# EXCHANGE RINGS, EXCHANGE EQUATIONS, AND LIFTING PROPERTIES

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Abstract. In this paper, we study exchange rings and clean rings $R$ with $2 \in \mathrm{U}(R)$ (or otherwise). Analogues of a theorem of Camillo and Yu characterizing clean and strongly clean rings with $2 \in \mathrm{U}(R)$ are obtained for such rings (as well as for exchange rings) using the viewpoint of exchange equations introduced in a recent paper of the authors. We also study a new class of rings including von Neumann regular rings in which square roots of one (instead of idempotents) can be lifted modulo left ideals, and conjecture that such rings are exchange rings. This conjecture holds for commutative rings, and would hold for all rings if it holds for semiprimitive rings of characteristic 2.

## 1. Introduction

Exchange rings were introduced in 1972 by Warfield [26] via the study of Crawley and Jónsson's exchange property of modules in [9]. Five years later, Nicholson [21] defined a ring $R$ to be *left suitable* if idempotents can be lifted modulo every left ideal $I \subseteq R$. (This means that, for any $r \in R$ with $r^2 - r \in I$, there exists an idempotent $e \in R$ such that $e - r \in I$.) Nicholson showed that left suitable rings are exactly Warfield's exchange rings. In particular, left suitable rings are the same as right suitable rings. A later paper of Nicholson [22] showed independently that $\mathrm{suit}_\ell(R)$, the set of left suitable elements in $R$, is always the same as $\mathrm{suit}_r(R)$, the set of right suitable elements in $R$. Here, $\mathrm{suit}_\ell(R)$ is defined to be the set of elements $a \in R$ for which there exists an idempotent $e \in Ra$ such that $1 - e \in R(1-a)$, and $\mathrm{suit}_r(R)$ is defined similarly. To simplify notations, we shall henceforth denote the set $\mathrm{suit}_\ell(R) = \mathrm{suit}_r(R)$ by $\mathrm{suit}\,(R)$. The beginning results in [21] have shown that $R$ is a suitable ring if and only if $R = \mathrm{suit}\,(R)$. Much later, some strong two-sided existential properties for idempotents in suitable (or exchange) rings were proved in [15].

In [21] and [23], Nicholson also introduced the set of *clean elements* $\mathrm{cn}\,(R)$ and the set of *strongly clean elements* $\mathrm{scn}\,(R)$. By definition, $a \in \mathrm{cn}\,(R)$ means that $a = e + u$ for some idempotent $e$ and some unit $u$ in $R$. If such a decomposition $a = e + u$ can be found with the additional property that $eu = ue$, we say that $a \in \mathrm{scn}\,(R)$. If $R = \mathrm{cn}\,(R)$ (resp. $R = \mathrm{scn}\,(R)$), $R$ is said to be a clean (resp. strongly clean) ring. In general, $\mathrm{scn}\,(R) \subseteq \mathrm{cn}\,(R) \subseteq \mathrm{suit}\,(R)$, so clean rings are always suitable rings.

In our recent paper [16], we initiated an "equational" study of the elements in the three sets above and proved the following result, where $\mathrm{U}(R)$ denotes the unit group of $R$, $\mathrm{idem}\,(R) = \{e \in R : \ e = e^2\}$, and $C_R(a) = \{r \in R : \ ra = ar\}$.

**Theorem 1.1.** *For any element $a$ in a ring $R$, the following hold.*
(1) $a \in \mathrm{suit}\,(R)$ *iff there exist* $f \in \mathrm{idem}\,(R)$ *and* $x \in R$ *such that* $xa - fx = 1$.
(2) $a \in \mathrm{cn}\,(R)$ *iff there exist* $f \in \mathrm{idem}\,(R)$ *and* $x \in \mathrm{U}(R)$ *such that* $xa - fx = 1$.

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(3) $a \in \mathrm{scn}\,(R)$ *iff there exist* $f \in \mathrm{idem}\,(R)$ *and* $x \in C_R(a)$ *such that* $xa - fx = 1$. *In this case,* $x$ *is automatically a unit in* $R$.

In [16], an equation of the form $xa - fx = 1$ where $f \in \mathrm{idem}\,(R)$ is called a (left) *exchange equation* for $a \in R$. In §2 of this paper, we exploit an idea of Camillo and Yu [6] in studying clean (and strongly clean) elements in a ring $R$ with $2 \in \mathrm{U}(R)$, whereby the role of idempotents is replaced by that of square roots of 1 in $R$. The pursuit of this idea *for suitable elements* $b \in \mathrm{suit}\,(R)$ leads to a new class of linear equations (which we'll call "left $\sqrt{1}$-exchange equations"), of the form $yb + \omega y = 1$, where $\omega^2 = 1$. The first main result in §2 is the following.

**Theorem 1.2.** *Let* $R$ *be a ring with* $2 \in \mathrm{U}(R)$, *and let* $a \in R$, $b := 2a - 1$. *Then* $a \in \mathrm{suit}\,(R)$ *if and only if* $b$ *satisfies a left* $\sqrt{1}$*-exchange equation.*

An expanded form of this result covering also the cases of $a \in \mathrm{cn}\,(R)$ and $a \in \mathrm{scn}\,(R)$ is presented in Theorem 2.1. In the rest of §2, we show that $\sqrt{1}$-exchange equations can be studied in their own right *over arbitrary rings* $R$ (without assuming $2 \in \mathrm{U}(R)$), and prove the following characterization result.

**Theorem 1.3.** *An element* $b \in R$ *satisfies a left* $\sqrt{1}$*-exchange equation if and only if the polynomial*

$$Q_b(y) := \big(1 - y\,(1+b)\big)\,\big(1 + (1-b)\,y\big) = 1 - by - yb - y^2 + yb^2y \tag{1.4}$$

*has a root in* $R$. *In this case,* $b$ *also satisfies a right* $\sqrt{1}$*-exchange equation.*

The proof of the left-right symmetry of $\sqrt{1}$-exchange equations here leads to a curious polynomial identity holding in any ring:

$$\big(1 - y\,(1+b)\big)\,\big(1 + (1-b)\,y\big) = \big(1 + y\,(1-b)\big)\,\big(1 - (1+b)\,y\big). \tag{1.5}$$

A precedent for such a polynomial identity has appeared earlier in [16] in the context of suitable elements and exchange equations; namely,

$$P_a(x) := (1 - xa)\,(1 + (1-a)\,x) = (1 + x\,(1-a))\,(1 - ax). \tag{1.6}$$

Indeed, a main result in [16] was that $a \in \mathrm{suit}\,(R)$ if and only if the polynomial $P_a(x)$ has a root in $R$. Theoretically, Theorem 1.3 is a "$\sqrt{1}$-analogue" of this result.

In §3, we switch our attention from exchange equations to lifting problems. The idea here is again that one tries to replace the role of idempotents by that of square roots of unity. In the same way as Nicholson had defined the lifting of idempotents modulo a 1-sided ideal $I \subseteq R$, we can define the notion of *lifting square roots of* 1 *modulo* $I$ to mean that, for any $b \in R$ such that $1 - b^2 \in I$, there exists $\omega \in R$ such that $\omega^2 = 1$ and $\omega - b \in I$. In the good case where $2 \in \mathrm{U}(R)$, the basic constructions of Camillo and Yu from [6] can be used to show that lifting squares roots of 1 modulo all left ideals is the same as lifting idempotents modulo such left ideals, which amounts to $R$ being an exchange ring. (This is proved in Theorem 3.2.) However, if nothing is assumed about $2 \in R$, the lifting of square roots of 1 modulo all left ideals emerges as a new viable property for us to work with. If $R$ has this property, we say that $R$ is a **left $\sqrt{1}$-lifting ring**. The polynomial $Q_b(y)$ in (1.4) is essentially constructed for the study of such rings. In Theorem 3.5, we give some characterizations for $R$ to be left $\sqrt{1}$-lifting; e.g. in terms of the solvability of equations of the form $Q_b(y)\,(1 - b^2) = 0$ in $R$. We show in Theorem 3.8 that *all von Neumann regular rings have such a property,* though a general exchange ring (or even a strongly $\pi$-regular ring) need not. For convenience of exposition, we raise in (3.4) the conjecture that all left $\sqrt{1}$-lifting rings are exchange rings. (If this was true, we would have produced a new interesting class of rings that is strictly between the class of von Neumann regular rings and the class of exchange rings.) While this conjecture might not be true for all rings, we show in Theorem 5.5 and Corollary 5.9 that it is sufficient to prove the conjecture in the case of semiprimitive rings of characteristic 2, and moreover, that the conjecture is true for all rings whose nilpotent elements lie in the Jacobson radical. In more explicit words, the latter means that, *if square roots of* 1 *can be lifted modulo every left ideal of a ring* $R$ *whose nilpotent elements all lie in its Jacobson radical, then* $R$ *is indeed an exchange ring.*

Section 4 is devoted to the consideration of the behavior of left $\sqrt{1}$-lifting rings with respect to the change of rings. Here, we begin to see that some features of the class of exchange rings are not shared by the class of left $\sqrt{1}$-lifting rings. Finally, in §5, we provide a characterization theorem (5.6) for certain rings $R$ to be left $\sqrt{1}$-lifting, in the crucial case where $2 \in \mathrm{rad}\,(R)$ (the Jacobson radical of $R$). For instance, if $R$ is a local ring or an abelian $\pi$-regular ring, then $R$ is left $\sqrt{1}$-lifting iff the "group of 1-units" $1 + \mathrm{rad}\,(R)$ is an elementary abelian 2-group.

Throughout this paper, $R$ denotes an arbitrary ring with $1 \in R$, and the notations introduced so far in this Introduction will be in force. The adjective "regular" (for rings and ring elements) will always be in the sense of being von Neumann regular. Other standard terminology and notations in ring theory follow those in [13] and [18]. Whenever it is more convenient, we'll use the widely accepted shorthand "iff" for "if and only if" in the text.

## 2. $\sqrt{1}$-Exchange Equations in Case $2 \in \mathrm{U}(R)$ (and Beyond)

While an element $a$ in a general ring $R$ is defined to be clean if $a = e + u$ for some $e \in \mathrm{idem}\,(R)$ and some $u \in \mathrm{U}(R)$, Camillo and Yu have shown in [6] that, in the case where $2 \in \mathrm{U}(R)$, the cleanness condition $a \in \mathrm{cn}\,(R)$ is equivalent to $2a - 1 = \omega + v$ for some $v \in \mathrm{U}(R)$ and some $\omega \in R$ with $\omega^2 = 1$. (Normally, $\omega$ denotes a cube root of unity. Here, we "borrow" the letter $\omega$ and use it for a square root of 1. This notation will be used consistently throughout §§2–3.) The case $a \in \mathrm{scn}\,(R)$ has a similar formulation, where we simply require additionally that $\omega v = v \omega$. While these alternative characterizations were well known (in case $2 \in \mathrm{U}(R)$) for clean and strongly clean elements, no analogue was known so far for Nicholson's suitable elements. With the new idea of using exchange equations, we can now fill this gap. Indeed, we have the following result which gives a uniform set of hierarchical "$\omega$-characterizations" for suitable, clean, and strongly clean elements in any ring in which 2 is a unit.

**Theorem 2.1.** *Let* $R$ *be a ring with* $2 \in \mathrm{U}(R)$. *For any* $a \in R$ *and* $b := 2a - 1$, *the following statements hold.*

(1) $a \in \mathrm{suit}\,(R)$ *iff there exist* $y, \omega \in R$ *with* $\omega^2 = 1$ *such that* $yb + \omega y = 1$.
(2) $a \in \mathrm{cn}\,(R)$ *iff there exist* $y \in \mathrm{U}(R)$ *and* $\omega \in R$ *with* $\omega^2 = 1$ *such that* $yb + \omega y = 1$.
(3) $a \in \mathrm{scn}\,(R)$ *iff there exist* $y \in C_R(b)$ *and* $\omega \in R$ *with* $\omega^2 = 1$ *such that* $yb + \omega y = 1$. *In this case,* $y$ *is automatically a unit in* $R$.

*Proof.* We shall only prove (1) here, since (2) and (3) can be proved in a similar way by using the same arguments below, applied in conjunction with part (2) and part (3) of Theorem 1.1 in §1.

First assume that $a \in \mathrm{suit}\,(R)$. By Theorem 1.1(1), there exist $x \in R$ and $f \in \mathrm{idem}\,(R)$ such that $xa - fx = 1$. The element $\omega := 1 - 2f$ satisfies $\omega^2 = 1$, so replacing $a$ by $(b+1)/2$ and $f$ by $(1-\omega)/2$ gives

$$1 = \frac{x\,(b+1)}{2} - \frac{(1-\omega)\,x}{2} = yb + \omega y \quad \text{for } y := x/2. \tag{2.2}$$

Conversely, assume that there is an equation $yb + \omega y = 1$ where $y \in R$ and $\omega^2 = 1 \in R$. For $f := (1-\omega)/2 \in \operatorname{idem}(R)$, we have

$$1 = y\,(2a-1) + (1-2f)\,y = (2y)\,a - f(2y). \tag{2.3}$$

This is an exchange equation for $a$, so Theorem 1.1(1) shows that $a \in \operatorname{suit}(R)$. □

In analogy with the term "(left) exchange equation" for $xa - fx = 1$ (where $f^2 = f$), we shall henceforth refer to $yb + \omega y = 1$ (where $\omega^2 = 1$) as a "*left* $\sqrt{1}$*-exchange equation*" for an element $b \in R$. (Right $\sqrt{1}$-exchange equations are defined similarly.) Applying Theorem 2.1 to all elements in a ring leads to the following result, in which the role of the condition $2 \in \mathrm{U}(R)$ is further clarified.

**Theorem 2.4.** *For any ring $R$, the following statements hold.*

(1) *$R$ is a suitable ring with $2 \in \mathrm{U}(R)$ iff for each $b \in R$ there is a left $\sqrt{1}$-exchange equation $yb + \omega y = 1$ with $y \in R$.*
(2) *$R$ is a clean ring with $2 \in \mathrm{U}(R)$ iff for each $b \in R$ there is a left $\sqrt{1}$-exchange equation $yb+\omega y = 1$ with $y \in \mathrm{U}(R)$.*
(3) *$R$ is a strongly clean ring with $2 \in \mathrm{U}(R)$ iff for each $b \in R$ there is a left $\sqrt{1}$-exchange equation $yb + \omega y = 1$ with $y \in C_R(b)$.*

*Proof.* Again, we'll only prove (1), and leave the similar proofs of (2) and (3) as exercises. For (1), the "only if" part follows from Theorem 2.1(1), since $2 \in \mathrm{U}(R)$ implies that the map $a \mapsto 2a - 1$ is a permutation of the set $R$. Conversely, assume the condition in (1) for every $b \in R$. Applying it to $b = 1$, we get two elements $y, \omega \in R$ with $\omega^2 = 1$ such that $y + \omega y = 1$. Left multiplying $1 = (1+\omega)\,y$ by $1-\omega$ gives $1 - \omega = (1-\omega^2)\,y = 0$, so $\omega = 1$. Thus, $1 = y + \omega y = 2y \Rightarrow 2 \in \mathrm{U}(R)$. Given this information, Theorem 2.1(1) shows that $R$ is an exchange ring. □

**Remark 2.5.** The "original" Camillo-Yu analogue of part (3) above would have been the following: *a ring $R$ is strongly clean with $2 \in \mathrm{U}(R)$ iff every element in $R$ has the form $\omega + v$ where $v \in \mathrm{U}(R)$, $\omega^2 = 1$, and $v\,\omega = \omega\,v$.* This fact was first stated and proved by Wang and Chen in [25, Proposition 5]

In the rest of this section, we shall make a quick study of the left $\sqrt{1}$-exchange equation $yb+\omega y = 1$ without assuming that $2 \in \mathrm{U}(R)$. It turns out that, *even for arbitrary rings,* a good part of the formalism we developed in [16] for the exchange equation $xa - fx = 1$ does have analogues in the new setting of $\sqrt{1}$-exchange equations. What we have learned about the exchange equations in [16] will serve as a valuable guide, although it should definitely not be assumed that "everything is the same" in the new case. After all, there is a rather significant difference between $f$ (with $f^2 = f$) and $\omega$ (with $\omega^2 = 1$). The following result on the existence and left-right symmetry of $\sqrt{1}$-exchange equations is an analogue of [16, (3.3), (3.4)].

**Theorem 2.6.** (1) *An element $b \in R$ satisfies a left $\sqrt{1}$-exchange equation $yb + \omega y = 1$ (for some $y, \omega \in R$, with $\omega^2 = 1$) iff the polynomial*

$$Q_b(y) := \big(1 - y\,(1+b)\big)\,\big(1 + (1-b)\,y\big) \tag{2.7}$$

*has a root in $R$. In this case, $b$ also satisfies a right $\sqrt{1}$-exchange equation.*
(2) *A central element $b \in R$ satisfies a left $\sqrt{1}$-exchange equation iff $b$ is the sum of a unit and a square root of* 1.

*Proof.* (1) First assume that $b$ satisfies a left $\sqrt{1}$-exchange equation $yb+\omega y = 1$. Left multiplying this equation by $\omega$, we get

$$\omega = (\omega y)\,b + y = (1-yb)\,b + y = b + y - yb^2. \tag{2.8}$$

(This shows that $\omega$ is actually uniquely determined by $b$ and $y$.) Substituting this back into the original left $\sqrt{1}$-exchange equation gives

$$0 = 1 - yb - (b + y - yb^2)\, y = 1 - by - yb - y^2 + yb^2 y. \tag{2.9}$$

The RHS of this equation is exactly the polynomial $Q_b(y)$ in (2.7). Indeed, by direct expansion of the RHS of (2.7), four of the nine terms cancel out to give the five terms on the RHS of (2.9).

Conversely, suppose there exists $y \in R$ such that $Q_b(y) = 0$. Taking hint from the work above, define $\omega := b + y - yb^2$. We have

$$1 - yb - \omega y = 1 - yb - (b + y - yb^2)\, y = Q_b(y) = 0, \tag{2.10}$$

so $y$ is a root of the left $\sqrt{1}$-exchange equation $yb + \omega y = 1$ *provided that* we have indeed $\omega^2 = 1$. This is easily checked as follows:

$$\begin{aligned} \omega^2 &= \omega\left(b + y\,(1 - b^2)\right) = \omega b + \omega y\,(1 - b^2) \\ &= (b + y - yb^2)\, b + (1 - yb)\,(1 - b^2) = 1. \end{aligned}$$

To see that, in the above setting, $b$ also satisfies a *right* $\sqrt{1}$-exchange equation, we simply note that the polynomial $Q_b(y)$ is left-right symmetric, in the sense that it remains the same if we write each of its monomial terms "from right to left". Or, in more detail, if we define $\omega' = b + y - b^2 y$, then we also have $(\omega')^2 = 1$, and $by + y\omega' = by + y\,(b + y - b^2 y) = 1$, just as before.

(2) Assume that $b \in R$ is central. If $b = u + (-\omega)$ where $u \in \mathrm{U}(R)$ and $(-\omega)^2 = 1$, then letting $y = u^{-1}$ gives $by = 1 - \omega y$, so we have $yb + \omega y = 1$. Conversely, if we have $yb + \omega y = 1$ for some $y, \omega \in R$ with $\omega^2 = 1$, then $(b + \omega)\, y = 1$. From (2.9) and the fact that $by = yb$, we see that $y \in \mathrm{U}(R)$. Thus, $b = y^{-1} + (-\omega)$, as desired. □

The remark in the proof of Theorem 2.6(1) on the symmetry of the polynomial $Q_b(y)$ in (2.7) implies, quite pleasantly, the following free algebra identity:

$$Q_b(y) = \big(1 - y\,(1 + b)\big)\,\big(1 + (1 - b)\, y\big) = \big(1 + y\,(1 - b)\big)\,\big(1 - (1 + b)\, y\big) \in \mathbb{Z}\,\langle b, y\rangle. \tag{2.11}$$

This is an analogue of an earlier free algebra identity

$$P_a(x) := (1 - xa)\,(1 + (1 - a)\, x) = (1 + x\,(1 - a))\,(1 - ax) \in \mathbb{Z}\,\langle a, x\rangle, \tag{2.12}$$

which we have first encountered in [16, (4.2)]. Not surprisingly, there is a close formal relationship between the two polynomials $P_a(x)$ and $Q_b(y)$, which we'll spell out explicitly in the following result. Note that the property (1) of $Q_b(y)$ below is the analogue of the formula $P_a(x) = P_{1-a}(-x)$ proved earlier in [16, (4.1)].

**Proposition 2.13.** (1) $Q_b(y) = Q_{-b}(-y)$. (2) *If $b \in R$ (in any ring $R$) has the form $2a - 1$ for some $a \in R$, then $Q_b(y) = P_a(2y)$ for every $y \in R$.*

*Proof.* (1) follows from the fact that all monomial terms of $Q_b(y) = 1 - by - yb - y^2 + yb^2 y$ (in $b$ and $y$) have *even* degrees. The property (1) here simply reflects the fact that the left $\sqrt{1}$-exchange equation $yb + \omega y = 1$ for $b$ can be written as a left $\sqrt{1}$-exchange equation for $-b$ (with a "new variable" $-y$).

(2) In the case where $2 \in \mathrm{U}(R)$, we know (from the proof of Theorem 2.1(1)) that the pair $(a, x)$ is "related" to the pair $(b, y)$ by the basic substitutions $a = (b + 1)/2$ and $x = 2y$. Therefore, if $b := 2a - 1$, we should expect $P_a(x)$ to formally transform into $Q_b(y)$, even without assuming that $2 \in \mathrm{U}(R)$. Indeed, a direct verification of the formula $Q_b(y) = P_a(2y)$ (using the factorized forms of these polynomials in (2.11) and (2.12)) is completely straightforward. □

**Example 2.14.** Let $R = \mathbb{M}_2(S)$ where $S$ is any ring, and let $B = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in R$ where $r \in \mathrm{U}(S)$. In [16, Corollary 3.7], we have shown that $B$ is a suitable element in $R$, with a "suitabilizer" $Y = \left(\begin{smallmatrix} 0 & r^{-1} \\ 0 & 0 \end{smallmatrix}\right)$; that is, $P_B(Y) = 0$. Using (2.13)(2) in a purely formal way, we can predict that $Q_B(Y) = 0$ also. To double-check this claim, we simply note that $I - Y\,(I + B) = \left(\begin{smallmatrix} 0 & * \\ 0 & 1 \end{smallmatrix}\right)$, and $I + (I - B)\,Y = \left(\begin{smallmatrix} 1 & * \\ 0 & 0 \end{smallmatrix}\right)$, so their product is $Q_B(Y) = 0$. Using the proof of Theorem 2.6 (especially the formula (2.8)), we can come up with an explicit left $\sqrt{1}$-exchange equation for $Y$; namely, $YB + \Omega Y = I$, for a highly nontrivial square root of unity

$$\Omega = B + Y - YB^2 = \begin{pmatrix} -r^{-1}sr & r^{-1}(1 - s^2) \\ r & s \end{pmatrix}. \tag{2.15}$$

We close this section by recording a useful application of some of our results to $\mathrm{C}^*$-algebras. Since the condition $2 \in \mathrm{U}(R)$ is automatic for any unital $\mathrm{C}^*$-algebra $R$, the result from [2, Theorem 7.2] that such $R$ is an exchange ring iff it has real rank zero leads to the following conclusions in view of what we have proved in this section.

**Theorem 2.16.** *For any unital* $\mathrm{C}^*$*-algebra* $R$*, the following statements are equivalent:*

(1) $R$ *is of real rank zero.*
(2) *For every* $b \in R$*, there exist* $y, \omega \in R$ *with* $\omega^2 = 1$ *such that* $yb + \omega y = 1$*.*
(3) *For every* $b \in R$*, the equation* $\big(1 - y\,(1 + b)\big)\big(1 + (1 - b)\,y\big) = 0$ *is solvable in* $R$*.*

The analogue of this result stated in terms of exchange equations (instead of $\sqrt{1}$-exchange equations) has appeared earlier in Theorem 5.15 of [16]. After proving Theorem 3.5 in the next section, we can add another equivalent condition to the above list; namely, (4) *for every* $b \in R$*, there exists* $y \in R$ *such that* $\big(b + y\,(1 - b^2)\big)^2 = 1$*.*

## 3. Lifting Square Roots of One

In this section, we turn our attention from exchange equations to lifting problems involving square roots of 1 (instead of idempotents). The introduction of the polynomials $Q_b(y)$ in §2 was largely in preparation for the study of the $\sqrt{1}$-lifting problem in this section. Recall from §1 that "lifting square roots of 1 modulo a left ideal $I \subseteq R$" means that, for every $b \in R$ with $1 - b^2 \in I$, there exists $\omega \in R$ with $\omega^2 = 1$ such that $\omega - b \in I$. In the best case where $2 \in \mathrm{U}(R)$, we have the Camillo-Yu bijection from $\mathrm{idem}\,(R)$ to the set of square roots of 1 in $R$ given by $e \mapsto 2e - 1$ (with inverse map $\omega \mapsto (1 + \omega)/2$). Using this bijection leads easily to the following basic result.

**Proposition 3.1.** *If* $2 \in \mathrm{U}(R)$ *and* $I \subseteq R$ *is a left ideal, the following two statements are equivalent:*

(1) *Idempotents lift modulo* $I$*.*
(2) *Square roots of* 1 *lift modulo* $I$*.*

*Proof.* (1) ⇒ (2). Let $b \in R$ be a "square root of 1 modulo $I$" (in that $1 - b^2 \in I$). Then $(1 + b)/2$ is an "idempotent modulo $I$", so (1) gives an idempotent $e \in R$ such that $e - (1 + b)/2 \in I$. We have now $(2e - 1) - b \in I$ where $(2e - 1)^2 = 1$, so (2) holds.

(2) ⇒ (1). Let $r \in R$ be an idempotent modulo $I$. Then $2r - 1$ is a square root of 1 modulo $I$, so by (2), there exists $\omega \in R$ such that $\omega^2 = 1$ and $\omega - (2r - 1) \in I$. Then $(1 + \omega)/2 - r \in I$ where $(1 + \omega)/2 \in \mathrm{idem}\,(R)$, so (1) holds. □

To simplify terminology, we'll say that a ring $R$ is "left $\sqrt{1}$-lifting" if square roots of 1 can be lifted modulo every left ideal of $R$. "Right $\sqrt{1}$-lifting" is defined similarly. For rings in which 2 is a unit, the result below shows that the notions of left or right $\sqrt{1}$-lifting do not really amount to anything new.

**Theorem 3.2.** *If* $2 \in \mathrm{U}(R)$*, the following three statements are equivalent:*

(1) $R$ *is an exchange ring.*
(2) $R$ *is left* $\sqrt{1}$*-lifting.*
(3) $R$ *is right* $\sqrt{1}$*-lifting.*

*Proof.* The equivalence (1) $\Leftrightarrow$ (2) follows Proposition 3.1 and Nicholson's classical result (in [21]) that exchange rings are precisely those rings in which idempotents lift modulo every left ideal. (1) $\Leftrightarrow$ (3) follows similarly since exchange rings are left-right symmetric. □

We offer some quick examples (and non-examples) for $\sqrt{1}$-lifting rings below, where the easy verifications for (A) and (B) are left to the reader. (Indeed, both (A) and (B) are special cases of a result on regular rings that we'll prove later in Theorem 3.8.)

**Example 3.3.** (A) Any division ring is left and right $\sqrt{1}$-lifting.

(B) Any Boolean ring is left and right $\sqrt{1}$-lifting.

(C) The strongly $\pi$-regular ring $R = \mathbb{Z}/16\,\mathbb{Z}$ is not $\sqrt{1}$-lifting. To see this, let $I = \overline{8}\,R$. For $b = \overline{3} \in R$, we have $1 - b^2 \in I$. But the only square roots of $1$ in $R$ are $\pm\overline{1}$ and $\pm\overline{7}$, none of which lifts $b = \overline{3}$. (This example shows, incidentally, that although idempotents always lift modulo nil ideals by [18, (21.28)], square roots of $1$ may not.) A closely related "non-lifting" example that is a discrete valuation domain is given by the localization $R$ of $\mathbb{Z}$ at the prime ideal $2\,\mathbb{Z}$ (taking $I$ here to be $8R$).

While Theorem 3.2 requires the assumption that $2 \in \mathrm{U}(R)$, it is of interest to ask how the conditions (1), (2), and (3) there are related if $2 \notin \mathrm{U}(R)$. In this case, (1) $\Rightarrow$ (2) no longer holds. This is shown by Example 3.3(C), since local rings are always exchange rings. The ("symmetry") question whether (2) $\Leftrightarrow$ (3) is much more subtle. A number of results to be proved in the balance of this paper will show that, for various classes of rings $R$ not subject to $2 \in \mathrm{U}(R)$, we do have (2) $\Leftrightarrow$ (3).

One final issue concerning the three conditions in Theorem 3.2 is whether (2) or (3) there might imply (1) for any ring $R$. For convenience of exposition, we raise somewhat tentatively the following:

**Conjecture 3.4.** *Any left* $\sqrt{1}$*-lifting ring is an exchange ring.*

We are not confident that this Conjecture is true in general. For instance, in light of the possible left-right asymmetry of $\sqrt{1}$-lifting, a more reasonable form of the conjecture would have been that (2) and (3) together (without $2 \in \mathrm{U}(R)$) would imply (1) in Theorem 3.2; that is, "any *left and right* $\sqrt{1}$-lifting ring is an exchange ring." The reason we chose to state our conjecture in the form of (3.4) is that there do exist many classes of rings for which that conjecture turns out to be true. This will be seen in the rest of this section, as well as later in §5. To embark on this work, we'll start with a basic characterization result for left $\sqrt{1}$-lifting rings where the polynomial $Q_b(y)$ introduced in (2.7) plays a crucial role.

**Theorem 3.5.** *For any ring* $R$*, the following four statements are equivalent:*

(A) $R$ *is left* $\sqrt{1}$*-lifting.*
(B) *For every* $b \in R$*, there exists* $\omega \in R$ *such that* $\omega^2 = 1$ *and* $\omega - b \in R\,(1 - b^2)$*.*
(C) *For every* $b \in R$*, there exists* $y \in R$ *such that* $\big(b + y\,(1 - b^2)\big)^2 = 1$*.*
(D) *For every* $b \in R$*, there exists* $y \in R$ *such that* $Q_b(y)\,(1 - b^2) = 0$*.*

*If* $\operatorname{char}(R) = 2$*, these statements are further equivalent to the following:*

(E) *For every* $r \in R$*, there exists* $y \in R$ *such that* $(r - yr^2)^2 = 0$*.*

*Proof.* (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C) are self-evident. On the other hand, the proof of (C) $\Leftrightarrow$ (D) takes a short calculation. Indeed, after transposing and factoring (and using a "leapfrog construction" in the sense of P. M. Cohn: see [8, p. 148]), the equation in (C) becomes

$$\begin{aligned} 0 &= [1-b-y\,(1-b^2)]\,[1+b+y\,(1-b^2)] \\ &= [1-y\,(1+b)]\,(1-b)\,[1+y\,(1-b)]\,(1+b) \\ &= [1-y\,(1+b)]\,[1+(1-b)\,y]\,(1-b)\,(1+b) \\ &= Q_b(y)\,(1-b^2), \end{aligned}$$

which is the equation in (D). For the rest, assume that $\mathrm{char}\,(R) = 2$. In this case, reading from the second line of the equations above transforms (C) into $[(1-y\,(1+b))\,(1+b)]^2 = 0$; that is, $[(1-yr)\,r]^2 = 0$, where $r := 1+b$. Hence (C) $\Leftrightarrow$ (E). ☐

**Remark 3.6.** From (2.8), as well as from (D) $\Rightarrow$ (B) above, we know that if $Q_b(y) = 0$ for some $y \in R$, then $\omega - b \in R\,(1-b^2)$ for some $\omega$ with $\omega^2 = 1$. However, the existence of such an $\omega$ would not imply the solvability of $Q_b(y) = 0$ in general, but would only guarantee the solvability of $Q_b(y)\,(1-b^2) = 0$. An explicit example for this is easy to name, by working in a nonzero ring $R$ with $\mathrm{char}\,(R) = 2$. If we take for instance $b = 1$, certainly $\omega - b \in R\,(1-b^2)$ is satisfied by $\omega = 1$. However, $Q_b(y) = 1 - 2y = 1 \neq 0$ for every $y \in R$.

We have nevertheless the following useful corollary of Theorem 3.5. For (2) below, recall that a ring $R$ is called *reversible* if $st = 0 \Rightarrow ts = 0$ in $R$; see [18, p. 201]. Part (2) of this corollary shows that left $\sqrt{1}$-lifting is equivalent to right $\sqrt{1}$-lifting for some classes of rings, including reversible rings. More such examples will be given in the rest of this section (as well as in §5).

**Corollary 3.7.** *For any left $\sqrt{1}$-lifting ring $R$, the following holds.*

(1) *Any factor ring of $R$ is left $\sqrt{1}$-lifting.*

(2) *If $\mathrm{ann}_\ell(1-b^2) \subseteq \mathrm{ann}_r(1-b^2)$ for all $b \in R$ (e.g. when $R$ is a reversible ring, or when all squares are central in $R$), then $R$ is also right $\sqrt{1}$-lifting.*

(3) *If $\mathrm{char}\,(R) = 2$ and $R$ is a reduced ring, then $R$ is a strongly regular ring.*[1]

*Proof.* (1) is clear since the condition (3.5)(C) is inherited by factor rings.

(2) For any $b \in R$, fix an element $y \in R$ such that $Q_b(y)\,(1-b^2) = 0$. The hypothesis of (2) implies that $(1-b^2)\,Q_b(y) = 0$. Applying the "right version" of Theorem 3.5 now shows that $R$ is *right* $\sqrt{1}$-lifting. The rest of (2) is clear.

(3) Under the assumptions in (3), Theorem 3.5 shows that, for every $r \in R$, there exists $y \in R$ such that $(r - yr^2)^2 = 0$. Since $R$ is reduced, we have $r = yr^2 \in R\,r^2$ (for all $r \in R$), so $R$ is a strongly regular ring according to [13, p. 28]. ☐

Another important consequence of Theorem 3.5 is the next result (3.8), part (2) of which gives the regular rings as a major class of left and right $\sqrt{1}$-lifting rings. This is a possibly surprising result since the feasibility of lifting $\sqrt{1}$ modulo one-sided ideals in regular rings of general characteristic has remained unnoticed in more than 80 years of history of the subject, while the feasibility of lifting idempotents for the same class of rings has been well known for some 40 years since the publication of [21].

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[1] Strongly regular rings are sometimes called *abelian regular rings*; see [13, §3].

**Theorem 3.8.** (1) *For any ring $R$, let $b \in R$ be such that one of $1 \pm b$ is a regular element. Then the equation $Q_b(y)\,(1-b^2) = 0$ has a solution in $R$.*

(2) *Any regular ring $R$ is a left and right $\sqrt{1}$-lifting ring.*

*Proof.* By (D) $\Rightarrow$ (A) in Theorem 3.5 (and left-right symmetry), it suffices to prove (1). For this, first assume that $1-b$ is regular. Then there exists $y \in R$ such that $1-b = (1-b)\,(-y)\,(1-b)$. Looking back at the third line in the array of equations in the proof of (3.5) (where the two middle factors multiply to become $(1-b) + (1-b)\,y\,(1-b)$), we see that $Q_b(y)\,(1-b^2) = 0$. The case where $1+b$ is regular also follows, upon recalling the equation $Q_b(y) = Q_{-b}(-y)$ in Proposition 2.13(1). □

**Example 3.9.** By part (2) above, all semisimple rings are (left and right) $\sqrt{1}$-lifting, and so is $R = \mathrm{End}_K(V)$ where $V$ is any right vector space over a division ring $K$. In the latter example, in the case where $\dim_K(V)$ is infinite, we get, for instance, the following kind of information which does not seem to be known before in the literature:

- *For any $\varphi \in R = \mathrm{End}_K(V)$, $\varphi^2 - 1$ has finite rank iff $\varphi$ is the sum of a reflection*[2] *and an endomorphism of finite rank.*

This results simply by applying $\sqrt{1}$-lifting to the ideal $I$ consisting of all finite rank endomorphisms in $R$. Similar results can be obtained if we take $I$ to be, instead, the ideal of $R$ consisting of all endomorphisms of rank less than some fixed infinite cardinal. (See [19, Exercise 3.16].)

Next, we offer another family of examples of left and right $\sqrt{1}$-lifting rings that is based on the "characteristic 2" theme of (3.5)(E) and (3.7)(3). These rather revealing examples will also be highly significant for some results to be developed later in §5, so we'll present them in the form of a self-contained proposition, as follows.

**Proposition 3.10.** *Let $R = \mathbb{T}_2(S)$ be the ring of upper-triangular $2 \times 2$ matrices over a ring $S$ with $\mathrm{char}\,(S) = 2$. Then $R$ is left $\sqrt{1}$-lifting iff $S$ is a strongly regular ring. In this case, $R$ is also right $\sqrt{1}$-lifting.*

*Proof.* It suffices to prove the "iff" statement, since it will clearly give the last part of the proposition on the ground of left-right symmetry.

First assume $S$ is strongly regular. To show that $R$ is left $\sqrt{1}$-lifting, (3.5)(E) (for characteristic 2) "requires" that for any $X \in R$, we try to find some $Y \in R$ such that $(X - YX^2)^2 = 0$. Since $R$ has Jacobson radical $J = \left(\begin{smallmatrix} 0 & S \\ 0 & 0 \end{smallmatrix}\right)$ so $R/J \cong S \times S$ is strongly regular, we can find $Y \in R$ such that $X - YX^2 \in J$. Then $(X - YX^2)^2 \in J^2 = 0$, as desired.

Conversely, assume that $R$ is left $\sqrt{1}$-lifting. To prove that $S$ is strongly regular, we want to show that $a \in S\,a^2$ for every $a \in S$. For $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & a \end{smallmatrix}\right) \in R$, (3.5)(E) yields a matrix $Y = \left(\begin{smallmatrix} b & c \\ 0 & d \end{smallmatrix}\right)$ such that $(X - YX^2)^2 = 0$. A short computation shows that $X - YX^2$ has the form $\left(\begin{smallmatrix} 0 & s \\ 0 & t \end{smallmatrix}\right)$ where $s = 1 - ba - ca^2$ and $t = a - da^2$. Since this matrix has square zero, we see that

$$0 = s\,t = (1 - ba - ca^2)\,(a - da^2). \tag{3.11}$$

Expansion of the RHS of (3.11) gives the desired conclusion that $a \in S\,a^2$. □

Next, we'll give a few more results on left $\sqrt{1}$-lifting rings of characteristic 2. To do this, we need to use the notion of a Drazin inverse from [11]. Recall that an element $r \in R$ is said to have *Drazin inverse* $y$ if $yr = ry$, $y = yry$, and $r^n = r^{n+1}y$ for some $n \geq 1$. (Such an element $y$ is known to be unique if it exists.) The following useful classical fact on Drazin inverses is extracted from [3] and [11].

[2] By a "reflection" here, we simply mean a vector space automorphism of order $\leq 2$.

**Lemma 3.12.** *Let* $R$ *be a ring for which there exists an integer* $n \geq 1$ *such that, for all* $r \in R$, $r^n \in R\,r^{n+1}$. *Then every* $r \in R$ *has a Drazin inverse* $y$ (*so* $R$ *is a strongly* $\pi$*-regular ring*), *and we have* $(r - yr^2)^n = 0$, *and* $r^n \in r^{n+1}R$.

*Proof.* To begin with, note that $R$ has nilpotence index $\leq n$. By a theorem of Azumaya [3, Theorem 4] on such rings, the hypothesis of this lemma implies that every element $r \in R$ has a Drazin inverse $y$. (This conclusion can also be deduced from Dischinger's theorem in [10]. However, Azumaya's result is a bit more elementary.) By the Fitting-Drazin theory (as developed in [11]), $r - yr^2$ is a nilpotent element, so $(r - yr^2)^n = 0$. Since $ry = yr$, this equation shows also that $r^n \in r^{n+1}R$, as claimed. □

**Proposition 3.13.** *If* $\text{char}\,(R) = 2$ *and* $r^2 \in R\,r^3$ *for every* $r \in R$, *then* $R$ *is a left and right* $\sqrt{1}$*-lifting ring.* (*This conclusion need not hold, however, without the assumption that* $\text{char}\,(R) = 2$.)

*Proof.* Consider any $r \in R$. Applying Lemma 3.12 for $n = 2$, we have $(r - yr^2)^2 = 0$ for some $y \in R$ such that $ry = yr$. As $\text{char}\,(R) = 2$, (E) $\Rightarrow$ (A) in Theorem 3.5 shows that $R$ is left $\sqrt{1}$-lifting. Since $r^2 \in r^3R$ holds too (for every $r$), $R$ is also right $\sqrt{1}$-lifting. To see that the assumption $\text{char}\,(R) = 2$ is essential, let $S = \mathbb{Z}/4\mathbb{Z}$ and let $R$ be the commutative local ring $S[x]/(x^2)$ of characteristic 4, with maximal ideal $J = (2,\, x)$. Clearly, $j^2 = 0$ for all $j \in J$, so $r^2 \in R\,r^3$ holds for every $r \in R$. *We claim that* $R$ *is not* $\sqrt{1}$*-lifting.* Indeed, for the ideal $I = 2R$, the element $\overline{1 + x}$ is a square root of 1 in $R/I$, and it does not lift to a square root of 1 in $R$ since any element in $(1 + x) + 2R$ has square $1 + 2x \neq 1$. (A "more general" way to see that $R$ is not $\sqrt{1}$-lifting is to use our future result Theorem 5.3. Since $1 + x \in 1 + J$ has square $1 + 2x \neq 1$, Theorem 5.3 implies that $R$ cannot be a $\sqrt{1}$-lifting ring.) □

Note that, for rings of characteristic 2, the *converse* of Proposition 3.13 is also false in general. For instance, let $R$ be any regular ring of characteristic 2 that is not strongly $\pi$-regular. By Theorem 3.8(2), $R$ is left and right $\sqrt{1}$-lifting, but certainly we cannot have $r^2 \in R\,r^3$ for all $r \in R$. On the other hand, if $R$ happens to be a *commutative* ring, it turns out that Proposition 3.13 does have a true converse (for rings of characteristic 2), as follows.

**Proposition 3.14.** *A commutative ring* $R$ *with* $\text{char}\,(R) = 2$ *is* $\sqrt{1}$*-lifting iff* $r^2 \in R\,r^3$ *for every* $r \in R$ (*which amounts to* $r^2$ *being regular for every* $r \in R$). *In this case,* $R$ *is a strongly* $\pi$*-regular ring, and hence a strongly clean* (*exchange*) *ring.*

*Proof.* We need only prove the "only if" part, so assume that $R$ is $\sqrt{1}$-lifting. By the implication (A) $\Rightarrow$ (E) in Theorem 3.5, for any $r \in R$, there exists $y \in R$ such that $(r - yr^2)^2 = 0$. Since $ry = yr$ (by assumption), this implies that $r^2 \in R\,r^3$. By Lemma 3.12, $R$ is a strongly $\pi$-regular ring. The proof is completed by recalling Nicholson's result [23, Theorem 1] that any strongly $\pi$-regular ring is a strongly clean (exchange) ring. □

While we are not able to prove Conjecture 3.4 (and the conjecture may be false in general), we have obtained some partial positive results in addition to that obtained in Proposition 3.14. To present these, we first give the following basic structural information on left $\sqrt{1}$-lifting rings that results fairly quickly from Theorem 3.5.

**Theorem 3.15.** (1) *Any left* $\sqrt{1}$*-lifting ring* $R$ *has a direct product decomposition* $R = A \times B$ *where* $A$ *is a ring with* $2 \in \text{U}(A)$, *and* $B$ *is a ring with* $8 = 0 \in B$.

(2) *If* $R$ *is left* $\sqrt{1}$*-lifting and* $(R, +)$ *has no* 2*-torsion, then* $2 \in \text{U}(R)$.

(3) *If an indecomposable ring* $R$ *is left* $\sqrt{1}$*-lifting, then either* $2 \in \text{U}(R)$ *or* $8 = 0 \in R$.

*Proof.* To prove (1), assume $R$ is left $\sqrt{1}$-lifting. We first prove that $2 \in R$ is $\pi$-regular; in fact, we'll show that 8 *is regular in* $R$. According to Theorem 3.5, for any $b \in R$, there exists $y \in R$ such that $\big(b + y\,(1 - b^2)\big)^2 = 1$. Taking $b = 3$, we get $9 - 48\,y + 64\,y^2 = 1$, so $8 \in 16\,R$. This implies that $8 \in 8^2 R$; that is, 8 is a regular element in $R$. Since 8 is central, a standard argument from the proof of [13, (1.14)] shows that $8R = eR$ for some *central* idempotent $e \in R$. We have now $R = A \times B$ where $A = eR$ and $B = (1 - e)\,R$. Clearly, $A$ is a ring with $2 \in \mathrm{U}(A)$, and $B$ is a ring with $8 = 0 \in B$. This proves (1). If we assume that $R$ has no 2-torsion, then $B = 0$, so $2 \in \mathrm{U}(A) = \mathrm{U}(R)$. Finally, (3) also follows (trivially) from (1). □

As a small application of part (3) above (and Theorem 3.5), we can offer some easy (commutative) examples and non-examples of local $\sqrt{1}$-lifting rings.

**Proposition 3.16.** *Let* $R = \mathbb{Z}/2^n\mathbb{Z}$ *where* $n \geq 1$*. Then* $R$ *is* $\sqrt{1}$*-lifting iff* $n \leq 3$*.*

*Proof.* The "only if" part follows from (3.15)(3). (Of course, it also follows from (3.3)(C) and (3.7)(1).) For the "if" part, it suffices to show (again in view of (3.7)(1)) that $R = \mathbb{Z}/8\,\mathbb{Z}$ is $\sqrt{1}$-lifting. We do this by checking the criterion in (3.5)(C); that is, for every $b \in R$, there exists $y \in R$ such that $\big(b + y\,(1 - b^2)\big)^2 = 1$. Indeed, if $b$ is odd, we can choose $y = 0$, and if $b$ is even, we can choose $y = 1$. □

Using Theorem 3.15(1), we can now make the following preliminary reduction of Conjecture 3.4 to the case of characteristic 2. For a somewhat stronger reduction, see Theorem 5.5 in §5.

**Proposition 3.17.** *If Conjecture* 3.4 *is true for all rings of characteristic* 2*, then it is true for all rings.*

*Proof.* If $R$ is left $\sqrt{1}$-lifting, write $R = A \times B$ as in Theorem 3.15(1). By Corollary 3.7(1), $A$ and $B$ are both left $\sqrt{1}$-lifting. Since $2 \in \mathrm{U}(A)$, Theorem 3.2 shows that $A$ is an exchange ring. On the other hand, if Conjecture 3.4 is true for all rings of characteristic 2, then the left $\sqrt{1}$-lifting ring $B/2B$ is also an exchange ring. As $(2B)^3 = 0$, this implies that $B$ is an exchange ring, and hence so is $R = A \times B$. □

Since Conjecture 3.4 does hold for *commutative* rings of characteristic 2 by Proposition 3.14, the reasoning in the proof of Proposition 3.17 gives the following.

**Corollary 3.18.** *Conjecture* 3.4 *is true for commutative rings; that is, any commutative* $\sqrt{1}$*-lifting ring is an exchange ring.*

This result will be further improved in Corollary 5.9 in §5.

## 4. Change of Rings for $\sqrt{1}$-Lifting Rings

In this section, we present a few results and examples to illustrate the behavior of the class $\mathfrak{F}$ of left $\sqrt{1}$-lifting rings with respect to the change of rings. For convenience, we shall write informally $R \in \mathfrak{F}$ to mean that $R$ is a left $\sqrt{1}$-lifting ring.

To begin with, it is clear that the family $\mathfrak{F}$ is closed with respect to arbitrary direct products and direct limits. By Corollary 3.7(1), $\mathfrak{F}$ is also closed with respect to the passage to factor rings. The next natural change of rings to consider is the passage to corner rings. Here, we have likewise a positive result below, the proof of which is inspired by some corner ring calculations for unit-regular rings in [20, §2].

**Theorem 4.1.** *If* $R \in \mathfrak{F}$, *then* $S := eRe \in \mathfrak{F}$ *for every* $e \in \text{idem}(R)$.

*Proof.* Consider any $b \in S$. Following the proof idea of the main theorem of [20], let $f = 1 - e$ and $c = f + b$. By Theorem 3.5, there exists $z \in R$ such that $\big(c + z(1 - c^2)\big)^2 = 1$. Since $fb = bf = 0$, $1 - c^2 = 1 - (f + b^2) = e - b^2 = e(e - b^2)$. Therefore,

$$e\big(c + z(1 - c^2)\big) = e\big(f + b + z\,e(e - b^2)\big) = b + y(e - b^2),$$

where $y := eze \in S$. As the RHS (and hence the LHS) of this equation is unchanged by right multiplication by $e$, we have

$$\big(b + y(e - b^2)\big)^2 = e\big(c + z(1 - c^2)\big)\,e\big(c + z(1 - c^2)\big) = e\big(c + z(1 - c^2)\big)^2 = e.$$

From (C) $\Rightarrow$ (A) in Theorem 3.5 (applied to $S$), it follows that $S \in \mathfrak{F}$. □

**Remark 4.2.** (A) With Theorem 4.1 in hand, one may wonder, for two complementary idempotents $e, f \in R$, whether $eRe$, $fRf \in \mathfrak{F}$ would imply that $R \in \mathfrak{F}$. The answer to this question is "no". Indeed, by taking $S$ to be a regular but not strongly regular ring (with $\text{char}(S) = 2$) in Proposition 3.10, we'll get a triangular ring $R = \mathbb{T}_2(S) \notin \mathfrak{F}$ which has two corner rings $eRe$, $fRf$ isomorphic to $S \in \mathfrak{F}$ where $e + f = 1$. This example shows yet another way in which $\mathfrak{F}$ differs from the class of exchange rings; namely, if $J$ is a nil (or even nilpotent) ideal in a ring $R$ and $R/J \in \mathfrak{F}$, we may *not* have $R \in \mathfrak{F}$. (A somewhat easier example showing the same thing would be $R = \mathbb{Z}/16\,\mathbb{Z}$, with $J = 8\,\mathbb{Z}/16\,\mathbb{Z}$; see Proposition 3.16.)

(B) After proving the corner ring result in Theorem 4.1, one would naturally ask whether the ring class $\mathfrak{F}$ is closed with respect to Morita equivalence. Later in Example 5.13, we'll show that the answer to this question is "no".

The next natural question about left $\sqrt{1}$-lifting rings is whether $R \in \mathfrak{F}$ would imply that $Z(R) \in \mathfrak{F}$, where $Z(R)$ denotes the center of $R$. The somewhat surprising answer to this question in Proposition 4.3 below shows a striking dichotomy between left $\sqrt{1}$-lifting rings $R$ with $2 \in \text{U}(R)$ and those with $\text{char}(R) = 2$.

**Proposition 4.3.** (1) *The center of a ring* $R \in \mathfrak{F}$ *with* $2 \in \text{U}(R)$ *need not be in* $\mathfrak{F}$.

(2) *If* $R \in \mathfrak{F}$ *and* $\text{char}(R) = 2$, *then* $Z(R) \in \mathfrak{F}$.

*Proof.* (1) Burgess and Raphael [5] produce an example of a clean (hence exchange) ring whose center is not exchange. By Theorem 3.2, $R \in \mathfrak{F}$. But $Z(R) \notin \mathfrak{F}$. Indeed, if $Z(R) \in \mathfrak{F}$, then since $2 \in \text{U}(Z(R))$, $Z(R)$ would be an exchange ring, again by Theorem 3.2. This would be a contradiction. (We recall, however, that if $R$ is a *regular* ring, then so is $Z(R)$ by [13, Theorem 1.14]. In this case, we do have $Z(R) \in \mathfrak{F}$ by Theorem 3.8(2).)

(2) Assume that $R \in \mathfrak{F}$ and $\text{char}(R) = 2$. For every $r \in Z(R)$, (A) $\Rightarrow$ (E) in Theorem 3.5 shows that $(r - yr^2)^2 = 0$ for some $y \in R$. As $r$ is central, this implies that $r^2$ is regular in $R$. Invoking again the standard argument in the proof of [13, (1.14)], we see that $r^2$ is regular in $Z(R)$. By Proposition 3.14 (in characteristic 2), this gives $Z(R) \in \mathfrak{F}$. □

Yet another interesting change of rings is the passage to *localizations* in the category of commutative rings. An example constructed by Anderson and Camillo in [1] led us directly to the following result, which shows once more the dichotomy (in the study of $\mathfrak{F}$) between the case $2 \in \text{U}(R)$ and the case $\text{char}(R) = 2$.

**Proposition 4.4.** (1) *The localization* $R_S$ *of a commutative ring* $R \in \mathfrak{F}$ *with* $2 \in \mathrm{U}(R)$ *at a multiplicative set* $S \subseteq R$ *need not be in* $\mathfrak{F}$.

(2) *If* $R \in \mathfrak{F}$ *is a commutative ring with* $\mathrm{char}\,(R) = 2$, *then* $R_S \in \mathfrak{F}$ *for every multiplicative set* $S \subseteq R$.

(3) *Let* $R$ *be a commutative ring of Krull dimension zero. If* $2 \in \mathrm{U}(R)$, *then* $R \in \mathfrak{F}$. *If* $\mathrm{char}\,(R) = 2$, *then* $R \in \mathfrak{F}$ *iff* $R$ *has nilpotence index* $\leq 2$.

*Proof.* To show (1), we make use of the same principle that was exploited in the proof of Proposition 4.3(1). According to [1], if $R = k\,[[x, y]]$ (where $k$ is any field) and $S \subseteq R$ is the multiplicative set given by the complement of $(x) \cup (y)$, then $R$ is an exchange ring, but the localization $R_S$ is not. If we take $k$ to be a field with $\mathrm{char}\,(k) \neq 2$, then 2 is a unit in both $R$ and $R_S$. Thus, by Theorem 3.2, we have $R \in \mathfrak{F}$, but $R_S \notin \mathfrak{F}$.

(2) For $S \subseteq R \in \mathfrak{F}$ as in (2), consider any element $r/s \in R_S$, where $r \in R$ and $s \in S$. By Proposition 3.14, $r^2 = yr^3$ for some $y \in R$. Then $(r/s)^2 = (yr^3)/s^2 = ys\,(r/s)^3 \in R_S(r/s)^3$. Applying Proposition 3.14 one more time, we see that $R_S \in \mathfrak{F}$.

(3) Here $R$ is a $\pi$-regular ring by [19, Exercise 4.15]; in particular, $R$ is an exchange ring. If $2 \in \mathrm{U}(R)$, Theorem 3.2 shows that $R \in \mathfrak{F}$. If $\mathrm{char}\,(R) = 2$, the desired conclusion will follow as a special case of Corollary 5.11 in §5. □

## 5. Characterizing Certain Classes of Left $\sqrt{1}$-Lifting Rings

In this section, we'll prove a characterization result for a large class of left $\sqrt{1}$-lifting rings, including for instance all commutative rings, all reduced rings, all local rings, and all abelian $\pi$-regular rings. The main tool to be used for getting such a characterization is $\mathrm{rad}\,(R)$, the Jacobson radical of a ring $R$. It is especially important to work with $1 + \mathrm{rad}\,(R)$, which is a normal subgroup of $\mathrm{U}(R)$; we shall refer to it as the "group of 1-units of $R$." This terminology is fairly commonly used, for instance in the theory of local rings. Here, we use it for arbitrary rings. For reasons that will be apparent a little bit later, we'll need to have a good understanding of the condition that $1 + \mathrm{rad}\,(R)$ be an elementary (abelian) 2-group (i.e., a vector space over $\mathbb{F}_2$). Our first result in this direction does not depend on the notion of left or right $\sqrt{1}$-lifting.

**Proposition 5.1.** *For any ring* $R$, *let* $J = \mathrm{rad}\,(R)$.

(1) $1 + J$ *is an elementary* 2*-group iff it is a group of exponent* $\leq 2$, *iff* $r^2 = 2\,r$ *for any* $r \in J$. (*In the case* $\mathrm{char}\,(R) = 2$, *both conditions amount to* $r^2 = 0$ *for any* $r \in J$.)

(2) *If* $1 + J$ *is an elementary* 2*-group, then for any* $r, s \in J$, *we have* $rs = sr = -rs$, *and* $r^3 = 0$. *Also,* $2\,J^2 = 4\,J = 0$.

*Proof.* (1) The first "iff" statement is well known, and the second "iff" statement follows from the squaring formula $(1 - r)^2 = 1 + (r^2 - 2\,r)$.

(2) Assuming the conditions in (1), consider any $r, s \in J$. Since $r^2 = 2\,r$, replacing $r$ by $-r$ gives $r^2 = -2r$; hence $4r = 0$ (proving that $4J = 0$), and $r^3 = 2r^2 = -4r = 0$. Also, since the elementary 2-group $1 + J$ is commutative, we have $(1 + r)\,(1 + s) = (1 + s)\,(1 + r)$, so $rs = sr$. On the other hand,

$$2\,(r + s) = (r + s)^2 = r^2 + s^2 + rs + sr = 2r + 2s + rs + sr, \tag{5.2}$$

so $rs = -sr = -rs$. Therefore, $2\,rs = 0$, which shows that $2J^2 = 0$. □

By injecting the notion of (left) $\sqrt{1}$-lifting into Proposition 5.1, we obtain a crucial result in (5.3) below. Note that, according to Theorem 3.15(1), the structure of a left $\sqrt{1}$-lifting ring may differ from that of an exchange ring with $2$ a unit "largely" in the case where $8 = 0 \in R$. Therefore, the assumption that $2 \in \operatorname{rad}(R)$ in the following result should not be too surprising.

**Theorem 5.3.** *Let $R$ be a left $\sqrt{1}$-lifting ring with $2 \in J := \operatorname{rad}(R)$. Then $1 + J$ is an elementary 2-group, and the conclusions on $J$ in* (5.1)(2) *all hold. In particular, $J$ is a nil ideal of nilpotence index $\leq 3$.*

*Proof.* Consider any 1-unit $b \in 1 + J$. Since $R$ is a left $\sqrt{1}$-lifting ring, there exists (by Theorem 3.5) an element $y \in R$ such that

$$[1 - y\,(1 + b)]\,[1 + (1 - b)\,y]\,(1 - b^2) = 0. \tag{5.4}$$

We have $1 - b \in J$, so adding $2b \in J$ gives also $1 + b \in J$. Therefore, the two left factors on the LHS of (5.4) are *units.* This implies that $b^2 = 1$, which completes the proof. □

With the help of Theorem 5.3, we arrive at the following reduction of Conjecture 3.4 that we have mentioned in the Introduction.

**Theorem 5.5.** *If* Conjecture 3.4 *is true for all semiprimitive rings of characteristic* 2*, then it is true for all rings.*

*Proof.* In view of Proposition 3.17, it suffices to prove Conjecture 3.4 for a (left $\sqrt{1}$-lifting) ring $R$ with $\operatorname{char}(R) = 2$. Let $J = \operatorname{rad}(R)$. The ring $R/J$ is semiprimitive with $\operatorname{char}(R/J) = 2$, and is left $\sqrt{1}$-lifting by Corollary 3.7(1). Thus, $R/J$ is hypothetically an exchange ring. Since $2 = 0 \in J$, Theorem 5.3 shows that $J$ is a nil ideal. From this, it follows (from [21, (1.5)]) that $R$ is an exchange ring. □

By Theorem 3.15, any left $\sqrt{1}$-lifting ring $R$ can be written as a direct product $A \times B$ where $A$ is a ring with $2 \in \mathrm{U}(A)$ (so it is an exchange ring by Theorem 3.2), and $B$ is a ring with $8 = 0 \in B$. Therefore, as we have indicated before, the understanding of the new features of a left $\sqrt{1}$-lifting ring $R$ rests on an analysis of its direct factor $B$. Assuming that $R = B$, $2$ is a central nilpotent element in $R$, so $2 \in \operatorname{rad}(R)$. In the balance of this paper, we'll write $\operatorname{nil}(R)$ for the set of nilpotent elements in a ring $R$. Our next goal is to find a characterization for a large class of left $\sqrt{1}$-lifting rings $R$ with $2 \in \operatorname{rad}(R)$; namely, those with the property that $\operatorname{nil}(R) \subseteq \operatorname{rad}(R)$. It turns out that all such rings are *strongly $\pi$-regular* (and hence exchange), and the class of such rings is left-right symmetric. More precisely, the main result in this section is as follows.

**Theorem 5.6.** *For any ring $R$ with $2 \in J := \operatorname{rad}(R)$, the following three statements are equivalent:*

(1) *$R$ is left $\sqrt{1}$-lifting, and $\operatorname{nil}(R) \subseteq J$.*
(2) *$R/J$ is strongly regular, and the group of 1-units $1 + J$ is an elementary 2-group.*
(3) *$R/J$ is strongly regular, and $1 + J = \{\, r \in R : r^2 = 1 \,\}$.*

*If* (1) *holds, then $R$ is also right $\sqrt{1}$-lifting, and it is a strongly $\pi$-regular ring of Drazin index $\leq 3$ (in the sense of* [11]*).*

*Proof.* (1) $\Rightarrow$ (3). By Theorem 5.3, the first condition in (1) implies that $1 + J$ is an elementary 2-group and that $J$ is a nil ideal. The latter, along with $\operatorname{nil}(R) \subseteq J$, shows that the left $\sqrt{1}$-lifting ring $R/J$ is a reduced ring (of characteristic 2), so Corollary 3.7(3) guarantees that $R/J$ is a strongly regular ring. Finally, let $r \in R$ be such that $r^2 = 1$. As $\operatorname{char}(R/J) = 2$, we have $(\overline{r - 1}\,)^2 = \overline{0} \in R/J$. Therefore, $\overline{r - 1} = \overline{0}$ (since $R/J$ is a reduced ring). This shows that $r \in 1 + J$.

(3) $\Rightarrow$ (2) is a tautology.

(2) $\Rightarrow$ (1). Under (2), $R/J$ is a reduced ring, so $\text{nil}\,(R) \subseteq J$. Recalling 3.5(C), it only remains to show that, for any $b \in R$, there exists $y \in R$ such that $\big(b + y\,(1 - b^2)\big)^2 = 1$. Given the second part of (2), it suffices to find $y \in R$ such that $b + y\,(1 - b^2) \in 1 + J$; that is, $1 - b - y\,(1 + b)\,(1 - b) \in J$. Letting $r = 1 - b$, this amounts to $r - y\,(2 - r)\,r \in J$, or simply $r + yr^2 \in J$ (since $2 \in J$). As long as $\overline{r}$ is a strongly regular element in $R/J$, we can surely find such an element $y \in R$. This proves (1).

Finally, suppose (1) (and hence (2)) holds. The "right version" of (2) $\Rightarrow$ (1) then shows that $R$ is *right* $\sqrt{1}$-lifting. For any $a \in R$, $R/J$ being strongly regular implies that $a - a^4 r \in J$ for some $r \in R$. Since $J$ is nil with nilpotence index $\leq 3$ (by Theorem 5.3), we have $(a - a^4 r)^3 = 0$. Expansion of the LHS of this equation shows that $a^3 \in a^4 R$. This proves that $R$ is strongly $\pi$-regular with Drazin index $\leq 3$. (Note that "three" is the best possible upper bound for the Drazin index of $R$, in view of the example $R = \mathbb{Z}/8\,\mathbb{Z}$ in Proposition 3.16.) □

**Remark 5.7.** In [17, (4.8)], it was claimed that, if $J$ is a nil ideal in *any* ring $R$ such that $R/J$ is strongly regular, then $R$ is strongly $\pi$-regular. However, the proof of this assertion in [17, (4.8)] contains a gap, so the truth of that result still remains uncertain. The argument in the last paragraph of the proof above shows that the result is true if $J$ is assumed to be nil with a nilpotence index $n < \infty$, as was already noted by Weixing Chen in [7, (3.15)]. In fact, if $R/J$ in this case is strongly $\pi$-regular with Drazin index $k$, our proof would show that $R$ is strongly $\pi$-regular with Drazin index $\leq kn$. The non-quantitative version of this statement can also be deduced from the theorem of Fisher and Snider in [12, (2.1)].

**Remark 5.8.** Since regular rings are (left and right) $\sqrt{1}$-lifting by Theorem 3.8(2), one may wonder whether, in the setting of Theorem 5.6, $R/J$ being regular (instead of strongly regular) and $1 + J$ being an elementary 2-group would already imply that $R$ is left $\sqrt{1}$-lifting. However, such an implication turns out to be false in general. Take, for instance, an upper triangular ring $R = \mathbb{T}_2(S)$ (as in (3.10)) where $S$ is a regular ring of characteristic 2 that is *not* strongly regular. The computation that $J = \left(\begin{smallmatrix} 0 & S \\ 0 & 0 \end{smallmatrix}\right)$ in (3.10) is still valid here, so the fact that $J^2 = 0$ (along with $\text{char}\,(S) = 2$) implies that $1 + J$ is an elementary 2-group. Since $S$ is not strongly regular (by design), it follows from Proposition 3.10 that $R$ is *not* left (or right) $\sqrt{1}$-lifting.

We record here a nice consequence of Theorem 5.6 which improves upon (3.18).

**Corollary 5.9.** *Conjecture* 3.4 *is true for any ring* $R$ *with* $\text{nil}\,(R) \subseteq \text{rad}\,(R)$*. In more detail, if* $R$ *is a left* $\sqrt{1}$*-lifting ring with* $\text{nil}\,(R) \subseteq \text{rad}\,(R)$*, then* $R$ *is also right* $\sqrt{1}$*-lifting, and it is an exchange ring.*

*Proof.* By Theorem 3.15(1), we have $R = A \times B$ where $A$ is a ring with $2 \in \mathrm{U}(A)$, and $B$ is a ring with $8 = 0 \in B$. Since $A \cong R/B$ is left $\sqrt{1}$-lifting, it is a right $\sqrt{1}$-lifting exchange ring by Theorem 3.2. On the other hand, $B$ is also left $\sqrt{1}$-lifting, and we have $2 \in \text{rad}\,(B)$ as well as $\text{nil}\,(B) \subseteq \text{rad}\,(B)$. Thus, Theorem 5.6 implies that $B$ is a right $\sqrt{1}$-lifting strongly $\pi$-regular ring, and hence an exchange ring. It follows that $R = A \times B$ is a right $\sqrt{1}$-lifting exchange ring, as claimed. □

A ring $R$ is said to be 2-*primal* if the set $\text{nil}\,(R)$ is contained in (and hence equal to) the prime radical of $R$; see [18, p. 195]. For such a ring, the inclusion $\text{nil}\,(R) \subseteq \text{rad}\,(R)$ in Theorem 5.6(1) is automatic. Thus, Theorem 5.6 would apply verbatim after dropping the condition $\text{nil}\,(R) \subseteq \text{rad}\,(R)$ from (5.6)(1). Also, for some 2-primal rings $R$, $R/\text{rad}\,(R)$ may be automatically strongly regular. For such rings, (5.6) can be further simplified. The following are two such instances.

**Corollary 5.10.** *Let* $R$ *be a local ring or an abelian* $\pi$*-regular ring. If* $2 \in J := \text{rad}\,(R)$*, then* $R$ *is left* $\sqrt{1}$*-lifting iff* $1 + J$ *is an elementary* 2*-group. In this case,* $R$ *is a strongly* $\pi$*-regular ring that is also right* $\sqrt{1}$*-lifting.*

*Proof.* First assume $R$ is local. Here, clearly $\text{nil}\,(R) \subseteq J$, and $R/J$ (being a division ring) is strongly regular, so the simplified "iff" conclusion follows from (5.6). (Of course, the "strongly $\pi$-regular" conclusion is obvious for any local ring with a nil maximal ideal.) Next, assume $R$ is abelian $\pi$-regular. By Hirano's results in [14] (see also [24] and [4]), we have $\text{nil}\,(R) = J$, and $R/J$ is strongly regular, so the simplified "iff" conclusion again follows from (5.6). (Here, the "strongly $\pi$-regular" conclusion is easily seen to hold without assuming $2 \in J$.) □

Of course, any commutative ring $R$ is 2-primal too, although $R/\text{rad}\,(R)$ may not be (strongly) regular in general. Nevertheless, we have the following variation on the theme of Corollary 5.10.

**Corollary 5.11.** *Let $R$ be a commutative ring with $2 \in J := \text{rad}\,(R)$. Then $R$ is a $\sqrt{1}$-lifting ring iff $R$ has Krull dimension zero and $1 + J$ is an elementary 2-group.*

*Proof.* First assume $R$ is $\sqrt{1}$-lifting. By $(1) \Rightarrow (2)$ in Theorem 5.6, $R/J$ is (strongly) regular, and $1 + J$ is an elementary 2-group. The latter implies that $J$ is a nil ideal, so $R$ has Krull dimension equal to that of $R/J$, which is zero. Conversely, assume that $R$ has Krull dimension zero and $1 + J$ is an elementary 2-group. By [19, Exercise 4.15], the former implies that $R/J$ is (strongly) regular. Thus, $(2) \Rightarrow (1)$ in Theorem 5.6 shows that $R$ is a $\sqrt{1}$-lifting ring. □

We conclude this paper by giving two concrete examples.

**Example 5.12.** Let $k$ be any field with $\text{char}\,(k) = 2$, and let $R$ be the commutative local ring $k\,[x, y, z]/(x^2, y^2, z^2)$. Certainly, every element in $J = \text{rad}\,(R) = (\overline{x}, \overline{y}, \overline{z})$ has square zero. By Corollary 5.10, $R$ is a $\sqrt{1}$-lifting ring. However, $J^3 \neq 0$. This example shows that in the setting of Theorem 5.3, although we have $2\,J^2 = 4\,J = 0$, we cannot expect the stronger conclusion $J^3 = 0$. By using "many" variables, we can get similar local $\sqrt{1}$-lifting examples of characteristic 2 in which $J^n \neq 0$ for all $n$.

**Example 5.13.** The property that a ring $R$ is left $\sqrt{1}$-lifting may not go up to the matrix rings $\mathbb{M}_n(R)$, even for $n = 2$. In fact, for the commutative $\sqrt{1}$-lifting ring $R$ in (5.12) above, the matrix $A = \left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)$ is in the Jacobson radical of $\mathbb{M}_2(R)$. However, $A^2 = \left(\begin{smallmatrix} 0 & xy \\ 0 & 0 \end{smallmatrix}\right)$ is nonzero. This shows that $\mathbb{M}_2(R)$ is *not* left $\sqrt{1}$-lifting, for otherwise Theorem 5.3 would have implied that $A^2 = 2A = 0$ (since $\text{char}\,\big(\mathbb{M}_2(R)\big) = 2$). Note that the same construction here also shows that $\mathbb{T}_2(R)$, the ring of $2 \times 2$ upper-triangular matrices over $R$, is not left $\sqrt{1}$-lifting. (Of course, we could have deduced this already from Proposition 3.10 since $R$ is not a strongly regular ring.)

To show that the left $\sqrt{1}$-lifting property may not go up to $2 \times 2$ matrix rings, the referee has also suggested the similar commutative example $R = \mathbb{Z}/8\mathbb{Z}$ (which is $\sqrt{1}$-lifting). For the matrix $B = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{M}_2(R)$ and any $Y \in \mathbb{M}_2(R)$, a straightforward computation shows that

$$\big((B + Y\,(I - B^2)\big)^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \neq I \in \mathbb{M}_2(R).$$

Thus the condition (C) in Theorem 3.5 fails, so $\mathbb{M}_2(R)$ is not a left $\sqrt{1}$-lifting ring. The fact that the left $\sqrt{1}$-lifting property is not preserved by Morita equivalence is another aspect in which the class of left $\sqrt{1}$-lifting rings differs from the class of exchange rings.

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