Theorem 0.1 (Stable Manifold Theorem) Let \( f \in \text{diff}^k(M) \) and \( \Lambda \) be a hyperbolic set for \( f \) with hyperbolic constants \( \lambda \in (0,1) \) and \( C \geq 1 \). Then there exists an \( r > 0 \) such that for each \( p \in \Lambda \) there exists two embedded disks \( W^s_r(p,f) \) and \( W^u_r(p,f) \) which are tangent to \( E^s_p \) and \( E^u_p \), respectively. There is an identification of these disks to \( E^s_p(r) \) and \( E^u_p(r) \) and \( W^s_r(p,f) \) is a graph of a \( C^k \) function \( \sigma^s_p : E^s_p(r) \rightarrow E^u_p(r) \) with \( \sigma^s_p(0_p) = 0_p \) and \( D(\sigma^s_p)_0 = 0 \). The sets \( W^s_r(p,f) \) and \( W^u_r(p,f) \) are not necessarily unique, but the sets \( W^s(p,f) \) and \( W^u(p,f) \) are unique.

Remark 0.2 Similar statements hold for the unstable manifold. Moreover, for \( \lambda < \lambda' < 1 \) and \( r > 0 \) sufficiently small

\[
W^s_r(p,f) = \{ q \in E^u_p(r) \times E^s_p(r) \mid f^j(q) \in E^u_p(r) \times E^s_p(r) \text{ and } d(f^j(q), f^j(p)) \leq C(\lambda')^j d(q,p) \text{ for all } j \geq 0 \}.
\]
Proposition 0.3 Let $\Lambda$ be a hyperbolic set for $f$. Then for each $p \in \Lambda$, $W^s(p, f)$ is an immersed copy of $E^s_p$ and $W^u(p, f)$ is an immersed copy of $E^u_p$.

Proof. We know that $W^s_r(p, f)$ is an embedded copy of $E^s_p(r)$ be a map. Since $f$ is a diffeomorphism we know that $f^j(W^s_r(p, f))$ is an embedded image of $E^s_p(r)$. Thus $W^s(p, f)$ is the union of embedded copies of a disk and $W^s(p, f)$ is an immersed copy. The proof for $W^u(p, f)$ is similar. □

We remark that the immersion can be very complicated (i.e. a homoclinic tangle). We will prove the Stable Manifold Theorem for a hyperbolic fixed point. The case of a periodic point is easily generalized. For a more general hyperbolic set see the proof in Katok and Hasselblatt or Robinson.

As a reminder a fixed point is hyperbolic if no eigenvalues of $Df_p$ lie on the unit circle. Then there exists a splitting $\mathbb{R}^n = E^s \oplus E^u$ such that there exists a $\lambda > 1$ and $\mu \in (0, 1)$ where each eigenvalue $\alpha$ whose generalized eigenspace lies in $E^u$ satisfies $|\alpha| > \lambda$ and each eigenvalue $\alpha$ whose generalized eigenspace lies in $E^s$ satisfies $|\alpha| < \mu$. Furthermore, there exists a $C > 0$ such that $\|Df^n_p|_{E^s}\| < C\mu^n$ and $\|Df^{-n}_p|_{E^u}\| < C\lambda^n$ for all $n > 0$. By suing an adapted metric we may suppose that $C = 1$.

1 Proof of the Stable Manifold Theorem

The Graph Transform (or Hadamard) Method for proof uses the following idea: Take a trial function $\sigma : E^s_p(r) \to E^u_p(r)$ that might give the stable manifold. Then define an appropriate operator $\Gamma$ on the set of all such trial functions. We show that $\Gamma$ is a contraction mapping and the unique fixed point gives the desired function.

Remark 1.1 The other common method is the Perron Method. This uses integral equations similar to how we showed the existence and uniqueness of solutions to ODE’s.

Outline: We first show that $W^s_r(0, f)$ is the graph of a Lipschitz function with Lipschitz constant less than some fixed $\alpha$. To do this we take $B_N = \bigcap_{j=0}^{N-1} f^j(E^u(r) \times E^s(r))$ and show $B_{N+1} = f(B_N) \cap B_0$ and $B_\infty = \bigcap_{j=0}^{\infty} B_j$ is the graph of a Lipschitz function. We then show the function is $C^1$ and prove a fact that will show it is $C^k$. 

2
1.1 Cone fields

A useful tool is the use of cone fields. For $\alpha > 0$ we define the stable and unstable cones by

$$
C_s(\alpha) = \{(v_s, v_u) \in E_s \oplus E_u \mid |v_u| \leq \alpha |v_s| \}
$$

$$
C_u(\alpha) = \{(v_s, v_u) \in E_s \oplus E_u \mid |v_s| \leq \alpha |v_u| \}
$$

Normally, $\alpha > 0$ is chosen sufficiently small so the cones are “thin”. A cone field for a set $\Lambda$ with a splitting $E_s \oplus E_u$ for each $x \in \Lambda$ is then the set of cones at each point $x \in \Lambda$ and $\alpha$ is a uniform constant (i.e. does not depend on $x$).

As a reminder the minimum (or conorm) of a linear map $A : \mathbb{R}^k \to \mathbb{R}^k$ is

$$
m(A) = \inf_{v \neq 0} \frac{|Av|}{|v|},
$$

This measures the minimum expansion of $A$. If $A$ is invertible, then $m(A) = \|A^{-1}\|^{-1}$. For a hyperbolic fixed point with $\|Df^n|_{E^u}\| < C\lambda^{-n}$ it follows that $m(Df^n|_{E^u}) > C\lambda^n$ for $n > 0$. Let $\pi_s : \mathbb{R}^n \to E^s$ and $\pi_u : \mathbb{R} \to E^u$. We may assume that the point $p$ is identified with the origin and $Df = \begin{bmatrix} A_{ss} & 0 \\ A_{su} & A_{uu} \end{bmatrix}$

where $A_{ss} = \pi_s(Df|_{E^s \times \{0\}})$ and $A_{uu} = \pi_u(Df|_{\{0\} \times E^u})$.

Since 0 is a hyperbolic fixed point there exists $0 < \mu < 1 < \lambda$ such that $m(A_{uu}) > \lambda > 1$ and $\|A_{ss}\| < \mu < 1$. We now define the norm on $\mathbb{R}^n$ to be the maximum component norms in $E^s$ and $E^u$. Let $B(r) = E^s(r) \times E^u(r)$.

We now want to specify how thin to make the cone fields. These estimates are needed so that in the set $B(r)$ the cones map into the interiors in a uniform way.

Fix $\alpha > 0$ sufficiently small and $\epsilon > 0$ sufficiently small such that

$$
\mu + \frac{\epsilon}{\alpha} + \epsilon \alpha + \epsilon < 1, \frac{\lambda}{\alpha} - 2\epsilon - \epsilon \alpha > 1, \text{ and } \lambda \alpha - 2\epsilon - \frac{\epsilon}{\alpha} > 1.
$$

Given such $\alpha$ and $\epsilon$ there exists an $r > 0$ such that for all $q \in B(r)$ the matrix

$$
Df_q = \begin{bmatrix} A_{ss}(q) & A_{su}(q) \\ A_{us}(q) & A_{uu}(q) \end{bmatrix}
$$

with

$$
\|A_{ss}(q)\| < \mu,
$$

$$
m(A_{uu}(q)) > \lambda,
$$

$$
\|A_{su}(q)\| < \epsilon,
$$

$$
\|A_{us}(q)\| < \epsilon, \text{ and } \|Df_q - Df_0\| < \epsilon.
$$
The next three lemmas give estimates on the $\alpha$ stable and unstable cone fields in $B(r)$ under the action $f$.

**Lemma 1.2** Let $q \in B(r)$. Then $Df_q C^u(\alpha) \subset C^u(\alpha)$.

**Proof.** Let $v = (v_s, v_u) \in C^u(\alpha) - \{0\}$ and $q \in B(r)$. Then

$$\frac{\|\pi_s Df_q v\|}{\|\pi_u Df_q v\|} = \frac{\|A_{su}(q)v_u + A_{ss}(q)v_s\|}{\|A_{uu}(q)v_u + A_{us}(q)v_s\|} \leq \frac{A_{su}(q)\|v_u\| + \|A_{ss}(q)\|\|v_s\|}{m(A_{uu}(q))\|v_u\| - \|A_{us}(q)\|\|v_s\|}$$

$$= \frac{A_{su}(q)\|v_u\| + \|A_{ss}(q)\|\|v_s/v_u\|}{m(A_{uu}(q)) - \|A_{us}(q)\|\|v_s/v_u\|} \leq \frac{\epsilon + \mu \alpha}{\lambda - \alpha \epsilon} = \alpha \left(\frac{\mu + \epsilon \alpha^{-1}}{\lambda \alpha^{-1} - \epsilon}\right) < \alpha.$$

\(\square\)

**Lemma 1.3** Let $q, q' \in B(r)$ and $q' \in \{q\} + C^u(\alpha)$. Then the following hold:

1. $|\pi_s \circ f(q') - \pi_s \circ f(q)| \leq \alpha (\mu + \epsilon \alpha^{-1}) |\pi_u(q' - q)|$,
2. $|\pi_u \circ f(q') - \pi_u \circ f(q)| \geq (\lambda - \epsilon \alpha) |\pi_u(q' - q)|$, and
3. $f(q') \in \{f(q)\} + C^u(\alpha)$.

**Proof.** Let $\psi(t) = q + t(q' - q)$ for $t \in [0, 1]$. Using the MVT we have

$$|\pi_s \circ f(q') - \pi_s f(q)| = |\pi_s \circ f \circ \psi(1) - \pi_s \circ f \circ \psi(0)|$$

$$\leq \sup_{0 \leq t \leq 1} |\frac{d}{dt} (\pi_s \circ f \circ \psi(t))|$$

$$\leq \sup_{0 \leq t \leq 1} |A_{su}(\psi(t))||\pi_u(q' - q)|$$

$$+ \sup_{0 \leq t \leq 1} |A_{ss}(\psi(t))||\pi_s(q' - q)|$$

$$\leq (\epsilon + \mu \alpha)|\pi_u(q' - q)|$$

$$= \alpha (\mu + \epsilon \alpha^{-1}) |\pi_u(q' - q)|.$$

For the second part let $q = (q_s, q_u)$, $q' = (q'_s, q'_u)$, and $z = (q_s, q'_u)$. So

- $q' - q = (q' - z) + (q - q)$,
- $z - q \in \{0\} \times E^u$, and
- $q' - z \in E^s \times \{0\}$. 
For $y \in \mathbb{E}^n(r)$ and $x \in \mathbb{E}^s(r)$ define $h(y) = \pi_u \circ f(q_s, y)$ and $g(x) = \pi_u \circ f(x, 0)$. Then $g(q_s) = \pi_u \circ f(q_s, 0) = h(0)$. Also, $Dg_x = A_{us}(x, 0)$ so
\[\|Dg_x\| \leq \epsilon.\] By the MVT we then have
\[\epsilon r \geq \epsilon |g_s| \geq |g(q_s)| = |h(0)|.\]
Then $Dh_y = A_u(u(q_s, +y)$ which is an isomorphism with minimum norm greater than $\lambda$ and $\|Dh_y - A_{uu}(0, 0)\| < \epsilon$. So $h|_{\mathbb{E}^n(r)}$ is onto
\[\mathbb{E}^n((\lambda - \epsilon)r - |h(0)|) \supset \mathbb{E}^u((\lambda - 2\epsilon)r) \supset \mathbb{E}^u(r).\]
So $h$ has a unique fixed point. Also, by the Inverse Function Theorem the derivative of $h^{-1}$ is given by $D(h^{-1})_{h(y)} = (Dh_y)^{-1}$ and has norm less than $\lambda^{-1}$. Therefore, the MVT says that $|h^{-1}(w_2) - h^{-1}(w_1)| \leq \lambda^{-1}|w_2 - w_1|$. Let $w_2 = h(q'_u) = \pi_u \circ f(z)$ and $w_1 = h(q_u) = \pi_u \circ f(q)$. So $h^{-1}(w_2) = q'_u$ and $h^{-1}(w_1) = q_u$. Then we have
\[\lambda|\pi_u(q' - q)| \leq |\pi_u \circ f(z) - \pi_u \circ f(q)|.\]
Also, we have
\[|\pi \circ f(q') - \pi \circ f(z)| \leq \sup \|A_{us}\||\pi_s(q' - q)| \leq \epsilon|\pi_s(q' - q)| \leq \epsilon \alpha|\pi_u(q' - q)|.\]
Combining these we have
\[|\pi_u \circ f(q') - \pi_u \circ f(q)| \geq |\pi_u \circ f(z) - \pi_u \circ f(q)| - |\pi_u \circ f(q') - \pi_u \circ f(q)| \geq (\lambda - \epsilon \alpha)|\pi_u(q' - q)|.\]
For (3) we see that
\[\frac{|\pi_s(f(q') - f(q))|}{|\pi_u(f(q') - f(q))|} \leq \alpha \frac{\mu + \epsilon \alpha^{-1}}{\lambda - \epsilon \alpha} < \alpha.\]
\[\square\]

**Lemma 1.4** Assume that $f(q_{-1}) = q$, $f(q'_{-1}) = q'$, $q' \in \{q\} + C^s(\alpha)$, and $q, q', q_{-1}, q'_{-1} \in B(r)$. Then
1. $q'_{-1} \in \{q_{-1}\} + C^s(\alpha)$, and
2. $|\pi_s(q'_{-1} - q_{-1})| \geq (\mu + \epsilon \alpha)^{-1}|\pi_s(q' - q)|$. 


Proof. The proof of (1) follows similarly to the previous lemma. For (2) we let $\psi(t) = q_{-1} + t(q'_{-1} - q_{-1})$ where $t \in [0, 1]$. Then

$$|\pi_s(q' - q)| = |\pi_s \circ f \circ \psi(t) - \pi_s \circ f \circ \psi(0)|$$

$$\leq \sup_{0 \leq t \leq 1} |\frac{d}{dt} \pi_s \circ f \circ \psi(t)|$$

$$\leq \sup \|A_{ss}(\psi(t))\| |\pi_s(q'_{-1} - q_{-1})| + \sup \|A_{su}(\psi(t))\| |\pi_s(q'_{-1} - q_{-1})|$$

$$\leq (\mu + \epsilon \alpha) |\pi_s(q'_{-1} - q_{-1})|.$$

\[ \square \]

Definition 1.5 An $\alpha$-unstable disk is a graph of a $C^1$ function $\varphi : E^u(r) \rightarrow E^s(r)$ such that $\|D\varphi_y\| \leq \alpha$ for all $y \in E^u(r)$.

The next lemma is the key ingredient to the proof of the Stable Manifold Theorem.

Lemma 1.6 Let $D^u_0$ be an $\alpha$-unstable disk over $E^u(r)$. Then

1. $D^n = f(D^u_0) \cap B(r)$ is an $\alpha$-unstable disk over $E^u(r)$ and $\text{diam}[\pi_u(f^{-1}(D^u_0) \cap D^u_r)] \leq (\lambda - \epsilon \alpha)^{-1}2r$.

2. Furthermore, let $D^n_n = f(D^u_{n-1}) \cap B(r)$. Then $D^n_n$ is an $\alpha$-unstable disk for $n \geq 1$ and $\text{diam}[\pi_u(f^{-1}(D^u_n))] \leq (\lambda - \epsilon \alpha)^{-n}2r$.

Proof. Since $D^u_0$ is the graph of a $C^1$ function $\varphi_0 : E^u(r) \rightarrow E^s(r)$ with $\|D\varphi_0\| \leq \alpha$. Let $\sigma_0 : E^u(r) \rightarrow B(r)$ be defined as $\sigma_0(y) = (\varphi_0(y), y)$. Define $h = \pi_u \circ f \circ \sigma_0$. Then

$$\|Dh - A_{uu}(0)\| = \|D(\pi_u \circ f \circ \sigma_0) - A_{uu}(0)\|$$

$$= \|A_{uu}(\sigma_0(y)) + A_{us}(\sigma_0(y))D(\varphi_0) - A_{uu}(0)\|$$

$$\leq \|A_{uu}(\sigma_0(y)) - A_{uu}(0)\| + \|A_{us}(\sigma_0(y))\| \|D(\varphi_0)\|$$

$$\leq \epsilon + \epsilon \alpha.$$

So $h$ is a $C^1$ local diffeomorphism and has an inverse and

$$h(E^u(r)) \supset \{h(0)\} + E^u((\lambda - \epsilon \epsilon \alpha)r) \supset E^u((\lambda - \epsilon - \epsilon \alpha)r - |h(0)|).$$

Notice that $h(0) = \pi_u \circ f(\varphi_0(0), 0)$ with $|\varphi_0(0)| \leq r$ and $\pi_u \circ f(0, 0) = 0$. By the MVT applied to $\pi_u \circ f$ at the points $(0, 0)$ and $(\varphi_0(0), 0)$ we have

$$|h(0)| \leq \sup(\|A_{uu}\|)|\varphi_0(0) - 0| \leq \epsilon r.$$
Then
\[ h(\mathbb{E}^u(r)) \supset \mathbb{E}^u((\lambda - \epsilon \alpha - \epsilon)r) \supset \mathbb{E}(r) \]
so \( h^{-1}(\mathbb{E}^u(r)) \subset \mathbb{E}^u(r) \). Define the map \( \sigma_1 = f \circ \sigma_0 \circ h^{-1}|_{\mathbb{E}^u(r)} \). Hence,
\[ \pi_u \circ \sigma_1 = \pi_u \circ f \circ \sigma_0 \circ h^{-1} = h \circ h^{-1} = id|_{\mathbb{E}^u(r)} \]
and this implies that \( \sigma_1 \) is a graph over \( \mathbb{E}^u(r) \). For \( v_u \in \mathbb{E}^u \) we have
\[
\|D(\pi_u \circ f \circ \sigma_0)_y v_u\| = \|A_{uu}(\sigma_0(y))v_u + A_{us}(\sigma_0(y))D(\varphi_0(y))D(\varphi_0)_y v_u\| \\
\geq \|A_{uu}(\sigma_0(y))v_u\| - \|A_{us}(\sigma_0(y))D(\varphi_0)_y v_u\| \\
\geq m(A_{uu}(\sigma_0(y)))v_u - \|A_{us}(\sigma_0(y))\|\|D(\varphi_0)_y\|v_u| \\
\geq (\lambda - \epsilon \alpha)v_u.
\]
Hence, \( m(Dh_y) \geq \lambda - \epsilon \alpha \) and \( \|D(h^{-1})_y\| \leq (\lambda - \epsilon \alpha)^{-1} \). and \( h^{-1}(y_2) - h^{-1}(y_1) \leq (\lambda - \epsilon \alpha)^{-1}y_2 - y_1 \).

Let \( \varphi_i(y) = \pi_s \circ \sigma_1(y) = \pi_s \circ f(\varphi_0(h^{-1}(y)), h^{-1}(y)) \). Then
\[
\|D(\varphi)_y\| \leq \|A_{uu}(\sigma_0 \circ h^{-1}(y))D(h^{-1})_y\| + \|A_{ss}(\sigma_0 \circ h^{-1}(y))D(\varphi_0)_{h^{-1}(y)}D(h^{-1})_y\| \\
\leq \epsilon(\lambda - \epsilon \alpha)^{-1} + \mu\alpha(\lambda - \epsilon \alpha)^{-1} \\
\leq \alpha \frac{\alpha + \mu\alpha}{\lambda - \epsilon \alpha} < \alpha.
\]
So now let \( D^u_1 = \sigma_1(\mathbb{E}^u(r)) \) and we have part (1). Part (2) follows inductively.

From the above lemma we have a graph transform.

**Proposition 1.7** \( W^s_r(0) = \bigcap_{j=0}^\infty f^{-j}(B(r)) \) is the graph of an \( \alpha \)-Lipschitz function \( \varphi^s : \mathbb{E}^s(r) \to \mathbb{E}^u(r) \) with \( \varphi^s(0) = 0 \).

**Proof.** Let \( S_n = \bigcup_{j=0}^n f^{-j}(B(r)) \) so \( W^s_r(0) = \bigcap_{n=0}^\infty S_n \). Fix \( x \in \mathbb{E}^s(r) \) and \( D^u_0(x) = \{x\} \times \mathbb{E}^u(r) \). By Lemma 1.6 we have \( D^u_n(x) \) is an unstable disk and
\[
\{x\} \times \mathbb{E}^u(r) \cap S_n = \bigcap_{j=0}^n f^{-j}(D^u_j(x)) \subset D^u_0(x)
\]
is a nested sequence of closed complete sets and the diameter is bounded above by \( (\lambda - \epsilon \alpha)^{-n}2r \). So intersection of \( \{x\} \times \mathbb{E}^u(r) \cap W^s_r(0) \) is a unique point. Hence, \( W^s_r(0) \) is a graph over \( \mathbb{E}^s \).

We now show the graph is Lipschitz. Let \( q \in W^s_r(0) \) and assume that \( q' \in W^s_r(0) \cap \{q\} + C^u(\alpha^{-1}) \). Then by Lemma 1.2 we get \( f^j(q) \in \{f^j(q')\} + \ldots + \)
$C^u(\alpha^{-1})$ and $|\pi_u(f^j(q) - f^j(q'))| \geq (\lambda - \epsilon \alpha^{-1})^j|\pi_u(q - q')|$. Since $(\lambda - \epsilon \alpha^{-1})^j \to \infty$ we get $f^j(q')$ or $j^i(q)$ leaves $B(r)$, but both are in the stable manifold, a contradiction. So $W^s_r(0) \cap \{q\} + C^u(\alpha^{-1}) = \{q\}$.

Now for $x \neq y$, $x, y \in \mathbb{E}^n(r)$ since $\sigma^u(x) = (x, \phi^s(\phi^s(x) + (y, \phi^s(y))) = \sigma^u(y)$ it follows that $\sigma^u(y)$ is not contained in $\{\sigma^u(x)\} + C^u(\alpha^{-1})$, so $\sigma^u(y) \in \{\sigma^u(x)\} + C^s(\alpha)$. Hence, $\text{Lip}(\pi_u \circ \sigma^u) \leq \alpha$. □

**Proposition 1.8** If $q \in W^s_r(0)$, then $|f^j(q)| \leq \alpha(\mu + \epsilon \alpha^{-1})^j|q|$ so $f^j(q) \to 0$ exponentially fast.

**Proof.** Since $W^s_r(0)$ is $\alpha$-Lipschitz we get $|\pi_u(f^j(q) - 0)| \leq \alpha|\pi_u(f^j(q) - 0)|$. So $f^j(q) \in \{0\} + C^s(\alpha)$. Applying Lemma 1.3 to $f^j(q)$ and $f^j(0) = 0$ we get

$$(\mu + \epsilon \alpha^{-1})|f^{j-1}(q)| \geq (\mu + \epsilon \alpha^{-1})|\pi_u \circ f^{j-1}(q)| \geq (\mu + \epsilon \alpha^{-1})|\pi_u(f^{j-1}(q) - f^{j-1}(0))| \geq |\pi_u(f^j(q) - f^j(0))|

Hence, $(\mu + \epsilon \alpha^{-1})^j|q| \geq |\pi_u \circ f^j(q)|$ and

$$|\pi_u \circ f^j(q)| \leq \alpha|\pi_u \circ f^j(q)| \leq \alpha(\mu + \epsilon \alpha^{-1})^j|q|.$$

So

$$|f^j(q)| \leq (\mu + \epsilon \alpha)^j|q| \max\{1, \alpha\}.$$

□

**Proposition 1.9** $W^s_r(0)$ is $C^1$ and tangent to $\mathbb{E}^s$ at $0$

To show this we need one more lemma. Let $C^{s,0}(q) = C^s(\alpha)$ and $C^{s,n}(q) = (Df^q)^{n-1}C^s(\alpha)$.

**Lemma 1.10** Let $q \in W^s_r(0)$. Then $C^{s,n}(q) \subset C^{s,n-1}(q) \subset \ldots \subset C^{s,0}(q)$ and there exists a positive angle between vectors in $C^{s,n+1}(q)$ and the complement of $C^{s,n}(q)$. Furthermore, $\cap_{n \geq 0} C^{s,n}(q)$ is a plane $P_q$ that is a graph of $L_q : \mathbb{E}^s \to \mathbb{E}^n$ that depends continuously on $q$.

**Proof.** from Lemma 1.2 we have $Df_j(q)C^u(\alpha^{-1}) \subset C^u(\alpha^{-1})$ so

$$C^{s,1}(f^j(q)) = [Df_{f^j(q)}]^{n-1}C^s(\alpha) \subset C^s(\alpha) = C^{s,0}(f^j(q)).$$
Inductively, we have $C^{s,n}(q) \subset \ldots \subset C^{s,0}(q)$. Applying Proposition 1.7 to the sequence of linear maps we get that $C^{s,n}(q)$ converges to a set that is a graph of a Lipschitz function and is a plane $P_q$. From estimates in Lemma 1.2 we can show the existence of the positive angle. This implies that for $q'$ near $q$ we have $P_{q'} \subset C^{s,n+1}(q') \subset C^{s,n}(q)$. So $P_q$ varies continuously with $q$. □

**Proof of Proposition 1.9.** We need to show that $\varphi^s$ is differentiable at $x_0 = \pi_s q$ with $D_r \varphi^s = L_q$ and $P_q = T_q W^s_r(0)$. Then $\varphi$ is $C^1$ since $P_q$ varies continuously. Also, $P_0 = \mathbb{E}^s \times \{0\}$ so $W^s_r(0)$ is tangent to $\mathbb{E}^s$ at 0.

Let $C^{s,n}(f^{n+1}(q), \delta) = C^{s,n}(f^{n+1}(q)) \cap \pi_s^{-1}(\mathbb{E}^s(\delta))$. Since $f^{-n-1}$ is differentiable there exists a $\delta > 0$ such that

$$f^{-n-1}(\{f^{n+2}(q)\} + C^{s,0}(f^{n+1}(q), \delta)) \subset \{q\} + C^{s,n}(q).$$

Since $\sigma^s$ and $f^{n+1}$ are continuous and $\varphi^s$ is $\alpha$-Lipschitz there exists an $\eta > 0$ such that if $|x - x_0| \leq \eta$, then

$$f^{n+1}(\sigma^s(x)) \in \{f^{n+1}(\sigma^s(x_0))\} + C^{s,0}(f^{n+1}(\sigma^s(s_0), \delta)).$$

So $\sigma^s(x) \in \{\sigma^s(x_0)\} + C^{s,n}(\sigma^s(x_0))$. Hence, given $n$ there exists an $\eta > 0$ such that $|x - x_0| < \eta$ implies that $\sigma^s(x) = \sigma^s(x_0) \in C^{s,n}(\sigma^s(x_0))$ so $\sigma^s$ is differentiable at $x_0$. □

**Proposition 1.11** $W^s_r(0)$ is $C^k$.

**Proof.** We have shown that this holds for $k = 1$. Assume that $k \geq 2$. Define $F(p, v) = (f(p), Df_p v)$. This $F$ is defined for $(p, v)$ with $p \in B(r)$ and $v \in T_p \mathbb{R}^n$. So $F$ is $C^{k-1}$ with $F(0, 0) = (0, 0)$ and

$$DF_{(p, v)}(p', v') = \begin{bmatrix} Df_p P' \ D^2 f_p(v, p') + Df_p v' \end{bmatrix}$$

Then

$$DF_{(0, 0)}(p', v') = \begin{bmatrix} Df_0 & 0 \\ 0 & Df_0 \end{bmatrix} \begin{bmatrix} p' \\ v' \end{bmatrix}.$$

So $(0, 0)$ is a hyperbolic fixed point. By induction $F$ has a $C^{k-1}$ stable manifold at $(0, 0)$. Thus, $TW^s_r(0)$ is $C^{k-1}$ and $W^s_r(0, 0)$ is $C^k$. □