Let $M$ be a $C^r$ manifold and $f \in \text{diff}^k(M)$ for $1 \leq k'$.r.

**Definition 0.1** A compact $f$-invariant set $\Lambda$ is hyperbolic if there exist constants $\lambda \in (0,1)$, $C > 0$ and subspaces $E^s_x \subset T_x M$ and $E^u_x \subset T_x M$ such that for all $x \in \Lambda$ the following holds:

1. $T_x M = E^s_x \oplus E^u_x$,
2. $\|df^n_x v^s\| \leq C \lambda^n \|v^s\|$ for all $v^s \in E^s_x$ and $n \geq 0$,
3. $\|df^{-n}_x v^u\| \leq C \lambda^n \|v^u\|$ for all $v^u \in E^u_x$ and $n \geq 0$, and
4. $df_x E^s_x = E^s_{f(x)}$ and $df_x E^u_x = E^u_{f(x)}$.

**Remark 0.2** The above allows $E^s$ or $E^u$ to be $\{0\}$.

The space $E^s_x$ is called the stable subspace and $E^u_x$ is the unstable subspace.

**Example 0.3** Let $f$ be a diffeomorphism and $p$ a hyperbolic fixed point. Then $df_p$ is a hyperbolic matrix and the splitting by the generalized eigenspaces gives the desired splitting. This example can be extended to periodic points. Other examples are the horseshoe, solenoid, and hyperbolic toral automorphisms.

**Definition 0.4** If $\Lambda = M$, then $f$ is an Anosov diffeomorphism.

**Proposition 0.5** Let $\Lambda$ be a hyperbolic set. Then $E^s_x$ and $E^u_x$ depend continuously on $x$. 
Proof. Let \( \{x_i\} \subset \Lambda \) such that \( x_i \to x_0 \) as \( i \to \infty \). By possibly using a subsequence we may assume that the dimension of the stable subspace is constant for the sequence. Let \( w_{1,i}, \ldots, w_{k,i} \) be an orthonormal basis of \( E^s_{x_i} \). Again by passing to a subsequence, if necessary, we have \( w_{j,i} \to w_{j,0} \in T_{x_0}M \) for each \( 1 \leq j \leq k \). Since condition (2) in the definition of a hyperbolic set is satisfied by each \( w_{j,i} \) and this is a closed condition we get that \( w_{j,0} \) satisfies property (2) for \( 1 \leq j \leq k \) and then is in \( E^s_{x_0} \). Hence, \( \dim E^s_{x_0} \geq k = \dim E^s_{x_i} \). Similarly, we can show that \( \dim E^u_{x_0} \geq n - kk = \dim E^u_{x_i} \) and continuity follows. \( \square \)

We now show there is a Riemannian metric defined in a neighborhood of a hyperbolic set such that \( C = 1 \). Such a metric is called a adapted metric.

**Proposition 0.6** If \( \Lambda \) is a hyperbolic set with constants \( C \) and \( \lambda \), then for all \( \epsilon > 0 \) sufficiently small there exists a Riemannian metric \( \langle \cdot, \cdot \rangle' \) in a neighborhood of \( \Lambda \) with hyperbolic constants \( C' = 1 \) and \( \lambda' = \lambda + \epsilon \) and \( E^s_x \) and \( E^u_x \) are \( \epsilon \)-orthogonal (i.e. \( \langle v^s, v^u \rangle' < \epsilon \) for all \( v^s \in E^s_x \) and \( v^u \in E^u_x \)).

**Proof.** For \( x \in \Lambda \), \( v^s \in E^s_x \) and \( v^u \in E^u_x \) set \( ||v^s'|| = \sum_{n=0}^{\infty} (\lambda + \epsilon)^{-n} ||df_x^n v^s|| \), and \( ||v^u'|| = \sum_{n=0}^{\infty} (\lambda + \epsilon)^{-n} ||df_x^{-n} v^u|| \). We get uniform convergence of each series for \( x \in \Lambda \) and

\[
||df_x f^s||' = \sum_{n=0}^{\infty} (\lambda + \epsilon)^{-n} ||df_x^{n+1} v^s|| = (\lambda + \epsilon)(||v^s|| - ||v^s||') < (\lambda + \epsilon)||v^s||'.
\]

Similarly, \( ||df_x^{-1} v^u|| < (\lambda + \epsilon)||v^u||' \). For \( v = v^u + v^s \) define \( ||v||' = \sqrt{(||v^s||')^2 + (||v^u||')^2} \) and \( \langle v, w \rangle' = \frac{1}{2}((||v + w||')^2 - (||v||')^2 - (||w||')^2) \). So \( E^s \) and \( E^u \) are \( \epsilon \)-orthogonal, \( \lambda' = \lambda + \epsilon \) and \( C' = 1 \) are hyperbolic constants. This can then be extended to a neighborhood of \( \Lambda \) (see for example Hirsch’s Differential Topology). \( \square \)