Lecture 2
Circle Homeomorphisms

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Take $S^1 = [0,1]/\sim$ where the relation identifies 0 and 1 (or $S^1 = \mathbb{R}/\mathbb{Z}$). Let λ be the Lebesgue measure on S^1. Let $R_\alpha(x) = (x+\alpha) \mod 1$, so R is rotation by an angle $2\pi\alpha$. The collection of such maps is an abelian group.

If $\alpha \in \mathbb{Q}$, then $\alpha = \frac{p}{q}$ and $R_\alpha^q(x) = x$ for all $x \in S^1$. Hence, every point is periodic of period q. If $\alpha \in \mathbb{R} - \mathbb{Q}$, then every positive or negative orbit is dense in S^1. (Proof is an exercise.) This implies that if α is irrational, then the only invariant set of S^1 is S^1. As a reminder a set is *minimal* if there are no proper closed non-empty invariant sets. Hence, S^1 is a minimal set for rotation by an irrational.

For a general orientation preserving homeomorphism f of S^1 to itself we call a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ a *lift* of f if $\pi \circ F = f \circ \pi$ where $\pi(t) = t \mod 1$. We also say that f is a *factor* of F.

Remark 0.1 *For F a lift of a circle homeomorphism we have the following:*

1. *F is monotonic increasing.*

2. $F(t+1) = F(t) + 1$.

3. *If $f = R_\alpha$, then $F(x) = x + \alpha$ is a lift and all lifts are of the form $F(x) = x + k + \alpha$ for some $k \in \mathbb{Z}$.*

4. *If F_1 and F_2 are lifts, then there exists $k \in \mathbb{Z}$ such that $F_1(x) = F_2(x) + k$.*

Let F be a lift of f and

$$\rho_0(F,t) = \lim_{n\to\infty} \frac{F^n(t) - t}{n}.$$

We will show that the limit exists and is independent of t. Furthermore, for two lifts we will show that

$$\rho_0(F_1) - \rho_0(F_2) \in \mathbb{Z}.$$

Definition 0.2 *Let f be an orientation circle homeomorphism and F a lift of f. The rotation number of f is*

$$\rho(f) = \rho_0(F, t) \mod 1.$$

Remark 0.3

1. *By definition*

$$\rho_0(F^k) = \lim_{n\to\infty} \frac{F^{nk}(t) - t}{n} = k\rho_0(F).$$

Hence, $\rho(f^k) = k\rho(f) \mod 1$.

2. *If $f = R_\alpha$, then $F = t + k + \alpha$ and*

$$\rho_0(F) = \lim_{n\to\infty} \frac{F^n(t) - t}{n} = \lim_{n\to\infty} \frac{n(k+\alpha)}{n} = k + \alpha.$$

3. *The rotation number then is the average rotation of f.*

Theorem 0.4 *Let f be an orientation preserving homeomorphism with lift F. Then*

1. *For $t \in \mathbb{R}$, $\rho_0(F, t)$ exists and is independent of t,*

2. *$\rho(f)$ is independent of F, and*

3. *$\rho(f)$ depends continuously on f.*

Proof. Suppose the limit exists for some t. Let $s \in \mathbb{R}$. Then there exists a unique $l \in \mathbb{Z}$ such that $t + l \leq s < t + l + 1$. We know that

$$F(t+l) = F(t) + l \leq F(s) < F(t+l+1) = F(t) + l + 1.$$

Inductively, we have

$$F^k(t) + l \leq F^k(s) < F^k(t+l+1) = F^k(t) + l + 1$$

for all $k \in \mathbb{N}$. This implies that

$$\frac{F^k(t)+l-(t+l+1)}{k} \leq \frac{F^k(s)-s}{k} < \frac{F^k(t)+l+1-(t+l)}{k}$$

and

$$\frac{F^k(t)-t}{k} - \frac{1}{k} \leq \frac{F^k(s)-s}{k} < \frac{F^k(t)-t}{k} + \frac{1}{k}.$$

Now we show the limit exists. For the first case suppose that f has a periodic point. Then there exists $t \in \mathbb{R}$ and $r, k \in \mathbb{Z}$ such that $F^k(t) = t + r$. For all $m \in \mathbb{N}$ we have $F^{mk}(t) = t + rm$. Hence, if $s \in \mathbb{Z}$, then $F^{mk+s}(t) = F^s(t+rm) = F^s(t) + rm$. For $n \in \mathbb{Z}$ we have $n = mk + s$ where $0 \leq s < k$. Therefore,

$$\frac{F^n(t)-t}{n} = \frac{F^{mk+s}(t)-t}{n} = \frac{F^s(t)-t}{mk+s} + \frac{rm}{mk+s}$$

and

$$\lim_{n\to\infty} \frac{F^n(t)-t}{n} = \lim_{n\to\infty} (\frac{F^s(t)-t}{mk+s} + \frac{rm}{mk+s}) = \frac{r}{k}$$

since

$$\frac{F^s(t)-t}{mk+s} = \frac{F^s(t)-t}{n}$$

goes to zero as $n \to \infty$.

Before proceeding with the proof we point out that this implies that if f has a periodic point the rotation number exists and is rational.

The second case is if f has no periodic point. We will show this implies that $\rho(f)$ is irrational. Since f has no periodic points we know that for all $k \in \mathbb{Z}$ there exists an $r \in \mathbb{Z}$ such that $r < F^k(0) < r+1$. Fix $k \in \mathbb{N}$. Then

$$r + F^{(m-1)k}(0) < F^{mk}(0) < F^{(m-1)k}(0) + r + 1.$$

Since F is increasing and $F(0) \leq F(x)$ and $F(0)+1 \geq F(x)$ for all $x \in [0,1]$ we have

$$rm < F^{mk}(0) < m(r+1).$$

Hence,

$$\frac{r}{k} < \frac{F^k(0)}{k} < \frac{r+1}{k} \text{ and } \frac{r}{k} < \frac{F^{mk}(0)}{mk} < \frac{r+1}{k}.$$

Then

$$|\frac{F^{mk}(0)}{mk} - \frac{F^k(0)}{k}| < \frac{2}{k}.$$

Similarly, if we think of m as fixed we get

$$|\frac{F^{mk}(0)}{mk} - \frac{F^k(0)}{k}| < \frac{2}{m}.$$

Hence,

$$|\frac{F^k(0)}{k} - \frac{F^m(0)}{m}| \leq \frac{2}{k} + \frac{2}{m}$$

and this is a Cauchy sequence. Therefore, the limit exists.

Part (2) is clear. To prove part (3) we fix $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $2/n < \epsilon$. Fix F a lift of f. We know there exists an $r \in \mathbb{Z}$ such that $r \leq F^n(0) < r+1$ and $r-1 < F^n(t) - t < r+1$ for all $t \in \mathbb{R}$.

For g near f in the C^0 topology a lift of G exists such that $r-1 < G^n(t) - t < r-1$. Then

$$F^{nk}(0) = F^{nk}(0) - 0 = \sum_{j=0}^{k-1} F^n \circ F^{jn}(0) - F^{jn}(0).$$

This implies that

$$k(r-1) < F^{nk}(0) < k(r+1) \text{ and } k(r-1) < G^{nk}(0) < k(r+1).$$

Since

$$\rho_0(F) = \lim_{k\to\infty} \frac{F^{kn}(0)}{kn} \text{ and } \rho_0(G) = \lim_{k\to\infty} \frac{G^{kn}(0)}{kn}$$

we have

$$\frac{r-1}{n} < \rho_0(F) < \frac{r+1}{n} \text{ and } \frac{r-1}{n} < \rho_0(G) < \frac{r+1}{n}.$$

Hence, $|\rho_0(F) - \rho_0(G)| < 2/n < \epsilon$. □

Theorem 0.5 *If $\rho(f) \in \mathbb{Q}$, then f has a periodic point.*

Proof. Let F_0 be a lift of f. So $\rho_0(F_0) = k + p/k$. So $F(t) = F_0(t) - k$ is a lift of F and $\rho_0(F) = p/k$. Hence, $\rho_0(F^q - p) = \rho_0(F^q) - p = q\rho_0(F) - p = 0$. Let $G = F^q - p$.

We will show that G has a fixed point. If $G(0) = 0$ we are done. Suppose $G(0) > 0$. If $G^n(0) \in (0,1)$ for all $n \in \mathbb{N}$. Then G has a fixed by continuity of G and the fact that G is strictly increasing.

Suppose there exists a $k > 0$ such that $G^k(0) > 1$. Then $G^{2k}(0) = G^k \circ G^k(0) > G^k(1) = G^k(0) + 1 > 2$. Inductively, we get $G^{jk}(0) > j$. Hence,

$$\frac{G^{jk}(0)}{jk} > \frac{1}{k} \text{ and } \rho_0(G) \geq \frac{1}{k}$$

a contradiction. If $G(0) < 0$ we use the same reasoning as above. □

Example 0.6 *$F(t) = t + \epsilon \sin(2\pi nt)$ for $\epsilon \in (0, 1/(2\pi n))$. The point x is fixed for f if $F(x) = x + \epsilon \sin(2\pi x) = x + p$ so $\epsilon \sin(2\pi x) = p$. Since $\epsilon < 1$ we have $p = 0$ and $x = j/2n$.*

We now state some results without proof. For proofs see Katok and Hasselblatt 11.1 and Robinson 2.8.

Theorem 0.7 *All periodic points of f have the same period.*

Proposition 0.8 *If $h : S^1 \to S^1$ is an orientation preserving homeomorphism, then $\rho(h^{-1}fh) = \rho(f)$.*

Theorem 0.9 *If $\rho(f) = \alpha$ is irrational, then f is semi-conjugate to R_α with semi-conjugacy at most 2 to 1 on $\omega(x)$ and preserving the orientation.*

Theorem 0.10 *If $\rho(f)$ is irrational, then*

1. *$\omega(x)$ is independent of x,*
2. *$\omega(x)$ is a minimal set,*
3. *$\omega(x)$ is either all of S^1 or a Cantor set,*
4. *if $\omega(x) = S^1$, then f is conjugate to R_α, and*
5. *if $\omega(x) \neq S^1$, then h collapses the closure of each open interval I in the complement of $\omega(x)$ to a point.*

Theorem 0.11 *(Denjoy) If f is a C^2 orientation preserving diffeomorphism of the circle to itself and $\rho(f)$ is irrational, then f has a dense orbit and is topologically conjugate to R_α.*

Theorem 0.12 *(Denjoy) For α irrational there exists a C^1 orientation preserving diffeomorphism of the circle to itself such that $\rho(f) = \alpha$ and $\omega(x) \neq S^1$.*