We noted that the stable and unstable manifolds depend continuously on one another (since the splitting is continuous) and there is a minimum positive angle between the subspace $E^s$ and $E^u$. Together with the smoothness of the stable and unstable manifolds this implies the following:

**Proposition 0.1** If $\Lambda$ is a hyperbolic set there exists an $\epsilon > 0$ such that for all $x, y \in \Lambda$ the set $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of at most one point, denoted $[x, y]$, and there exists a $\delta > 0$ such that whenever $d(x, y) < \delta$ the point $[x, y]$ exists. Furthermore, there exists $C_p > 1$ such that if $d(x, y) < \delta$ and $x, y \in \Lambda$, then $d^s(x, [x, y]) \leq C_p d(x, y)$ and $d^u([x, y], y) \leq C_p d(x, y)$.

The next definition gives a class of hyperbolic sets that is the best understood and most used.

**Definition 0.2** A hyperbolic set $\Lambda$ is locally maximal (or isolated) for $f \in \text{Diff}(M)$ if there exists a neighborhood $U$ of $\Lambda$ in $M$ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$. Such a neighborhood is called an isolating neighborhood.

We will see that the next definition is equivalent to local maximality.

**Definition 0.3** A hyperbolic set $\Lambda$ has a local product structure if there exists $\delta > 0$ and $\epsilon > 0$ such that for any points $x, y \in \Lambda$ where $d(x, y) < \delta$ the set $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of one point contained in $\Lambda$. 
Examples of hyperbolic sets that have a local product structure (and as we will see are then locally maximal) are the horseshoe, solenoid, and Anosov diffeomorphisms. An example of a hyperbolic set that does not have a local product structure is the following: Let \( \Lambda \) consist of a hyperbolic fixed point together with the orbit of a transverse homoclinic point. We showed in the last lecture that this set is hyperbolic and does not possess a local product structure.

**Proposition 0.4** A hyperbolic set \( \Lambda \) is locally maximal if and only if it has a local product structure.

**Proof.** We may assume that \( \Lambda \) has an adapted metric on. Assume that \( U \) is an isolating neighborhood of \( \Lambda \). Then there exists an \( \epsilon > 0 \) such that \( W^s_\epsilon(x) \cap W^u_\epsilon(y) \) consists of one point. For \( \epsilon \) sufficiently small we may assume that \( \bigcup x \in \Lambda B_\epsilon(x) \subset U \). Then there exists a \( \delta > 0 \) such that \( [x, y] = W^s_\epsilon(x) \cap W^u_\epsilon(y) \) consists of one point and that all iterates of this point stay within \( \epsilon \) of \( \Lambda \). Hence, \( z \in \bigcup_{n \in \mathbb{Z}} f^n(U) = \Lambda \) and \( \Lambda \) has a local product structure.

Now assume that \( \Lambda \) has a local product structure. Fix constants \( \delta > 0 \) and \( \epsilon > 0 \) from the local product structure. Fix \( \alpha \in (0, \delta/3) \) such that if \( x \in \Lambda \) and \( z \in W^u_\alpha(x) \), then \( f(z) \in W^u_{\delta/3}(f(x)) \).

Assume that \( x_0 \in \Lambda \) and \( z \in W^u_\alpha(x_0) \) such that for all \( n > 0 \) there exists a point \( y_n \in \Lambda \) and \( d(f^n(z), y_n) < \alpha/c_p \), where \( C_p \) is defined as in Proposition 0.1. Then \( d(f(x_0), y_1) \leq d(f(x_0), f(z)) + d(f(z), y_1) < \delta/3 + \alpha/C_p < \delta \). Let \( x_1 = W^s_\epsilon(y_1) \cap W^u_\epsilon(f(x_0)) \in \Lambda \) and \( f(z) \in W^u_\alpha(x_1) \).

Similarly, we have a point \( x_2 = [y_2, f(x_1)] \in \Lambda \) and \( f^2(z) \in W^u_\alpha(x_2) \). Inductively, we then have points \( x_n \in [y_n, f(x_{n-1})] \in \Lambda \) and \( f^n(z) \in W^u_\alpha(x_n) \) for all \( n > 0 \). Let \( z_n = f^{-n}(x_n) \). We then have \( z_n \to z \) as \( n \to \infty \). Since \( \Lambda \) is closed we know that \( z \in \Lambda \). Similarly, if \( z \in W^s(x_0) \) and \( f^{-n}(z) \) stays close to \( \Lambda \) for all \( n > 0 \), then \( z \in \Lambda \).

Now assume that \( O(z) \) stays close to \( \Lambda \). Then \( \Lambda \cup O(z) \) is a hyperbolic set (this can be proved by methods from the last lecture). So there exists \( x_n \in \Lambda \) for all \( n \in \mathbb{Z} \) such that \( d(f^n(z), x_n) < \alpha \). If \( y = [z, x_0] \), then \( O^+(y) \) stays close to \( \Lambda \). If \( y' = [x_0, z] \), then \( O^-(y') \) stays close to \( \Lambda \). By the previous arguments we know that \( y, y' \in \Lambda \) Hence, \( z = [y, y'] \) is contained in \( \Lambda \) and \( \Lambda \) is locally maximal. □