KEY

Math 334 Test 2
Instructor: Scott Glasgow
Sections: 3
Dates: March 8-10.
Instructions: Use your own paper, be very neat, be painfully clear (use lots of English sentences), use large fonts and, so, lots of paper, and enjoy!

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1)
   a. Calculate $L[y_1]$ given $L[y] := t^2 y'' + 5ty' + 4y$ and $y_1(t) := t^{-2} (t \neq 0)$.
   5pts.
   
   b. Calculate $L[vy_1]$, $v = v(t)$ an arbitrary (differentiable, etc.) function.
   5pts.
   
   c. Find a nonzero solution $v$ to the differential equation $L[vy_1] = 0$ valid for $t > 0$.
   5pts.
   
   d. Given the above, write the general solution $y$ to $L[y] = 0$ valid for $t > 0$. (That is, do not use the Euler equation analysis/ansatz. But you should use that to check your answer.)
   5pts.
   
   e. Check your work, i.e. demonstrate that your expression solves the O.D.E. If it doesn’t, discover your error and correct.
   10pts.

Solution:

   a. $L[y_1] = t^2(y_1'') + 5t(y_1') + 4y_1 = t^2(-2y_1 + 3t^{-3}y_1') + 5t(-2y_1 + 3t^{-3}y_1'') + 4y_1 =$
   $6t^{-2} - 10t^{-2} + 4t^{-2} = 0$.
   
   b. $L[vy_1] = t^2(vy_1'') + 5t(vy_1') + 4vy_1 = t^2(v''y_1 + 2vy_1' + vy_1'') + 5t(v'y_1 + vy_1') + 4vy_1 =$
   $t^2(y_1'') + 5ty_1' + 4y_1)v + t^2y_1'v'' + 2t^2y_1'v' + 4ty_1v' =
   L[y_1]v + t^2t^{-2}v'' + (2t^2(-2t^{-3}) + 5tt^{-2})v' = 0 \cdot v + v'' + (-4t^{-1} + 5t^{-1})v' = v'' + t^{-1}v'$. 

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c. $0 = L[vy] = v^2 + t^{-1}v' \iff v'' = -t^{-1} \iff \log v' = -\log t \iff v' = t^{-1} \iff v = \log t$.

d. Thus, for $t > 0$, the general solution $y$ to $L[y] = 0$ may be expressed as $y = C_1 y_1 + C_2 y_1 v = C_1 t^{-2} + C_2 t^{-2} \log t$.

2)

a) Solve the initial value problem $L y = y'' - 2y' + y = te^t + 4, \ y(0) = y'(0) = 1$ by any (valid) means.

b) Check your work, i.e. demonstrate that your expression solves the I.V.P. If it doesn’t, discover your error and correct.

10pts.

Solution:

I will use the operator factorization and multiple integrating factors technique that I have advertised: write

$$L = D^2 - 2D + 1 = (D - 1)(D - 3) = e^D e^{-t} D e^{-t} = e^D D e^{-t}.$$ 

Thus

$$L y = e^D D e^{-t} y = te^t + 4 \iff D e^{-t} y = t + 4 e^{-t} \iff$$

$$D e^{-t} y = \frac{1}{2} t^2 - 4 e^{-t} + C_1 \iff e^{-t} y = \frac{1}{6} t^3 + 4 e^{-t} + C_1 t + C_2 \iff$$

$$y = \frac{1}{6} t^3 e^t + 4 + C_1 te^t + C_2 e^t \Rightarrow$$

$$y' = \left(\frac{1}{2} t^2 + \frac{1}{2} t^2\right) e^t + C_1 (t + 1) e^t + C_2 e^t.$$

The initial data specifies that

$$y(0) = 1 = 4 + C_2 \iff C_2 = -3$$

$$y'(0) = 1 = C_1 + C_2 = C_1 - 3 \iff C_1 = 4,$$

giving

$$y = \frac{1}{6} t^3 e^t + 4 + 4 t e^t - 3 e^t$$

$$= \left(\frac{1}{6} t^3 + 4t - 3\right) e^t + 4.$$
3) By any (valid) means, find the general solution to the O.D.E.

\[ Ly = y'' - 2y' + y = \frac{e'}{1 + t^2} \left( = e' \left( \tan^{-1}(t) \right) \right) = e' \left( t \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) \right), \]

in case you find these last representations useful—ignore otherwise). Note that undetermined coefficients will not at all work!

10pts.

b) Check your work, i.e. demonstrate that your expression solves the O.D.E. If it doesn’t, discover your error and correct.

10pts.

Solution:

a) Given the hint, the most efficient means would be to factor the operator as in problem 2):

\[ Ly = e' D^2 e^{-y} = e' \left( t \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) \right)'' \Leftrightarrow D^2 e^{-y} = \left( t \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) \right)'' \Leftrightarrow e^{-y} = t \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 \Leftrightarrow y = e' \left( t \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 \right). \]

b) We have

\[ y = e' \left( t \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 \right) \Rightarrow \]

\[ y' = e' \left( t \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 + \tan^{-1}(t) + \frac{t}{1 + t^2} - \frac{1}{2} \frac{2t}{1 + t^2} + C_1 \right) \]

\[ = e' \left( (t + 1) \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 + C_1 \right) \Rightarrow \]

\[ y'' = e' \left( (t + 1) \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 + C_1 + \tan^{-1}(t) + \frac{t + 1}{1 + t^2} - \frac{1}{2} \frac{2t}{1 + t^2} + C_1 \right) \]

\[ = e' \left( (t + 2) \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 + 2C_1 + \frac{1}{1 + t^2} \right), \]

so that
\[ Ly = y'' - 2y' + y = e^t \left( (t + 2) \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 + 2C_1 + \frac{1}{1 + t^2} \right) \\
-2 \left( (t + 1) \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 + C_1 \right) + t \tan^{-1}(t) - \frac{1}{2} \ln(1 + t^2) + C_1 t + C_2 \right) \\
= e^t \left( (t + 2 - 2t - 2 + t) \tan^{-1}(t) - \frac{1}{2} (1 - 2 + 1) \ln(1 + t^2) + (1 - 2 + 1)C_1 t + (1 - 2 + 1)C_2 + (2 - 2)C_1 + \frac{1}{1 + t^2} \right) \\
= e^t \frac{1}{1 + t^2}. \]

4) Find a non-trivial, polynomial solution to the O.D.E.
\[(x^4 - 3x^2 + 2)y'' - (12x^2 - 6)y = 0,\]
by seeking a solution in the form of a power series about \( x_0 = 0 \), and noting that the series can be made to truncate in a finite number of terms for special choices of the free parameters. Normalize your polynomial solution \( y \) so that \( y(0) = 1 \).

10pts.

b) Check your work, i.e. demonstrate that your polynomial solves the O.D.E. If it doesn’t, discover your error and correct.

10pts.

Solution:

Write \( y = \sum_n a_n x^n \), were \( a_{n<0} = 0 \), to get
\[
0 = \sum_n \left( x^4 - 3x^2 + 2 \right) a_n x^n - (12x^2 - 6)a_n x^n \\
= \sum_n \left( x^4 - 3x^2 + 2 \right) a_n n(n-1)x^{n-2} - 12a_n x^n + 6a_n x^n \\
= \sum_n a_{n-2}(n - 2)(n - 3)x^n - 3a_n n(n-1)x^n + 2a_{n+2}(n + 2)(n + 1)x^n - 12a_{n-2}x^n + 6a_n x^n \\
= \sum_n \left\{ a_{n-2}(n - 2)(n - 3) - 3a_n n(n-1) + 2a_{n+2}(n + 2)(n + 1) \right\} x^n \\
= \sum_n \left\{ (2n)(n+1)a_{n+2} - 3(n(n-1) - 2)a_n + (n(n-3) - 12)a_{n-2} \right\} x^n \\
= \sum_n \left\{ (2n)(n+1)a_{n+2} - 3(n^2 - n - 2)a_n + (n^2 - 5n - 6)a_{n-2} \right\} x^n \\
= \sum_n \left\{ (2n)(n+1)a_{n+2} - 3(n+1)(n-2)a_n + (n+1)(n-6)a_{n-2} \right\} x^n \\
= \sum_n \left\{ (n+1)(2n+2)a_{n+2} - 3(n-2)a_n + (n-6)a_{n-2} \right\} x^n, \]
uniformly in $x$ (over an interval containing $x = 0$). This (with other information 
given) implies that for each $n$

$$a_{n+2} = \frac{3(n-2)a_n - (n-6)a_{n-2}}{2(n+2)},$$

where again we recall that $a_{n<0} = 0$. Note that certain coefficients in the recurrence 
relation vanish at $n = 2$ and $n = 6$, and that the recurrence relation only relates even 
terms in the series to other even terms. So setting $a_0 = 1$ in order to get the 
normalization required, and setting $a_1 = 0$, we note that all odd terms vanish, and, 
ultimately, that there are only a finite number of even terms: starting with $n = 0$, we get

$$a_2 = \frac{3(-2)a_0 - (0-6)a_{-2}}{2(2+2)} = \frac{3(-2) \cdot 1 - (0-6) \cdot 0}{2(2+2)} = -\frac{3}{2},$$

$$a_4 = a_{2+2} = \frac{3(2-2)a_2 - (2-6)a_{2-2}}{2(2+2)} = \frac{4a_0}{2 \cdot 4} = \frac{1}{2},$$

$$a_6 = a_{4+2} = \frac{3(4-2)a_4 - (4-6)a_{4-2}}{2(4+2)} = \frac{3 \cdot 2a_4 + 2a_2}{2(4+2)} = 0,$$

$$a_8 = a_{6+2} = \frac{3(6-2)a_6 - (6-6)a_{6-2}}{2(6+2)} = \frac{3(6-2) \cdot 0 - 0 \cdot a_4}{2(6+2)} = 0,$$

so that all subsequent terms are zero. We have then that our polynomial solution is

$$y = \sum_{n=0}^{\infty} a_{2n} x^{2n} = a_0 + a_2 x^2 + a_4 x^4 = 1 - \frac{3}{2} x^2 + \frac{1}{2} x^4.$$

5)

a) Find a (closed-form) general solution of the O.D.E.

$$x^2 y'' + 2xy' - 6y = 0.$$  

5 pts.

b) Check your work, i.e. demonstrate that your expression solves the O.D.E. If it 
doesn’t, discover your error and correct.

5 pts.

**Solution:**

The equation is an Euler equation, for which the ansatz $y = x^r$ is productive:

$$0 = x^2 x'' + 2xx' - 6x$$

$$= (r(r-1) + 2r - 6)x^r \iff$$

$$0 = r^2 - r + 2r - 6 = r^2 + r - 6 = (r-2)(r+3)$$

$$\iff r = 2, -3,$$
which leads to the general solution
\[ y = C_1 x^2 + C_2 x^{-3}. \]

6) Find a general expression for the Wronskian of any two solutions to the O.D.E.
\[ 0 = t^2 y'' - (t^2 + 2t) y' + (t + 2) y. \]

5 pts.

**Solution:**

Write the equation as
\[ 0 = y'' - \left(1 + \frac{2}{t}\right)y' + \left(\frac{t + 2}{t^2}\right)y =: y'' + py' + qy, \]
for which Abel’s theorem then claims that
\[ W[y_1, y_2](t) = Ce^{-\int(pd)dt} = Ce^{\int\left(\frac{2}{t} - \frac{1}{t^2}\right)dt} = Ce^{t^2\ln t} = Ct^2 e^t, \]
for any two solutions \( y_1, y_2 \) of this equation.

7) a) Find the general solution of the O.D.E.
\[ y'' + y = \sec(t) \]
by any (valid) means. Note that undetermined coefficients will not work.

10pts.

b) Check your work, i.e. demonstrate that your expression solves the O.D.E. If it doesn’t, discover your error and correct.

10pts.

**Solution:**

For variation of parameters make the ansatz that
\[ y = u_1 \cos(t) + u_2 \sin(t), \]
and, additionally, that
\[ \cos(t)u_1' + \sin(t)u_2' = 0, \]
to get
\[ y' = -\sin(t)u_1 + \cos(t)u_2 \Rightarrow \]
\[ y'' = -\cos(t)u_1 - \sin(t)u_2 - \sin(t)u_1' + \cos(t)u_2' = -y - \sin(t)u_1' + \cos(t)u_2' \]
\[ \Leftrightarrow y'' + y = -\sin(t)u_1' + \cos(t)u_2' = \sec(t). \]

So we have
\[
\begin{bmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{bmatrix}
\begin{bmatrix}
u'_1 \\ u'_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
\sec(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
u'_1 \\ u'_2
\end{bmatrix}
= 
\begin{bmatrix}
\cos(t) & -\sin(t) \\
\sin(t) & \cos(t)
\end{bmatrix}
\begin{bmatrix}
0 \\
\sec(t)
\end{bmatrix}
= 
\begin{bmatrix}
-\sin(t) \\
\cos(t)
\end{bmatrix}
\]\n
and ultimately the general solution
\[
y = u_1 \cos(t) + u_2 \sin(t) = \\
(\ln|\cos t| + C_1)\cos(t) + (t + C_2)\sin(t)
\]
\[
= C_1 \cos(t) + C_2 \sin(t) + \cos(t) \ln|\cos t| + t \sin(t).
\]

8) a) Find the value of \( y_0 \) so that the following I.V.P. has its solution approaching zero as \( t \to \infty \):
\[
y'' - y' - 2y = 0; y(0) = y_0, y'(0) = 2.
\]
4 pts.

b) Check your work, i.e. demonstrate that your expression solves the I.V.P. found, and approaches zero as \( t \to \infty \). If it doesn’t, discover your error and correct.
4 pts.

Solution:
The characteristic equation is
\[0 = r^2 - r - 2 = (r - 2)(r + 1) \iff r = 2, -1.\]
Only a solution associated with \( r = -1 \) can approach zero for large times. Thus the desired solution is of the form
\[y = Ce^{-t} \Rightarrow \]
\[y' = -Ce^{-t}.\]
Using the initial data we get
\[y(0) = y_0 = C,\]
and then
\[y'(0) = 2 = -C = -y_0 \iff y_0 = -2.\]