Theorem 0.1 (Spectral Decomposition) Let $M$ be a Riemannian manifold, $f \in \text{Diff}(M)$, and $\Lambda$ a locally maximal hyperbolic set for $M$. Then there exist disjoint closed sets $\Lambda_1, \ldots, \Lambda_m$ such that $\text{NW}(f|\Lambda) = \bigcup_{i=1}^m \Lambda_i$ and a permutation $\sigma$ such that $f(\Lambda_i) = \Lambda_{\sigma(i)}$. Furthermore, if $f^k(\Lambda_i) = \Lambda_i$, then $f^k$ is topologically mixing restricted to $\Lambda_i$.

Proof. We define a relation $\sim$ on the periodic points of $f$ restricted to $\Lambda$ by $x \sim y$ if $W^s(x) \cap W^u(y) \neq \emptyset$ and $W^s(y) \cap W^u(x) \neq \emptyset$. This relation is trivially reflexive and symmetric. Now suppose that $x \sim y$ and $y \sim z$. By the Inclination Lemma we see that $x \sim z$. We know by the Local Product Structure that if periodic points are uniformly $\delta$-close that they are related. By compactness of $\Lambda$ we then have equivalence classes under $\sim$ that are pairwise disjoint and uniformly $\delta$-apart. Hence, there are only finitely many equivalence classes. Let $\Lambda_i$ be the closure of one of the equivalence classes.

So there exists disjoint closed sets $\Lambda_1, \ldots, \Lambda_m$. Furthermore, we know that the $\Lambda_i$ are permuted by $f$.

We claim that $\text{NW}(f|\Lambda) = \overline{\text{Per}}(f|\Lambda)$. Let $U$ be an isolating neighborhood of $\Lambda$. Fix $\epsilon > 0$ small such that $\bigcap_{x \in \Lambda} B_{\epsilon/3}(x) \subset U$. Fix $x \in \text{NW}(f|\Lambda)$. Then there exists an $N \in \mathbb{N}$ such that $f^N(B_{\epsilon/3}(x)) \cap B_{\epsilon/3}(x) \neq \emptyset$. So there exists some $y \in B_{\epsilon/3}(x) \cap \Lambda$ such that $d(f^N(y), y) < 2\epsilon/3$. By the Anosov Closing Lemma there exists a periodic point $z$ such that $d(f^n(z), f^n(y)) < 2\epsilon/3$ for all $n \in \{0, \ldots, N-1\}$. Hence, $z \in \text{Per}(f|\Lambda)$ and we know that $d(x, z) \leq d(x, y) + d(y, z) \leq \epsilon/3 + 2\epsilon/3 = \epsilon$. So $\text{NW}(f|\Lambda) = \overline{\text{Per}}(f|\Lambda)$ as claimed.

We now show that $f^k|\Lambda_i$ is topologically mixing. If $p \in \Lambda_i$ is periodic and $p \sim q$, then there exists a point $z \in W^u(p) \cap W^s(q)$. Let $N$ be the com-
mon period of $p$ and $q$. So $W^u(p)$ accumulates on $W^u(q)$ by the Inclination Lemma. So $W^u(p)$ is dense in $\Lambda_i \cap \text{Per}(f|_\Lambda)$ hence in $\Lambda_i$.

Let $U$ and $V$ be open in $\Lambda_i$. So there exist periodic points $p \in U$. Since $U$ is open there exists some $\delta > 0$ such that $W^u_\delta(p) \subset U$. So there exists some $m_0 \in \mathbb{N}$ such that $V \cap \bigcup_{i=0}^{m_0} f^i(W^u_\delta(p)) \neq \emptyset$. Since $f^j(W^u_\delta(p))$ is a neighborhood of $f^k(p)$ in $W^u(f^k(p))$ there exists $m_1, \ldots, m_{n-1} \in \mathbb{N}$ such that $V \cap \bigcup_{i=0}^{m_{n-1}} f^i(W^u_\delta(p)) \neq \emptyset$. Take $M = \max_{0 \leq i \leq n-1} m_i(n+1)$. Then for $m \geq M$ we have $V \cap \bigcup_{i=0}^{m} f^i(W^u_\delta(p)) \neq \emptyset$ and $V \cap f^m(U) \neq \emptyset$. □

**Corollary 0.2** If $\Lambda$ is a locally maximal hyperbolic set and mixing, then $\Lambda = \text{Per}(f|_\Lambda)$, $\Lambda \subset W^s(p)$, and $\Lambda \subset W^u(p)$ for all $p \in \text{Per}(f|_\Lambda)$.

**Corollary 0.3** If $\Lambda$ is a locally maximal hyperbolic set, then $f|_\Lambda$ is topologically transitive if and only if $\sigma$ is cyclic.

**Corollary 0.4** If $f$ is Anosov and $\text{NW}(f) = M$, then $f$ is topologically mixing.

**Definition 0.5** A diffeomorphism $f$ is Axiom A if $\text{NW}(f)$ is hyperbolic and $\text{NW}(f) = \text{Per}(f)$.

**Remark 0.6** Axiom A implies that $\text{NW}(f|_{\text{NW}(f)}) = \text{NW}(f)$. This is not true in general (exercise), even if $\text{NW}(f)$ is hyperbolic.

**Definition 0.7** If $f$ is Axiom A, then a transitive component $\Lambda_i$ in the Spectral Decomposition Theorem are called the basic sets.

For $f$ Axiom A and $\Lambda_i$ and $\Lambda_j$ basic sets we define $\Lambda_j \ll \Lambda_k$ if $(W^u(\Lambda_j) - \Lambda_j) \cap (W^s(\Lambda_k) - \Lambda_k) \neq \emptyset$. A **k-cycle** is a sequence of basic sets $\Lambda_{j_1}, \ldots, \Lambda_{j_k}$ with

$$\Lambda_{j_1} \ll \ldots \ll \Lambda_{j_k} \ll \Lambda_{j_1}.$$

**Definition 0.8** A diffeomorphism $f$ is Axiom A with no cycles if $f$ is Axiom A and there are no cycles among basic sets.

**Definition 0.9** A diffeomorphism is structurally stable if there exists a neighborhood $\mathcal{N}$ of $f$ in $\text{Diff}(M)$ such that all $g$ in $\mathcal{N}$ are topologically conjugate to $f$ on $M$. 

2
Definition 0.10 A diffeomorphism $f$ satisfies Axiom A plus strong transversality if $f$ is Axiom A and $W^s(x) \cap W^u(y) = W^s(x) \cap W^u(y)$ for all $x, y \in NW(f)$.

The next theorem is a remarkable achievement in mathematics. The proof was proven in part by J. Robbin (’71), C. Robinson (’76), and R. Mañé (’88).

Theorem 0.11 (Structural Stability) A diffeomorphism $f$ is structurally stable if and only if $f$ satisfies Axiom A plus strong transversality.