1 Manifolds

The idea of a manifold is that locally the manifold “looks like” Euclidean space. We then glue the local pictures together. To get a smooth picture we use “differentiable glue”.

Definition 1.1 A $C^r$ $n$-dimensional manifold $M$ is a second countable metric space with a collection of homeomorphisms $\varphi_\alpha : V_\alpha \subset \mathbb{R}^n \to U_\alpha \subset M$ for $\alpha$ in some indexing set $A$ such that

1. $\varphi(V_\alpha) = U_\alpha$,
2. $\{U_\alpha\}_{\alpha \in A}$ is an open cover of $M$, and
3. if $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\varphi_{\alpha,\beta} = \varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \subset V_\alpha \to \varphi_\beta^{-1}(U_\alpha \cap U_\beta) \subset V_\beta$$

is a $C^r$ diffeomorphism between open sets of $\mathbb{R}^n$.

One of the maps $\varphi_\alpha$ is called a coordinate chart.

Example 1.2 Let $M = S^n$. Let $D^n$ be the open unit ball in $\mathbb{R}^n$. For $1 \leq j \leq n$ define $\varphi_j^\pm : D^n \to S^n$ by

$$\varphi_j^\pm(y_1, \ldots, y_n) = (y_1, \ldots, y_{j-1}, \pm \sqrt{1 - y_1^2 - \cdots - y_{n-1}^2}, y_j, \ldots, y_n).$$
Example 1.3 Another method is to use level set of a function. Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a $C^r$ function for some $r \geq 1$. Assume that $c \in \mathbb{R}$ is a value such that for all $p \in F^{-1}(c)$, $DF_p \neq 0$. Let $M = F^{-1}(c)$. This set is then a $C^r$ manifold (exercise). (Hint: The Implicit Function Theorem shows there exists a neighborhood $U_p$ of $p$ and a $C^r$ function $\sigma_p : V_p \subset \mathbb{R}^n \to \mathbb{R}$ such that the graph of $\sigma_p$ is onto $U_p$. These graphs give $M$ the structure of a $C^r$ manifold.)

Definition 1.4 Let $M$ and $N$ be two $C^r$ manifolds for some $r \geq 1$. Assume that $f : M \to N$ is continuous. We say that $f$ is $r$ times differentiable, or $C^r$, provided for each $p \in M$ and coordinate charts $\varphi_\alpha : V_\alpha \to U_\alpha \subset M$ and $\varphi_\beta : V_\beta \to U_\beta \subset N$ at points $p$ and $f(p)$, respectively, $\varphi_\beta^{-1} \circ f \circ \varphi_\alpha$ is differentiable at $C^r$ at $\varphi_\alpha^{-1}(p)$.

Remark 1.5 If $\varphi_\beta^{-1} \circ f \circ \varphi_\alpha$ is $C^r$ and $\varphi_{\alpha'}$ and $\varphi_{\beta'}$ are two other coordinate charts at $p$ and $f(p)$, respectively, then $\varphi_{\beta'}^{-1} \circ f \circ \varphi_{\alpha'}$ is $C^r$ at $\varphi_{\alpha'}^{-1}(p)$ since $\varphi_{\alpha,\alpha'}$ and $\varphi_{\beta,\beta'}$ are $C^r$.

The set of $C^r$ maps from $M$ to $N$ is denoted $C^r(M,N)$. A $C^r$ map from $M$ to $M$ is a diffeomorphism provided it is 1-1, onto, and the derivative at each point is nonsingular for the map and its inverse. The set of all $C^r$ diffeomorphisms on $M$ is denoted $\text{diff}^r(M)$.

2 Tangent Space

We start by looking at Euclidean space.

Definition 2.1 A tangent vector at $p$ is a pair $(p,v) = v_p$ where $p,v \in \mathbb{R}^n$. The set of all tangent vectors at $p$ is denoted $T_p\mathbb{R}^n$ and is called the tangent space at $p$. The tangent space is a vector space where $(p,v) + (p,w) = (p,v+w)$. The disjoint union of the tangent space or tangent bundle is denoted $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.

For a manifold the structure will be more involved. Let $\gamma : (-\delta,\delta) \subset \mathbb{R} \to M$ be a $C^1$ curve with $\gamma(0) = p$. Assume that $\varphi_\alpha : V_\alpha \to U_\alpha$ is a coordinate chart at $p$. Then $\varphi_\alpha^{-1} \circ \gamma(t)$ is $C^1$ and the tangent vector determined at $p$ by $\gamma$ is $(\varphi_\alpha^{-1} \circ \gamma)'(0) = v_p^\alpha$. If $\varphi_\beta : V_\beta \to U_\beta$ is another coordinate chart. Then $v_p^\beta = (\varphi_\beta^{-1} \circ \gamma)'(0)$ and $v_p^\beta = D(\varphi_\beta \circ \varphi_\alpha)_p v_p^\alpha$. So we can identify $v_p^\alpha \sim v_p^\beta$ by an
equivalence relation. Two curves have the same germ if for any coordinate chart \((\varphi^{-1}_\alpha \circ \gamma_1)'(0) = (\varphi^{-1}_\alpha \circ \gamma_2)'(0)\). A tangent vector is an equivalence class of curves with the same germ. The set of tangent vectors at \(p\) is denoted \(T_p M\). The set \(T_p M\) is a vector space with a canonical identification to \(\mathbb{R}^n\) where \(n = \dim(M)\). The tangent bundle of \(M\) is \(TM = \bigcup_{p \in M} T_p M\). For a set \(S \subseteq M\) the set \(T_S M = \bigcup_{p \in S} T_p M\) is the tangent bundle of \(S\) with respect to \(M\).

**Definition 2.2** If \(f : M \to N\) is \(C^1\) we define the derivative of \(f\) at \(p\) as the linear map from \(T_p M\) to \(T_{f(p)} N\) such that for any coordinate chart \(\varphi_\alpha : V_\alpha \to U_\alpha \subseteq M\) and \(\varphi_\beta : V_\beta \to U_\beta \subseteq N\), \(D(\varphi_\beta^{-1} \circ f \circ \varphi_\alpha)_p v^\alpha_p = w^\beta_{f(p)}\).

### 3 Immersions and Embeddings

**Definition 3.1** A \(C^r\) map \(f : M \to N\) is an immersion provided the derivative of \(f\) at each point is an isomorphism. The image of a 1-1 immersion is called an immersed submanifold. If an immersion is a homeomorphism and such that the preimage of every compact set in \(N\) is a compact set in \(M\), then it is called an embedding and its image is called an embedded submanifold.

It is sometimes useful to think of \(n\)-dimensional manifold \(M\) as an embedded manifold in \(\mathbb{R}^N\) for some \(N \geq n\). (By Whitney’s Embedding Theorem this is always possible.) We will do this at times.

### 4 Topology on \(\text{diff}^r(M)\)

We will start with easier examples. Let \(f : \mathbb{T}^n \to \mathbb{T}^n\), then there exists a lift \(F : \mathbb{R}^n \to \mathbb{R}^n\) such that \(\pi \circ F = f \circ \pi\) where \(\pi\) is the projection from \(\mathbb{R}^n\) to \(\mathbb{T}^n\) and \(F(x + j) = F(x) + j\) for all \(j \in \mathbb{Z}^n\) and all \(x \in \mathbb{R}^n\). If \(f, g : \mathbb{T}^n \to \mathbb{T}^n\) are two \(C^r\) maps, then

\[
d_r(f, g) = \sup \{d(f \circ \pi(x), g \circ \pi(x)), \|D^i F_x - D^i G_x\|, x \in \mathbb{R}^n \text{ and } 1 \leq i \leq r\}.\]

For \(f, g : M \to \mathbb{R}^N\) let \(\varphi_j\) be a finite collection of coordinate charts with \(\bigcup_{j=1}^J U_j\) covering \(M\) and \(C_j \subseteq U_j\) compact sets such that \(\bigcup_{j=1}^J C_j = M\). Then

\[
d_r(f, g) = \sup \{d(|f(x) - g(x)|, \|D^i (f \circ \varphi_j)_{\varphi^{-1}_j(x)} - D^i (g \circ \varphi_j)_{\varphi^{-1}_j(x)}\|, x \in C_j, 1 \leq j \leq J \text{ and } 1 \leq i \leq r\}.\]
For the general case $f, g : M \to N$ one uses a finite set of coordinate charts on $M$ and $N$ such that $\bigcup_{j=1}^{J} U_j = M$, $\bigcup_{k=1}^{K} V_k = N$, $\varphi_j : U_j \to M$, and $\psi_k : V_k \to N$. Then

$$d_r(f, g) = \sup \{ d(\| f(x) - g(x) \|),$$

$$\| D^i(\psi^{-1}_k \circ f \circ \varphi_j^{-1}) - D^i(\psi^{-1}_k \circ g \circ \varphi_j^{-1}) \|,$$

$$x \in C_j, 1 \leq j \leq J, 1 \leq k \leq K, \text{ and } 1 \leq i \leq r \}.$$

## 5 Riemannian Metrics and the Exponential Map

**Definition 5.1** A Riemannian metric is an inner product on each tangent space

$$\langle \cdot, \cdot \rangle_p : T_p M \times T_p M \to \mathbb{R}$$

that is a symmetric, positive definite bilinear form that varies continuously in $p$.

This can be obtained by embedding into $\mathbb{R}^N$ and using the Euclidean inner product.

A manifold $(M, \varphi, \langle \cdot, \cdot \rangle)$ is a Riemannian manifold. So a Riemannian manifold consists of a topological space, a differentiable structure, and an assigned inner product. The norm on $T_p M$ is given by $\| v_p \| = \langle v_p, v_p \rangle^{1/2}$.

Let $p \in M$ and $U \subset T_p M$ be a sufficiently small open set containing $(p, 0)$. The map $\exp : U \to M$ given by $\exp(q, v) = \gamma(|v|, q, \frac{v}{|v|})$ where $\gamma(|v|, q, \frac{v}{|v|})$ is the curve passing through $q$ at time $|v|$ with velocity $\frac{v}{|v|}$ such that the curve passes through $p$ at time $t = 0$ and $\frac{d}{dt}(\frac{d\gamma}{dt}) = 0$ for all $t \in [0, |v|]$. (Such a curve is called a geodesic and represents the straight lines on the manifold.)

Generally, we look at $\exp_p : B_\epsilon(0) \subset T_p M \to M$ by $\exp_p(v) = \exp(p, v)$.

**Remark 5.2** For $\epsilon > 0$ sufficiently small the map $\exp_p : B_\epsilon(0) \to M$ is a diffeomorphism. For larger $\epsilon$ this map may not be 1-1.