KEY

Math 334 Midterm I
Winter 2012
section 002
Instructor: Scott Glasgow

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1. Solve the following initial value problem:

\[
\frac{dy}{dt} = 1 + t + y + ty, \quad y(2) = 0.
\]  

(1.1)

Also, what is the value of this solution at \( t = -4 \)? (I.e., what is \( y(-4) \)?)

10 points

**Solution**

The equation separates to

\[
\frac{dy}{1 + y} = (1 + t)\,dt,
\]

(1.2)

which, with the initial data in (1.1), yields the integral statement

\[
\int_0^y \frac{dy'}{1 + y'} = \int_2^t (1 + t')\,dt',
\]

(1.3)

which, after some work, gives

\[
\log(1 + y) = \log(1 + y) - \log(1 + 0) = \log(1 + y')\bigg|_0^y = \\
\int_0^y \frac{dy'}{1 + y'} = \int_2^t (1 + t')\,dt' \\
= t' + \frac{1}{2}t^2\bigg|_2^t = t + \frac{1}{2}t^2 - \left(2 + \frac{1}{2}2^2\right) = t + \frac{1}{2}t^2 - 4 \\
= \frac{1}{2}(t^2 + 2t - 8) = \frac{1}{2}(t - 2)(t + 4),
\]

(1.4)

i.e.,

\[
y(t) = \exp\left(\frac{1}{2}(t - 2)(t + 4)\right) - 1.
\]

(1.5)

Thus \( y(-4) = \exp\left(\frac{1}{2}(-4 - 2)(-4 + 4)\right) - 1 = 1 - 1 = 0 = y(2) \).
2. Prove that the following differential equation is exact and then find an expression for its general solution.

\[
\left(2xy^3 + 3x^2y^4 + y^6\right)dx + \left(3x^2y^2 + 4x^3y^3 + 5y^4 + 6xy^5\right)dy = 0.
\] (1.6)

14 points

Solution

The equation (1.6) is, by definition, exact if the left-hand side is the differential of a (continuously differentiable) function (of two variables \(x\) and \(y\), in some simply-connected region of the \(x\)-\(y\) plane, etc., etc.), i.e. if there is a function \(\psi(x, y)\) such that

\[
d\psi(x, y) = \left(2xy^3 + 3x^2y^4 + y^6\right)dx + \left(3x^2y^2 + 4x^3y^3 + 5y^4 + 6xy^5\right)dy.
\] (1.7)

But we have, by definition,

\[
d\psi(x, y) = \psi_x(x, y)dx + \psi_y(x, y)dy,
\] (1.8)

so that the equation (1.7) is the (potentially) over-determined system of PDE’s

\[
\psi_x(x, y) = 2xy^3 + 3x^2y^4 + y^6, \quad \text{and} \quad \psi_y(x, y) = 3x^2y^2 + 4x^3y^3 + 5y^4 + 6xy^5.
\] (1.9)

This over-determined pair of equations is consistent (or integrable) iff \(\left(\psi_x\right)_y = \left(\psi_y\right)_x\), i.e. iff

\[
\left(2xy^3 + 3x^2y^4 + y^6\right)_y = \left(3x^2y^2 + 4x^3y^3 + 5y^4 + 6xy^5\right)_x.
\] (1.10)

(1.10) holds true, so that the equation (1.6) is indeed exact, because either side of (1.10) is \(6xy^2 + 12x^2y^3 + 6y^5\).

As for developing the function \(\psi(x, y)\), and then (an expression for) the general solution of (1.6), one notes that the equations (1.9) demand, respectively, that

\[
\psi(x, y) = \int \left(2xy^3 + 3x^2y^4 + y^6\right)dx = x^2y^3 + x^3y^4 + xy^6 + f(y),
\]

and

\[
\psi(x, y) = \int \left(3x^2y^2 + 4x^3y^3 + 5y^4 + 6xy^5\right)dy = x^2y^3 + x^3y^4 + y^5 + xy^6 + g(x),
\] (1.11)

for some initially rather arbitrary functions \(f(y)\) and \(g(x)\). The two statements (1.11) are compatible iff

\[
x^2y^3 + x^3y^4 + xy^6 + f(y) = x^2y^3 + x^3y^4 + y^5 + xy^6 + g(x) \Leftrightarrow f(y) - y^5 = g(x) - 0,\]
implies both sides of the last equation are independent of both \( x \) and \( y \). As far as finding the general solution of (1.6) is concerned, without loss of generality we can choose \( f(y) - y^5 = g(x) - 0 = 0 \) so that (1.11) becomes (“in either case”)

\[
\psi(x, y) = x^2 y^3 + x^3 y^4 + xy^6 + y^5 = y^3 (1 + xy) \left( x^2 + y^2 \right).
\] (1.12)

(1.12) is NOT the general solution to the (exact) differential equation (1.6). It is not even a specific solution. Rather (1.12) defines a “potential (function) for the solution.” Using it one notes that (1.6) can be written as

\[
d\psi(x, y) = d \left( y^3 (1 + xy) \left( x^2 + y^2 \right) \right) = 0,
\] (1.13)

the general solution to which is clearly

\[
y^3 (1 + xy) \left( x^2 + y^2 \right) = C.
\] (1.14)

3. By using one of the estimates from Picard’s proof of the Fundamental Theorem of First Order ODE’s, show that there exists a solution \( y = \phi(t) \) to the IVP

\[
\frac{dy}{dt} = \frac{3 t^2}{2} \left( 1 + y^2 \right), \quad y(0) = 0,
\] (1.15)

at least throughout the interval

\[
t \in [-h, h] = [-1, 1].
\] (1.16)

Hint: If you do not remember the estimate, do the following to jog your memory. Instead of (1.15), and following Picard, write (for some \( h > 0 \) to be determined)

\[
\phi(t) = \frac{3}{2} \int_0^t s^2 \left( 1 + \phi^2(s) \right) ds, \quad |t| \leq h,
\] (1.17)

and, so, deduce that

---

1 Note that the solution of (1.15) actually persists throughout the interval \( t \in \left( -\frac{\pi}{1.3}, \frac{\pi}{1.3} \right) = (-1.46, 1.46) \), since the solution has the formula \( \tan \left( \frac{t^3}{2} \right) \), and since the domain of (the relevant instance of) the tangent function is \( t \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \).
\[ |t| \leq h \Rightarrow |\phi(t)| \leq \frac{3h}{2} \int_0^h s^2 \left( 1 + \phi^2(s) \right) ds \leq \frac{3h^3}{2} s^2 \left( 1 + \phi^2(s) \right) ds . \quad (1.18) \]

Now demand that \( h \) is the biggest number that is still small enough so that, for some \( Y > 0 \), imposing \( |\phi(s)| \leq Y \) (for each \( s \in [-h, h] \)) on the right hand side of (1.18) certainly ensures that \( |\phi(t)| \leq Y \) (for each \( t \in [-h, h] \)) on the left of (1.18).

By doing this you will get an \( h \) that depends on \( Y \), i.e. you will get an \( h(Y) \). Now find

\[ h := \max_{Y>0} h(Y) . \quad (1.19) \]

This should be the number indicated in (1.16), i.e., the result of (1.19) should be the number 1.

10 points

Solution

Demanding \( |\phi(s)| \leq Y \) (for each \( s \in [-h, h] \)) on the right hand side of (1.18) gives there that

\[ |t| \leq h \Rightarrow |\phi(t)| \leq \frac{3h}{2} \int_0^h s^2 \left( 1 + \phi^2(s) \right) ds \leq \frac{3h^3}{2} s^2 \left( 1 + Y^2 \right) ds = \frac{h^3}{2} \left( 1 + Y^2 \right) . \quad (1.20) \]

So now we certainly get \( |\phi(t)| \leq Y \) (for each \( t \in [-h, h] \)) provided we choose \( h \) small enough so that \( \frac{h^3}{2} \left( 1 + Y^2 \right) \leq Y \), the largest such \( h \) accomplishing this being

\[ h(Y) = \left( \frac{2Y}{1+Y^2} \right)^{1/3} . \quad (1.21) \]

So then we get the \( h \) indicated in (1.16) by noting that

\[ h := \max_{Y>0} h(Y) = \max_{Y>0} \left( \frac{2Y}{1+Y^2} \right)^{1/3} = \left( \frac{2 \cdot 1}{1+1^2} \right)^{1/3} = \left( \frac{2}{2} \right)^{1/3} = 1 . \quad (1.22) \]

We could get the maximum indicated in (1.22) by using the relevant tools from calculus.
4. Suppose we have 2 continuous solutions $\phi(t)$ and $\psi(t)$, $t \in [-h,h] = [-1,1]$, to the integral equation indicated in (1.17), i.e. suppose that both

$$\phi(t) = \frac{3}{2} \int_0^t s^2 \left(1 + \phi^2(s)\right) ds, \text{ and}$$

$$\psi(t) = \frac{3}{2} \int_0^t s^2 \left(1 + \psi^2(s)\right) ds$$

for each $t \in [-1,1]$. Then we could note that the difference $\phi(t) - \psi(t)$ of these two solutions obeys

$$\phi(t) - \psi(t) = \frac{3}{2} \int_0^t \left[s^2 \left(1 + \phi^2(s)\right) - s^2 \left(1 + \psi^2(s)\right)\right] ds = \frac{3}{2} \int_0^t s^2 \left(\phi^2(s) - \psi^2(s)\right) ds$$

and

$$= \frac{3}{2} \int_0^t s^2 (\phi(s) + \psi(s)) (\phi(s) - \psi(s)) ds,$$

(1.24)

and, for $t \in [-1,1]$, we could get the estimate

$$\left|\phi(t) - \psi(t)\right| = \left|\frac{3}{2} \int_0^t s^2 (\phi(s) + \psi(s)) (\phi(s) - \psi(s)) ds\right|$$

$$= \frac{3}{2} \int_0^t s^2 |\phi(s) + \psi(s)| |\phi(s) - \psi(s)| ds$$

$$\leq \frac{3}{2} \int_0^t \max_{[0,1]} |\phi(\tau) + \psi(\tau)| |\phi(s) - \psi(s)| ds$$

$$= \frac{3}{2} \int_0^t K |\phi(s) - \psi(s)| ds = \frac{3}{2} K \int_0^t |\phi(s) - \psi(s)| ds$$

(1.25)

where evidently

$$0 \leq K := \max_{[0,1]} \tau^2 |\phi(\tau) + \psi(\tau)| < \infty,$$

(1.26)

the last inequality in (1.26) holding because we have a continuous function on the bounded interval $\tau \in [0,1]$. So now define the new function
\[ U(t) := \int_{0}^{t} |\phi(s) - \psi(s)| \, ds \]  

(1.27)

for each \( t \in [-1,1] \), and note that

\[
\begin{align*}
U(t) &\geq 0, t \in [0,1], \\
U(0) &= 0,
\end{align*}
\]

(1.28)

and also then note that (1.25) can then be written as

\[
U'(t) = |\phi(t) - \psi(t)| \leq \frac{3}{2} K \int_{0}^{t} |\phi(s) - \psi(s)| \, ds = \frac{3}{2} K U(t)
\]

(1.29)

for \( t \in [0,1] \), i.e., we get

\[
U'(t) \leq \frac{3}{2} K U(t), t \in [0,1].
\]

(1.30)

Use (1.28) and (1.30) to show that

\[
U(t) = 0,
\]

(1.31)

for \( t \in [0,1] \) and, so, deduce that \( \phi(t) = \psi(t), t \in [0,1] \), i.e., deduce that there is at most one continuous solution to the integral equation (1.17) for \( t \in [0,1] \). (You could also show \( U(t) = 0 \) for \( t \in [-1,0] \) by a related but different argument.)

15 points

Solution

From (1.30) we have that, for each \( t \in [0,1] \),

\[
U'(t) - \frac{3}{2} K U(t) \leq 0,
\]

(1.32)

and then that

\[
e^{-3/2Kt} U'(t) - \frac{3}{2} e^{-3/2Kt} K U(t) \leq 0
\]

(1.33)
for \( t \in [0,1] \). But then since
\[
e^{-3/2Kt}U''(t) - \frac{3}{2} e^{-3/2Kt} Ku(t) = \frac{d}{dt} \left( e^{-3/2Kt} U(t) \right),
\]
(1.34)

(1.33) is
\[
\frac{d}{dt} \left( e^{-3/2Kt} U(t) \right) \leq 0,
\]
(1.35)
again for \( t \in [0,1] \). Together with \( U(0) = 0 \) (see (1.28)), (1.35) gives, for each \( t \in [0,1] \),
\[
e^{-3/2Kt} U(t) = e^{-3/2Kt} U(t) - 0 = e^{-3/2Kt} U(t) - e^{-3/2Ks} U(0) = e^{-3/2Ks} U(s) \right|_{s=0} = \]
\[
\int_0^t \frac{d}{ds} \left( e^{-3/2Ks} U(s) \right) ds \leq \int_0^t 0 ds
\]
\[
= 0,
\]
i.e. (1.35) gives
\[
e^{-3/2Kt} U(t) \leq 0,
\]
(1.37)
or, equivalently,
\[
U(t) \leq 0
\]
for each \( t \in [0,1] \). Between (1.38) and \( U(t) \geq 0 \) (from (1.28)), we must have \( U(t) = 0 \). But then, using (1.27), we have
\[
0 = \frac{d}{dt} U(t) = \frac{d}{dt} \int_0^t \left[ \phi(s) - \psi(s) \right] ds = \left[ \phi(t) - \psi(t) \right]
\]
(1.39)
for \( t \in [0,1] \), which then gives \( \phi(t) = \psi(t) \) for \( t \in [0,1] \).

5. After a time \( t = 0 \), a solution of constant concentration of 1 gram solute per liter solvent enters a (perfect) stirring tank at a constant rate of 6 liters per minute. The well-stirred mixture exits the tank at a constant rate of 4 liters per minute. Suppose the solute takes no volume in solution. If the tank contains 10 liters of fluid at a time \( t = 0 \), write down a (self-contained) differential equation for the time evolution of the grams of solute \( Q(t) \) accumulated in the tank at time \( t \), one that is valid for as long as the tank is not overflowing. Then, assuming there are
10 grams of solute in the tank at \( t = 0 \), give an expression for the grams \( Q(t) \) of solute accumulated in the tank at time \( t \) (by solving the relevant IVP).

**15 points**

**Solution**

By stoichiometric / “unit-canceling” / “chain-rule” reasoning, one has

\[
\frac{dQ}{dt} = \left( \frac{dQ}{dt} \right)_{\text{total}} - \left( \frac{dQ}{dt} \right)_{\text{out}} = \left( \frac{dQ}{dt} \right)_{\text{in}} - \left( \frac{dQ}{dt} \right)_{\text{out}} = \left( \frac{dV}{dt} \right)_{\text{in}} - \left( \frac{dV}{dt} \right)_{\text{out}}
\]

\[
= C_{\text{in}} R_{\text{in}} - C_{\text{out}} R_{\text{out}},
\]

\[
= R_{\text{in}} C_{\text{in}} - R_{\text{out}} \frac{Q}{V} = 6 \cdot 1 - 4 \frac{Q}{V} = 6 - 4 \frac{Q}{V},
\]

where the fluid tank volume \( V = V(t) \) is specified by

\[
\frac{dV}{dt} = R_{\text{in}} - R_{\text{out}} = 6 - 4 = 2, \ V(0) = V_0 = 10,
\]

the latter (trivial) initial value problem having the unique solution

\[
V = V_0 + t(R_{\text{in}} - R_{\text{out}}) = 10 + t \cdot 2 = 10 + 2t.
\]

Thus the required, “self-contained” differential equation is

\[
\frac{dQ}{dt} = R_{\text{in}} C_{\text{in}} - R_{\text{out}} \frac{Q}{V_0 + t(R_{\text{in}} - R_{\text{out}})}
\]

\[
= 6 - \frac{4}{10 + 2t} Q
\]

\[
\iff
\]

\[
\frac{dQ}{dt} + \frac{2}{5 + t} Q = 6.
\]

We solve the initial value problem which is ODE (1.43) together with initial data

\[
Q(0) = 10.
\]

An integrating factor for the ODE (1.43) is, according to the standard theory,
\[ \mu = \exp \int \frac{2}{5+t} dt = \exp \left( 2 \log (5+t) \right) = (5+t)^2. \] (1.45)

Use of the integrating factor (1.45) in (1.43) gives
\[
\frac{d \left( (5+t)^2 Q \right)}{dt} = (5+t)^2 \frac{dQ}{dt} + (5+t)^2 \frac{2}{5+t} Q = 6(5+t)^2 = \frac{d}{dt} \left( 2(5+t)^3 \right). \] (1.46)

Integration of (1.46) using relevant limits (and dummy variables) gives
\[
(5+t)^2 Q(t) - 2 \cdot 5^3 = (5+t)^2 Q(t) - 5^2 \cdot 10 = (5+t)^2 Q(t) - (5+0)^2 Q(0) = (5+s)^2 Q(s) \bigg|_0^t = 
\int_0^t d \left( (5+s)^2 Q(s) \right) = \int_0^t d \left( 2(5+s)^3 \right) = 2(5+s)^3 \bigg|_0^t = 2(5+t)^3 - 2(5+0)^3 = 2(5+t)^3 - 2 \cdot 5^3,
\] (1.47)
or, equivalently,
\[ Q(t) = 2(5+t) = 10 + 2t. \] (1.48)

6. Show that the following differential equation, equation (1.49), is not exact, but can be rendered exact by multiplication by an integrating factor that is only a function of $x$ or only a function of $y$. Find an expression of the general solution of the differential equation.
\[ (2xy + 3x^2y^3 + y^4)dx + (3x^2 + 4x^3y + 5y^2 + 6xy^3)dy = 0. \] (1.49)

15 points

**Solution**

(1.49) is not exact since
\[
\psi_{xy} \neq (\psi_x)_y = (2xy + 3x^2y^3 + y^4)_y = 2x + 6x^2y + 4y^3
\neq 6x + 12x^2y + 6y^3 = (3x^2 + 4x^3y + 5y^2 + 6xy^3)_x = (\psi_y)_x = \psi_{yx}. \] (1.50)
By theorem we know that, with an integrating factor \( \mu \), (1.49) can be made exact. We note from (1.50) that

\[
\left( 2xy + 3x^2y^2 + y^4 \right)_y - \left( 3x^2 + 4x^3y + 5y^2 + 6xy^3 \right)_x = 2x + 6x^2y + 4y^3 - \left( 6x + 12x^2y + 6y^3 \right)
\]

\[
= -4x - 6x^2y - 2y^3
\]

\[
= -2 \left( 2x + 3x^2y + y^3 \right)
\]

which divides the first term in (1.49), (which is \( y \left( 2x + 3x^2y + y^3 \right) \)), the remaining factor \((-y/2\)) being only a function of \( y \). Thus we suspect the existence of an integrating factor only depending on \( y \). At any rate, with the use of such a factor, ODE (1.49) becomes

\[
\left( 2xy + 3x^2y^2 + y^4 \right) \mu(y)dx + \left( 3x^2 + 4x^3y + 5y^2 + 6xy^3 \right) \mu(y)dy = 0, \quad (1.52)
\]

and exactness demands that

\[
0 = \left( \left( 2xy + 3x^2y^2 + y^4 \right) \mu(y) \right)_y - \left( \left( 3x^2 + 4x^3y + 5y^2 + 6xy^3 \right) \mu(y) \right)_x
\]

\[
= \left( 2x + 6x^2y + 4y^3 \right) \mu(y) + \left( 2xy + 3x^2y^2 + y^4 \right) \mu'(y) - \left( 6x + 12x^2y + 6y^3 \right) \mu(y)
\]

\[
= -2 \left( 2x + 3x^2y + y^3 \right) \mu(y) + y \left( 2x + 3x^2y + y^3 \right) \mu'(y)
\]

\[
= \left( 2x + 3x^2y + y^3 \right) \left( -2\mu(y) + y\mu'(y) \right)
\]

\[
\Rightarrow
\]

\[
y\mu'(y) = 2\mu(y).
\]

Thus, as suspected, there is an integrating factor depending only on \( y \). A solution of the last differential equation in (1.53) is given by

\[
\mu(y) = y^2, \quad (1.54)
\]

in which case (1.52) becomes

\[
0 = \left( 2xy + 3x^2y^2 + y^4 \right) y^2dx + \left( 3x^2 + 4x^3y + 5y^2 + 6xy^3 \right) y^2dy
\]

\[
= \left( 2xy^3 + 3x^2y^4 + y^6 \right) dx + \left( 3x^2y^2 + 4x^3y^3 + 5y^4 + 6xy^5 \right) dy, \quad (1.55)
\]

which is the exact equation considered in problem 2. Thus the solution is the same as in problem 2, namely
\[ x^3 y^3 + x^3 y^4 + xy^6 + y^5 = y^3 (1 + xy)(x^2 + y^2) = C. \] (1.56)

7. Find a linear, first order, ordinary differential equation with the property that every solution \( y(t) \) of it approaches the function \( f(t) = 1 + t^2 \) arbitrarily closely as \( t \to +\infty \). Note that the (too) simple equation

\[ y'(t) = f'(t) = (1 + t^2)' = 2t \] (1.57)

does not work since the general solution of (1.57) is

\[ y(t) = \int 2tdt = C + t^2, \] (1.58)

giving

\[ \lim_{t \to +\infty} \left( y(t) - f(t) \right) = \lim_{t \to +\infty} \left( C + t^2 - \left(1 + t^2 \right) \right) = C - 1, \] (1.59)

which is not zero for every possible choice of \( C \).

10 points

Solution

Introduce a general solution of the form

\[ y(t) = 1 + t^2 + Ce^{-at} \] (1.60)

with \( a > 0 \) to get

\[ \lim_{t \to +\infty} \left( y(t) - f(t) \right) = \lim_{t \to +\infty} \left( 1 + t^2 + Ce^{-at} - \left(1 + t^2 \right) \right) = \lim_{t \to +\infty} Ce^{-at} = 0 \] (1.61)

for every choice of \( C \), as demanded by the problem. Thus, to get a first order ODE with the required property we differentiate (1.60) with respect to \( t \) and eliminate \( C \) between (1.60) and this new result. Differentiating (1.60) gives

\[ y'(t) = 2t - aCe^{-at}, \] (1.62)

and elimination of \( C \) between (1.60) and (1.62) gives the required first order ODE, namely
\[ y'(t) = 2t - aCe^{-at} = 2t - a\left( y(t) - (1 + t) \right) \]
\[ = -ay(t) + a + 2t + at^2. \] (1.63)

8. Solve the following initial value problem. State the properties of the solution as $t \to +\infty$ for all possible choices of the initial value $y_0$.

\[ y'(t) = -y(t) + (1+t)^2, \ y(0) = y_0. \] (1.64)

15 points

Solution

The ODE in (1.64) can be written as

\[ y'(t) + 1y(t) = (1+t)^2. \] (1.65)

(1.65) suggests the integrating factor

\[ \mu(t) = \exp \int 1 \, dt = e^t, \] (1.66)

which renders the ODE (1.65) as

\[
\frac{d}{dt} e^t y(t) = 
\]
\[ e^t y'(t) + e^t y(t) = (1+t)^2 e^t = \frac{d}{dt} \left( (1+t)^2 e^t \right) - e^t \frac{d}{dt} (1+t)^2 
\]
\[ = \frac{d}{dt} \left( (1+t)^2 e^t \right) - 2e^t (1+t) 
\]
\[ = \frac{d}{dt} \left( (1+t)^2 e^t - 2e^t (1+t) \right) + 2e^t \frac{d}{dt} (1+t) 
\]
\[ = \frac{d}{dt} \left( (1+t)^2 e^t - 2e^t (1+t) + 2e^t \right) 
\]
\[ = \frac{d}{dt} \left( (1+t^2) e^t \right) 
\] (1.67)

which, with the initial data specified in (1.64), integrates to
\[ e' y(t) - y_0 = e' y(t) - 1 \cdot y_0 = e' y(t) - e^0 y(0) = e^s y(s) \bigg|_0^t = \]
\[ \int_0^t e' y(s) = \int_0^t d \left( 1 + s^2 \right) e^s \]
\[ = \left( 1 + t^2 \right) e^t = \left( 1 + t^2 \right) e^t - \left( 1 + 0^2 \right) e^0 = \left( 1 + t^2 \right) e^t - 1, \]
or, equivalently,
\[ y(t) = 1 + t^2 + e^{-t} \left( y_0 - 1 \right). \] 

Note then the differential equation in (1.64) gives a solution to problem 7, which indicates the desired properties.

9. Carefully state the (nonlinear) existence and uniqueness theorem for a single first order ODE.

15 points

Solution

Consider the initial value problem
\[ y'(t) = f(t, y), \quad y(t_0) = y_0. \] 

Suppose \( f(t, y) \) and \( f_y(t, y) \) are both continuous in an open rectangle \((t_{-1}, t_{+1}) \times (y_{-1}, y_{+1})\) containing the point \((t_0, y_0)\). Then there exists an \( h > 0 \) such that (1.70) has a unique, continuously differentiable solution \( y = \phi(t) \) persisting over the \( t \) interval \((t_0 - h, t_0 + h)\) (potentially much smaller than the interval \((t_{-1}, t_{+1})\)).

10. Carefully state the linear existence and uniqueness theorem for a single first order ODE. Explain in general terms how it is proven.

15 points

Solution

Consider the initial value problem
\[ y'(t) = p(t)y + q(t), \quad y(t_0) = y_0. \]
Suppose $p(t)$ and $q(t)$ are both continuous in an open interval $(t_{-1}, t_{+1})$ containing the point $t_0$. Then (1.71) has a unique, continuously differentiable solution $y = \phi(t)$ persisting over the $t$ interval $(t_{-1}, t_{+1})$.

The theorem is proven by explicitly integrating (1.71), using an integrating factor, the various theorems of calculus, including that the integral of a continuous function exists, etc.