant\beta} pe_{i,\beta} = p$. The summands in this decomposition are pairwise orthogonal idempotents, since p is central, as R is Abelian. Set $e'_{i,\beta-1} = pe_{i,\beta} \in Rx_{\beta-1}x_i$. This finishes the construction of the claimed idempotents in this case, and it remains to show that $\sum_{i\geqslant\beta}\left(e_{i,0} + \sum_{j<\beta} e'_{i,j}\right) = 1$ with all of the summands on the left-hand side orthogonal. We start with the orthogonality claim. By the inductive assumption, it suffices to verify that $e'_{i,\beta-1} = pe_{i,\beta}$ is orthogonal to all of the other summands, and this follows since p is orthogonal to all of the summands which remain from the previous case and $e_{i,\beta}$ is orthogonal to all of the newly constructed idempotents. Finally we compute

$$\sum_{i\geqslant\beta}\left(e_{i,0} + \sum_{j<\beta} e'_{i,j}\right) = \sum_{i\geqslant\beta}\left(e_{i,0} + \sum_{j<\beta-1} e'_{i,j}\right) + \sum_{i\geqslant\beta} e'_{i,\beta-1} = (1-p) + p = 1.$$

Case 2: Suppose β is a limit ordinal. In this case there are no idempotents to construct, as all the necessary idempotents for this case were constructed for previous ordinals. But we still have to verify that $\sum_{i\geqslant\beta}\left(e_{i,0} + \sum_{j<\beta} e'_{i,j}\right) = 1$. We compute

$$\sum_{i\geqslant\beta}\left(e_{i,0} + \sum_{j<\beta} e'_{i,j}\right) = \lim_{\gamma\to\beta}\sum_{i\geqslant\gamma}\left(e_{i,0} + \sum_{j<\gamma} e'_{i,j}\right) = \lim_{\gamma\to\beta} 1 = 1.$$

We have thus constructed the elements we need in all cases, but particularly in the case $\beta = \alpha$ we have a sum $e_{\alpha,0} + \sum_{j<\alpha} e'_{\alpha,j} = 1$. As observed previously, the left-hand side lies in Rx_α. □

4. The Main Theorem and Related Results

With all of the machinery built up in the previous section we are now ready to prove:

Theorem 7. *Let R be an Abelian exchange ring with a left linear Hausdorff topology. Let $\{x_i\}_{i\in I}$ be a summable family, all of whose left multiples are summable. If $\sum_{i\in I} x_i = 1$ then there are orthogonal idempotents $e_i \in Rx_i$ with $\sum_{i\in I} e_i = 1$.*

Proof. Well-ordering I, we may assume that I is a cardinal $\aleph$, and we work by induction on $\aleph \geqslant 2$. The case when $\aleph$ is a finite cardinal follows from the fact that R is an exchange ring. So we may assume $\aleph$ is infinite, and assume that the claim is true for all smaller cardinals in every ring satisfying the assumptions of the theorem.

Suppose $\{x_i\}_{i\in\aleph}$ is a summable family of elements in R as hypothesized. Set $y_i = \sum_{j>i} x_j$ and $y'_i = \sum_{j\geqslant i} x_j$ for each $i \in \aleph$. We will inductively construct idempotents $e_i \in Rx_i$ and $f_i \in Ry_i$ for each $i \in \aleph$ so that

$$\sum_{k\leqslant i} e_k + f_i = 1$$

and the summands are pairwise orthogonal idempotents.

From $\sum_{i\in\aleph} x_i = 1$ we have $x_0 + y_0 = 1$. By Nicholson's characterization of exchange rings, there are orthogonal idempotents $e_0 \in Rx_0$ and $f_0 \in Ry_0$ which sum to 1. This finishes the base case. Now suppose by induction that the idempotents have been constructed for all ordinals less than $\alpha \in \aleph$.

Case 1: Suppose α is a successor ordinal. We then have $\sum_{k\leqslant\alpha-1} e_k + f_{\alpha-1} = 1$. Write $f_{\alpha-1} = r_{\alpha-1}(x_\alpha+y_\alpha)$. Set $g_1 = \sum_{k\leqslant\alpha-1} e_k$, $g_2 = r_{\alpha-1}x_\alpha$ and $g_3 = r_{\alpha-1}y_\alpha$. By Lemma 5, there are idempotents $e_\alpha \in Rg_2 \subseteq Rx_\alpha$ and $f_\alpha \in Rg_3 \subseteq Ry_\alpha$ so that $g_1 + e_\alpha + f_\alpha = 1$ and the summands are orthogonal idempotents. From the fact that e_α and f_α are orthogonal to g_1, we deduce they are also orthogonal to e_k for all $k < \alpha$. This finishes the inductive construction in this case.

Case 2: Suppose α is a limit ordinal. Setting $f'_\alpha = 1 - \sum_{k<\alpha} e_k$, we see that f'_α is an idempotent, and orthogonal to e_k for each $k < \alpha$. Note that for each $\beta < \alpha$ we have the equality $f'_\alpha f_\beta f'_\alpha = f'_\alpha$, as seen by multiplying the equation $\sum_{k\leqslant\beta} e_k + f_\beta = 1$ by the central element f'_α and noting

$$f'_\alpha e_j = \left(1 - \sum_{k<\alpha} e_k\right) e_j = e_j - e_j - \sum_{k<\alpha,\ k\neq j} e_k e_j = 0.$$

Write $f_\beta = r_\beta y_\beta$ for some $r_\beta \in R$. We then have a sum

$$\left(\sum_{\beta<k<\alpha} f'_\alpha r_\beta f'_\alpha x_k\right) + f'_\alpha r_\beta f'_\alpha y'_\alpha = f'_\alpha.$$

Since $\alpha \in \aleph$ (hence $|\alpha| < \aleph$) and $f'_\alpha R f'_\alpha$ is an Abelian exchange ring, the induction hypothesis applies to the ring $f'_\alpha R f'_\alpha$ and the ordinal α. Thus, we can left multiply each of the summands in the previous offset equation, and obtain orthogonal idempotents summing to f'_α (the identity in the corner ring $f'_\alpha R f'_\alpha$) for any $\beta < \alpha$. This is exactly the set-up in Proposition 6, except that we are working in the corner ring, so we may conclude that $f'_\alpha \in f'_\alpha R f'_\alpha y'_\alpha f'_\alpha$. Write $f'_\alpha = r'_\alpha y'_\alpha$ for some $r'_\alpha \in f'_\alpha R f'_\alpha$. We then have

$$\sum_{k<\alpha} e_k + r'_\alpha x_\alpha + r'_\alpha y_\alpha = 1.$$

We finish this case just as in Case 1.

The inductive construction is now complete. Notice that

$$\sum_{i\in\aleph} e_i = \lim_{\alpha\to\aleph}\left(\sum_{k\leqslant\alpha} e_k\right) + \lim_{\alpha\to\aleph} f_\alpha = \lim_{\alpha\to\aleph}\left(\left(\sum_{k\leqslant\alpha} e_k\right) + f_\alpha\right) = \lim_{\alpha\to\aleph} 1 = 1,$$

which completes the theorem. ∎

Theorem 8. *Let R be an exchange ring with a left linear, Σ-complete, Hausdorff topology. If $R/J(R)$ is Abelian, then R is a full exchange ring.*

Proof. Let $\{x_i\}_{i\in I}$ be a summable family in R, summing to 1. Set $\overline{R} = R/J(R)$. By Lemma 2 and the previous theorem, we can find orthogonal idempotents $\epsilon_i \in \overline{Rx_i}$ summing to 1. Since R is an exchange ring, we can strongly lift each of the ϵ_i to idempotents $e_i \in Rx_i$. Notice that $\sum_{i\in I} \overline{e_i} = \overline{1}$, hence $u = \sum_{i\in I} e_i$ is a unit. By Lemma 4, $\{u^{-1}e_i\}_{i\in I}$ is a set of orthogonal idempotents, summing to 1, with $u^{-1}e_i \in Rx_i$, which is what we set out to construct. □

Theorem 9. *If M is a square-free module with the finite exchange property then M has the full exchange property.*

Proof. We know that $E = \mathrm{End}(M)$ is an exchange ring, which has a left linear, complete, Hausdorff topology given by the finite topology. Further, E/Δ is Abelian, hence $E/J(E)$ is Abelian by Lemma 3. By the previous theorem, E is a full exchange ring in the finite topology, hence M has the full exchange property. □

The previous theorem actually applies to the larger class of modules whose complements are essentially unique, by [11, Lemma 15].

Corollary 10. *Quasi-continuous modules with finite exchange have full exchange.*

5. An Interesting Observation

The ideas used to prove the main theorem were initially motivated by an observation that square-free modules with finite exchange automatically possess an infinite version of the (C_3) property, which we record here. Recall that (C_3) says that a sum of two non-intersecting summands is a summand:

$$A, B \subseteq^{\oplus} M \text{ and } A \cap B = (0) \Longrightarrow A \oplus B \subseteq^{\oplus} M.$$

It turns out that if a module possesses the property that any union of summands is a summand, then the module in question is a direct sum of indecomposable modules [8, Section 2.3]. Thus, the following proposition may come as a surprise. In particular, note that the summability hypothesis is all that prevents the module in question from reducing to a direct sum of indecomposables.

Proposition 11. *Let M_Λ be a square-free module with finite exchange, and set $E = \mathrm{End}(M_\Lambda)$. If $\{e_i\}_{i\in I}$ is a summable set of idempotents in E with $e_i(M) \cap e_j(M) = (0)$ whenever $i \neq j$, then $\bigoplus_{i\in I} e_i(M)$ is a summand of M.*

Proof. We begin by first proving that the e_i are orthogonal modulo $J(E)$. To see this, first note that since M is square-free we know E/Δ is Abelian. Let $i \neq j$ and let $m \in M$, $m \neq 0$. We have the equality $e_ie_j(m) = e_je_i(m) + \delta(m)$ for some $\delta \in \Delta$. Using the definition of Δ, there exists some element $\lambda \in \Lambda$ with $m\lambda \neq 0$ but $\delta(m\lambda) = 0$. In particular we have the equality $e_ie_j(m\lambda) = e_je_i(m\lambda)$. But $e_i(M) \cap e_j(M) = (0)$, hence $e_ie_j(m\lambda) = 0$. Since $m \in M$ was arbitrary, and recalling that $\Delta \subseteq J(E)$ by [11, Lemma 11], we have $e_ie_j \in \Delta \subseteq J(E)$.

Set $f = 1 - \sum_{i\in I} e_i$. Since the e_i are orthogonal modulo $J(E)$, and $J(E)$ is a closed ideal, we see that f is an idempotent modulo $J(E)$. Using the finite exchange property, there is an idempotent $g^2 = g \in E$ with $f - g \in J(E)$. Now $\{e_i\}_{i\in I} \cup \{g\}$ is a summable set of idempotents, orthogonal modulo $J(E)$, with $u = \left(\sum_{i\in I} e_i\right) + g$ a unit (since the sum is congruent to 1 modulo $J(E)$). [11, Lemma 8] implies that $\{e_iu^{-1}\}$ is a summable family of orthogonal idempotents, which finishes the proof. □

Corollary 12 (cf. [11, Lemma 16]). *If M_Λ is a square-free module with finite exchange then M has (C_3).*

Proof. A finite set of idempotents is always summable. □

We end with the following example, showing that injective, square-free modules with (finite) exchange are not always direct sums of indecomposable modules.

Example 13. Let $\mathcal{P}$ be the set of integer primes, and set $R = \prod_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}$. The ring R is a product of fields, and hence an exchange ring. In particular, R_R has finite exchange. Similarly, R_R is injective (see for example [5, Corollary 3.11B]).

Next, we demonstrate that R_R is square-free. Suppose to the contrary that there are two nonzero submodules $X_1, X_2 \subset R_R$ with $X_1 \cap X_2 = (0)$ and $X_1 \cong X_2$. Let $f : X_1 \to X_2 \subset R$ be such an isomorphism. Due to the fact that R_R is injective, we can extend f to a map $F : R \to R$, and we set $F(1) = r$. In particular, f is given by left multiplication by r. But X_1, being a right ideal in R is also a left ideal in R, as R is commutative. Hence $(0) \neq X_2 = f(X_1) = rX_1 \subseteq X_1$. This contradicts the hypothesis that $X_1 \cap X_2 = (0)$, thus proving that R_R is square-free. Note however that the Abelian group $R_{\mathbb{Z}}$ is not square-free. In fact, the $\mathbb{Z}$-submodules generated by $(1+2\mathbb{Z}, 0+3\mathbb{Z}, 1+5\mathbb{Z}, 0+7\mathbb{Z}, \ldots)$ and $(0+2\mathbb{Z}, 1+3\mathbb{Z}, 0+5\mathbb{Z}, 1+7\mathbb{Z}, \ldots)$ are both isomorphic to $\mathbb{Z}_{\mathbb{Z}}$.

Finally, let e_i be the element of R with 0 in every coordinate, except with 1 in the coordinate corresponding to the ith prime. Notice that e_iR is isomorphic to a field, hence indecomposable, for each $i > 0$. On the other hand, given any nonzero submodule $M_R \subseteq R_R$ then $e_iM = e_iR$ for some $i > 0$. In particular, the set $\{e_iR\}$ contains all indecomposable submodules of R, but $\bigoplus_{i\in\mathbb{Z}_{>0}} e_iR = \bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z} \subsetneq R$.

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