Lecture 1
Recurrence

December 12, 2007

Definition 0.1 A discrete-time dynamical system consists of a set \( X \) and a map \( f : X \to X \).

For \( n \in \mathbb{N} \) the map \( f^n = f \circ f \circ \ldots \circ f \) where the number of compositions is \( n \). If \( f^{-1} \) exists we can define \( f^n(x) \) for \( n \in \mathbb{Z} \) in a similar way.

Definition 0.2 A continuous-time dynamical system consists of \( X \) and \( f^t : X \to X \) for \( t \in \mathbb{R} \).

In general a dynamical system consists of a group \( G \) space \( X \) and map \( \Phi : G \times X \to X \) such that

1. \( \Phi(e, x) = x \) where \( e \) is the identity element of \( G \).
2. \( \Phi(g_2, \Phi(g_1, x)) = \Phi(g_2g_1, x) \) for all \( g_1, g_2 \in G \) and \( x \in X \).

So in general \( \Phi(g, \cdot) \) is a map from \( X \) to itself. Depending on the situation the map may be assumed to be continuous, endomorphism, homeomorphism, or smooth.

We will be primarily interested in discrete dynamical systems this semester. However, many statements work for continuous-time systems and more general dynamical systems.

The positive (negative) orbit of \( x \) is \( O_f^+(x) = \bigcup_{n \geq 0} f^n(x) \) (\( O_f^-(x) = \bigcup_{n \leq 0} f^n(x) \)). The orbit of \( x \) is \( O_f(x) = O_f^+(x) \cup O_f^-(x) \). A point \( x \) is periodic if \( f^n(x) = x \) for some \( n \in \mathbb{N} \). We define the period of \( x \) to be the smallest such \( n \). If the period of \( x \) is 1 we say \( x \) is a fixed point.
We denote the periodic points of period \( n \) as \( \text{Per}_n(f) \). The fixed points are denoted \( \text{Fix}(f) \).

For \( A \subset X \) we denote \( f^n(A) = \{ y \in X \mid y = f^n(x) \text{ for some } x \in A \} \) where \( n \in \mathbb{N} \). The pre-image for \( A \) under \( f^n \) is \( f^{-n}(A) = \{ x \in X \mid f^n(x) \in A \} \).

For \( f : X \to X \) and \( g : Y \to Y \) a semiconjugacy from \( (Y,g) \) to \( (X,f) \) is a surjective map \( \pi : Y \to X \) such that \( f \circ \pi = \pi \circ g \).

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Y \\
\downarrow{\pi} & & \downarrow{\pi} \\
X & \xrightarrow{f} & X
\end{array}
\]

If \( \pi \) is injective, then \( \pi \) is a conjugacy (topological conjugacy). The dynamical system \( (X,f) \) is said to be a factor of \( (Y,g) \), and \( (Y,g) \) is an extension of \( (X,f) \). \( \pi \) is also called a factor map or projection.

An extension \( (Y,g) \) of \( (X,f) \) is called a skew product over \( (X,f) \) if \( Y = X \times F \) and \( \pi \) is a projection onto the first factor. \( Y \) is then a fiber bundle over \( X \) with projection \( \pi \) and fibers \((x,\cdot)\).

**Example 0.3** As an example let \( Y = \mathbb{R} \times S^1 \), \( X = \mathbb{R} \), \( g(x,y) = (2x, y + \cos x) \), and \( f(x) = 2x \).

A set \( A \subset X \) is forward invariant if \( f(A) \subset A \). A set \( A \subset X \) is backward invariant if \( f^{-1}(A) \subset A \). A set \( A \subset X \) is invariant if it is forward and backward invariant.

**Definition 0.4** A point \( y \) is (forward) asymptotic to \( p \) if \( d(f^n(x), f^n(y)) \to 0 \) as \( n \to \infty \). If \( x \) is a periodic point of period \( k \), then \( y \) is asymptotic to \( x \) if \( d(x, f^k(y)) \to 0 \) as \( n \to \infty \).

The stable set of \( x \) is the set of all asymptotic points to \( x \) and is denoted \( W^s(x) \). Similarly, we can define the backward asymptotic points and \( W^u(x) \) as the collection of all points backward asymptotic to \( x \).

**Theorem 0.5** Assume \( f : \mathbb{R} \to \mathbb{R} \) is \( C^1 \), \( p \) is fixed, and \( |f'(p)| < 1 \), then \( p \) is an attracting fixed point. If \( f^n(p) = p \) and \( |(f^n)'(p)| < 1 \), then \( p \) is an attracting periodic point.
Proof. Since $|f'(p)| < 1$ there exists an interval $[p - \epsilon, p + \epsilon]$ and $\lambda \in (0, 1)$ such that $|f'(x)| < \lambda$ for all $x \in [p - \epsilon, p + \epsilon]$. By the Mean Value Theorem there is a $z$ between $x$ and $p$ such that

$$|f(x) - p| = |f(x) - f(p)| = f'(z)|x - p| \leq \lambda|x - p| < |x - p|.$$ 

Then

$$|f^j(x) - p| \leq \lambda^j|x - p| \to 0 \text{ as } j \to \infty.$$ 

Similarly, we can show the results for periodic points. \(\square\)

Definition 0.6 A point $y$ is an $\omega$-limit point for $x$ if $\lim_{k \to \infty} d(f^{-n_k}(x), y) = 0$ for some subsequence $f^{-n_k}(x)$ of $f^n(x)$. The $\omega$ limit set of $x$ is the set of all $\omega$-limit points of $x$ and is denoted $\omega(x)$.

Similarly, we can define an $\alpha$-limit point and $\alpha$-limit set, denoted $\alpha(x)$, for backward iterates of $f$.

Theorem 0.7 Let $f : X \to X$ be continuous and $X$ be a complete metric space. Then

1. For all $x \in X$, $\omega(x) = \bigcap_{N \geq 0} \left( \bigcup_{n \geq N} \{f^n(x)\} \right)$. If $f^{-1}$ exists, then $\alpha(x) = \bigcap_{N \leq 0} \left( \bigcup_{n \leq N} \{f^n(x)\} \right)$.

2. If $f^j(x) = y$ for some $j \in \mathbb{Z}$, then $\omega(x) = \omega(y)$. Also, $\alpha(x) = \alpha(y)$ if $f^{-1}$ exists.

3. For all $x \in X$, $\omega(x)$ is closed and positively invariant.

4. If $\mathcal{O}^+(x)$ is contained in a compact set, then $\omega(x) \neq \emptyset$, is compact, and $d(f^n(x), \omega(x)) \to 0$ as $n \to \infty$.

5. If $D \subset X$ is closed and positively invariant and $x \in D$, then $\omega(x) \subset D$.

6. If $y \in \omega(x)$, then $\omega(y) \subset \omega(x)$.

Proof. Let $y \in \omega(x)$. Then $y \in \bigcup_{n \geq N} \{f^n(x)\}$ for all $N > 0$ by the definition of an omega-limit point. If

$$y \in \bigcap_{N \geq 0} \bigcup_{n \geq N} \{f^n(x)\},$$
then for all $N \geq 0$ there exists a $k_N > k_{N-1}$ with $k_N \geq N$ and

$$d(f^{k_N}(x), y) < \frac{1}{N}.$$ 

Therefore, $y \in \omega(x)$.

Part (2) is clear. For (3) we know that $\omega(x)$ is closed by (1). To see invariance let $y \in \omega(x)$. Then there exists a subsequence $n_k$ such that $d(f^{n_k}(x), y) \to 0$ as $k \to \infty$. For $j \in \mathbb{N}$ we have $d(f^{n_k+j}(x), f^j(y)) \to 0$ as $k \to \infty$ by continuity of $f$. Hence, $f^j(y) \in \omega(x)$.

For part (4) we know that $\mathcal{O}^+(x)$ is contained in a compact set. Then $\omega(x)$ is compact and nonempty since it is the intersection of a nested set of compact sets. Assume that $d(f^n(x), \omega(x))$ does not go to 0. Then there exists a $\delta > 0$ and subsequence $n_k$ such that $d(f^{n_k}(x), \omega(x)) > \delta$. Since $f^{n_k}(x)$ is bounded there is a convergent subsequence, a contradiction.

Part (5) follows from (1). For part (6) we know that $\omega(y) \subset \omega(x)$ by positive invariance of $\omega(x)$. □

**Definition 0.8** The Birkhoff center is $B(f) = \{x \mid x \in \omega(x)\}$. The limit set of $f$ is $L(f) = (\bigcup_{x \in X} \omega(x))$.

**Definition 0.9** A point $p \in X$ is non-wandering if for all neighborhoods $U$ of $p$ there exists some $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. The set of all non-wandering points is denoted $\Omega(f)$. A point is wandering if it is not non-wandering.

**Definition 0.10** An $\epsilon$-chain from $x$ to $y$ of length $n$ is a sequence $\{x = x_0, \ldots, x_n = y\}$ such that $d(f(x_{i-1}), x_i) < \epsilon$ for all $1 \leq j \leq n$.

**Definition 0.11** A point is chain recurrent if for all $\epsilon > 0$ there is an $\epsilon$-chain from $x$ to $x$. The set of all chain recurrent points is denoted $R(f)$.

**Claim 0.12** $\text{Per}(f) \subset B(f) \subset L(f) \subset \Omega(f) \subset R(f)$.

Proof is an exercise.