Let $\Lambda$ be a locally maximal hyperbolic set and $\epsilon, \delta > 0$ constants for the local product structure where $\delta < \epsilon/2$ and $\epsilon$ is also a constant for expansion on $\Lambda$. For $A \subset \Lambda$ denote $\text{int}(A)$ as the interior of $A$ relative to $\Lambda$ and $\partial A$ as the boundary in $\Lambda$.

Let $W^s(x, A) = W^s(x) \cap A$ and $W^u(x, A) = W^u(x) \cap A$. For $0 < \eta \leq \epsilon$ we set $W^s_\eta(x) = \{y \in W^s_\epsilon(x) | d(x, y) \leq \eta\}$ and $\text{int}(W^s_\eta(x)) = \{y \in W^s_\epsilon(x) | d(x, y) < \eta\}$.

**Proposition 0.1** There exists a constant $\rho \in (0, \delta/2)$ such that for all $x \in A$ the bracket is a homeomorphism of

$$(\text{int}(W^u_\rho(x)) \cap \Lambda) \times (\text{int}(W^s_\rho(x)) \cap \Lambda)$$

onto an open neighborhood of $x$ in $\Lambda$.

**Proof.** Let $V_\delta(x) = \{y \in \Lambda | d(x, y) < \delta\}$. Define

$$\Pi_s : V_\delta(x) \to W^s_\epsilon(x, \Lambda) \text{ by } y \mapsto [x, y], \text{ and}$$

$$\Pi_u : V_\delta(x) \to W^u_\epsilon(x, \Lambda) \text{ by } y \mapsto [y, x].$$

The bracket is uniformly continuous on compact neighborhoods of the diagonal in $\Lambda \times \Lambda$ so there exists a $\rho \in (0, \delta/2)$ such that for all $x, y, z \in \Lambda$, $d(x, y) < \rho$, and $d(x, z) < \rho$ implies that $d(x, [y, z]) < \delta$. So if

$$(y, z) \in (\text{int}(W^u_\rho(x)) \cap \Lambda) \times (\text{int}(W^s_\rho(x)) \cap \Lambda),$$
then $\Pi_s$ and $\Pi_u$ are defined and $(\Pi_u, \Pi_s)$ is the inverse for $[\cdot, \cdot]$. So the bracket is a homeomorphism onto the open set

$$\Pi_s^{-1}(\text{int}(W^s_\rho(x)) \cap \Lambda) \cap \Pi_u^{-1}(\text{int}(W^u_\rho(x)) \cap \Lambda).$$

☐

**Definition 0.2** A subset $R$ of $\Lambda$ is a rectangle of $\text{diam}(R) < \delta$ provided that for all $x, y \in R$ the point $[x, y]$ is defined and contained in $R$. A rectangle is proper if $\text{int}(R) = R$.

From the previous proposition we have the following result

**Proposition 0.3** If $R$ is a rectangle and $\text{diam}(R) < \rho$ with $x \in R$, then

$$\text{int}(R) = [\text{int}(W^u(x, R)), \text{int}(W^s(x, R))],$$

$$\partial R = [\partial W^u(x, R), W^s(x, R)] \cap [W^u(x, R), \partial W^s(x, R)].$$

**Corollary 0.4** Let $R$ be a rectangle and $\text{diam}(R) < \rho$. Then

1. $x \in \text{int}(R)$ if and only if $x \in \text{int}(W^u(x, R)) \cap \text{int}(W^s(x, R))$,

2. if $x \in \text{int}(R)$, then

$$\text{int}(W^s(x, R)) = W^s(x, \text{int}(R))$$

$$\text{int}(W^u(x, R)) = W^u(x, \text{int}(R))$$

and

3. if $R$ is closed, then $\partial R = \partial^s R \cup \partial^u R$, where

$$\partial^s R = \{x \in R \mid x \notin \text{int}(W^u(x, R))\} = \{x \in R \mid W^s(x, R) \cap \text{int}(R) = \emptyset\}$$

and

$$\partial^u R = \{x \in R \mid x \notin \text{int}(W^s(x, R))\} = \{x \in R \mid W^u(x, R) \cap \text{int}(R) = \emptyset\}.$$

**Proof.** Parts (1) and (2) are clear from the previous arguments. For part (3) we see that for all $z \in R$ we have $[z, y] = y$ whenever $y \in W^s(z, R)$ and $[y, z] = y$ whenever $y \in W^u(z, R)$. Hence, 

$$\left[\{z\} \cap \text{int}(W^u(z, R)), \text{int}(W^s(z, R))\right] = \text{int}(R) \cap \text{int}(W^u(z, R))$$

$$= \text{int}(R) \cap W^s(z, R),$$

and

$$\left[\{z\} \cap \text{int}(W^u(z, R)), \text{int}(W^s(z, R))\right] = \text{int}(R) \cap (\text{int}(W^u(z, R)))$$

$$= \text{int}(R) \cap W^u(z, R).$$

☐
Definition 0.5 A Markov Partition of $\Lambda$ for $f$ is a finite collection of rectangles $\{R_1, ..., R_n\}$ with the diameter of each rectangle less than $\rho$ such that the following are satisfied:

1. if $x \in \text{int}(R_i)$, $f(x) \in \text{int}(R_j)$, then $f(W^s(x, R_i)) \subset W^s(f(x), R_j)$, and
2. if $x \in \text{int}(R_i)$, $f(x) \in \text{int}(R_j)$, then $f^{-1}(W^u(x, R_i)) \subset W^u(f^{-1}(x), R_j)$.

Let $A$ be the matrix of 0’s and 1’s such that $a_{ij} = 1$ if $f(\text{int}(R_i)) \cap \text{int}(R_j)$ and 0 else. Let $\Sigma_A$ be the SFT associated with $A$. Define $h : \Sigma_A \rightarrow \Lambda$ to be the itinerary map. This will be finite-to-one and hence have the same entropy as $f|_\Lambda$.

Theorem 0.6 Every locally maximal hyperbolic set has a Markov Partition.

Before proving the above theorem we list another related result

Theorem 0.7 (F.) For every hyperbolic set $\Lambda$ and a neighborhood $V$ of $\Lambda$ there exists a hyperbolic set $\tilde{\Lambda} \subset \bigcap_{n \in \mathbb{Z}} f^n(V)$ such that $\Lambda \subset \tilde{\Lambda}$ and $\tilde{\Lambda}$ has a Markov partition.

Proof. Let $\beta \in (0, \rho/2)$ where $\rho < \delta/2 < \epsilon$. From the Shadowing Lemma there exists some $\alpha \in (0, \beta)$ such that every bi-infinite $\alpha$-pseudo orbit in $\Lambda$ is $\beta$-shadowed by a unique point in $\Lambda$. Since $\Lambda$ is compact there exists some $\gamma \in (0, \alpha/2)$ such that for all $x, y \in \Lambda$, $d(x, y) < \gamma$ we have $d(f(x), f(y)) < \alpha/2$.

Let $p_1, ..., p_k$ be a set of $\gamma$-dense points (so $\bigcup_{i=1}^k B_\gamma(p_i) \supset \Lambda$). Define $A$ a $k \times k$ matrix by $a_{ij} = 1$ if $f(p_i) \in B_\alpha(p_j)$ and $a_{ij} = 0$ else. Let $\Sigma_A$ be the SFT associated with $A$. For all $s \in \Sigma_A$ the sequence $\{p_{s_i}\}_{i \in \mathbb{Z}}$ is an $\alpha$-pseudo orbit and $\beta$-shadowed by a unique $x = \theta(s)$. This defines a map $\theta : \Sigma_A \rightarrow \Lambda$.

Proposition 0.8 The map $\theta$ is a semi-conjugacy of $\sigma_A$ and $f$.

Proof. If $x$ $\beta$-shadows the pseudo orbit $\{p_{s_i}\}$, then $f(x)$, $\beta$-shadows $\{p_{(s)s_i}\}$ so $f\theta = \theta\sigma$. Continuity follows from the Shadowing Lemma and expansiveness of $\Lambda$. Indeed, suppose that $s^n \rightarrow l$, $t^n \rightarrow l$, and $\theta(s^n)$ and $\theta(t^n)$ have different limits, denoted $s$ and $t$. We know for all $i \in \mathbb{Z}$ and all $n \in \mathbb{N}$ that $d(f^i(\theta(s^n)), p_{s_i}) < \beta$. So passing to the limit we have for all $i \in \mathbb{Z}$ that

$$d(f^i(s), p_i) \leq \beta, \text{ and } d(f^i(t), p_i) \leq \beta.$$
Hence, for all $i \in \mathbb{Z}$ we have $d(f^i(s), f^i(t)) \leq 2\beta < \delta/2 < \epsilon$, but $\epsilon$ is an expansive constant for $\Lambda$, a contradiction. Hence, $\theta$ is continuous. \hfill $\Box$

To continue to proof of the theorem we will see that $\theta(C_i^0)$ is a rectangle and the image of the rectangles cross correctly. However, they may overlap.

**Proposition 0.9** The map $\theta$ is a morphism of the local product structure. More precisely, for all $t \in \Sigma_\Lambda$,

\[
\theta(W^s_{1/3}(t)) \subset W^s_\epsilon(\theta(t)),
\]

\[
\theta(W^u_{1/3}(t)) \subset W^u_\epsilon(\theta(t)),
\]

if $d(t, t') < 1/2$, then $d(\theta(t), \theta(t')) < \rho \leq \delta$, and $\theta([t, t']) = [\theta(t), \theta(t')]$.

**Proof.** Note that $d(t, t') < \text{frm[0]}/2$ implies that $t_0 = t'_0$. Also we have

\[
d(\theta(t), \theta(t')) \leq d(\theta(t), p_{t_0}) + d(p_{t_0}, \theta(t')) \\
\leq 2\beta < \rho.
\]

So $[\theta(t), \theta(t')]$ is defined. Let $t \in \Sigma_\Lambda$ and $t' \in W^u_{1/3}(t)$. Then $t'_i = t_i$ for all $i \geq 0$. Hence, for all $i \geq 0$ we have

\[
d(f^i(\theta(t)), f^i(\theta(t'))) \leq d(f^i(\theta(t))p_i) + d(p_i, f^i(\theta(t'))) \\
\leq 2\beta < \epsilon.
\]

So $\theta(t') \in W^s_\epsilon(\theta(t))$. Similarly, we can show that $\theta(W^u_{1/3}(t)) \subset S^u_\epsilon(\theta(t))$. From the definition of the bracket we then have

\[
\theta([t, t']) = \theta(W^s_{\text{frm}[0]/3}(t) \cap W^u_{1/3}(t')) \\
\subset W^s_\epsilon(\theta(t)) \cap W^u_\epsilon(\theta(t')) \\
= [\theta(t), \theta(t')] .
\]

$\Box$

**Lemma 0.10** Let $1 \leq i \leq k$ and $t \in C_i = \{t' \in \Sigma_\Lambda | t_0' = i\}$. Then $T_i = \theta(C_i)$ is a closed rectangle, $\text{diam}(T_i) < \rho$,

\[
\theta(W^s(t, C_i)) = W^s_\epsilon(\theta(t), T_i), \quad \text{and}
\]

\[
\theta(W^u(t, C_i)) = W^u_\epsilon(\theta(t), T_i).
\]
Proof. We know that $C_i$ is a rectangle of diameter less than or equal to $1/2$ in $\Sigma_A$. Then the previous proposition says that $T_i$ is a rectangle and $\text{diam}(T_i) < \rho$. The set $T_i$ is closed since $C_i$ is compact and $\theta$ is continuous. Furthermore, $W^s(t, C_i) = W^s_{1/3}(t)$ so

$$
\theta(W^s(t, C_i)) \subset W^s_{\epsilon}(\theta(t)) \cap T_i = W^s(\theta(t), T_i) \text{ and } \\
\theta(W^u(t, C_i)) \subset W^u_{\epsilon}(\theta(t)) \cap T_i = W^u(\theta(t), T_i).
$$

Let $x \in W^s(\theta(t), T_i)$ and $s \in \theta^{-1}(x) \cap C_i$. Then

$$
\theta([t, s]) = [\theta(t), \theta(s)] = [\theta(t), x] = x
$$

and $W^s(\theta(t), T_i) \subset \theta(W^s(t, C_i))$. Likewise, $W^u(\theta(t), T_i) \subset \theta(W^u(t, C_i))$. □

**Proposition 0.11** Is $s \in C_i$, $\sigma(s) \subset C_j$, and $x = \theta(s)$, then

$$
f(W^s(x, T_i)) \subset W^s(f(x), T_j), \text{ and } \quad f^{-1}(W^u(f(x), T_j)) \subset W^u(x, T_i).
$$

**Proof.** From the previous lemma we have

$$
f(W^s(x, T_i)) = f(W^s(\theta(s), T_i)) \\
= f(\theta(W^s(s, C_i))) \\
= \theta\sigma(W^s(s, C_i)) \\
= W^s(\theta(\sigma s), T_j) \\
= W^s(f(x0, T_j)).
$$

Similarly, we get the result for the unstable direction. □

**Corollary 0.12** Let $x \in T_i$. Then the following are satisfied:

1. There exists $1 \leq j \leq k$ such that $f(x) \in T_j$, $f(W^s(x, T_i)) \subset W^s(f(x), T_j)$, and $f^{-1}(W^u(f(x), T_j)) \subset W^u(x, T_i)$.

2. There exists $1 \leq m \leq k$ such that $f^{-1}(x) \in T_m$, $f(W^s(f^{-1}(x), T_m)) \subset W^s(x, T_i)$, and $f^{-1}(W^u(f^{-1}(x), T_m)) \subset W^u(f^{-1}(x), T_m)$.

**Proof.** Let $s \in C_i$ such that $\theta(s) = x$, $s_1 = j$, and $s_{-1} = m$. Then the result follows from the previous proposition. □
We now want to subdivide the $T_i$s so they do not overlap. If $\text{int}(T_i) \cap \text{int}(T_j) \neq \emptyset$ we let

$$\tau_{ij}^1 = \{ x \in T_i \mid W^*(x, T_i) \cap T_j \neq \emptyset, W^*(x, T_i) \cap T_j = \emptyset \},$$

$$\tau_{ij}^2 = \{ x \in T_i \mid W^*(x, T_i) \cap T_j \neq \emptyset, W^*(x, T_i) \cap T_j = \emptyset \},$$

$$\tau_{ij}^3 = \{ x \in T_i \mid W^*(x, T_i) \cap T_j = \emptyset, W^*(x, T_i) \cap T_j \neq \emptyset \},$$

and

$$\tau_{ij}^4 = \{ x \in T_i \mid W^*(x, T_i) \cap T_j = \emptyset, W^*(x, T_i) \cap T_j = \emptyset \}.$$

**Lemma 0.13** The $\tau_{ij}^m$ form a partition of $T_i$, $\tau_{ij}^1 = T_i \cap T_j$, and $\tau_{ij}^1, \tau_{ij}^1 \cup \tau_{ij}^2, \tau_{ij}^1 \cup \tau_{ij}^3$ are closed.

**Proof.** The partition follows from the previous results. Since $T_i$ and $T_j$ are closed we see that $\tau_{ij}^1$ is closed. The other two conditions also follow from the closure of $T_i$ and $T_j$. □

Now let $T_{ij}^m = \tau_{ij}^m$.

**Lemma 0.14** The sets $T_{ij}^m$ satisfy the following:

1. $T_{ij}^1 = T_i \cap T_j$,
2. $\text{int}(T_{ij}^l) \cap T_{ij}^m = \emptyset$ if $l < m$ or $(l, m) = (3, 2)$,
3. the $T_{ij}^m$ have disjoint interiors,
4. $T_{ij}^l \subset \tau_{ij}^l \cup \left( \bigcup_{m=1}^4 \partial T_{ij}^m \right)$, and
5. $T_{ij}^m$ is a rectangle and $\text{diam}(T_{ij}^m) < \rho$.

**Proof.** Numbers (1)-(3) are clear from the definitions and previous results. (For (2) a picture is sufficient.)

Since the interiors are disjoint we know that

$$\text{int}(T_{ij}^l) - \left( \bigcup_{m=1}^4 \partial T_{ij}^m \right) \subset T_i - \bigcup_{m \neq l} T_{ij}^m \subset \tau_{ij}^l.$$

So (4) follows.

For (5) we let the closure of the interior of the rectangles of diameter be less than $\rho$ is a rectangle by continuity of the bracket. So we need to show that $\tau_{ij}^m$ is a rectangle. Since $\tau_{ij}^m$ is closed under the bracket and the diameter is less than $\rho$ we know that $\tau_{ij}^m$ is a rectangle. □

6
For \( Z = \Lambda - \bigcup_{m=1}^{4} \bigcup_{i,j=1}^{k} \partial T_{ij}^{m} \) the previous lemma shows that \( Z \) is open and dense in \( \Lambda \) and

\[
Z \cap T_{ij}^{m} = Z \cap \text{int}(T_{ij}^{m}) = Z \cap \tau_{ij}^{m}.
\]

Let \( x \in Z \) and

\[
K(x) = \{ T_{i} \mid x \in T_{i} \},
\]

\[
K^{*}(x) = \{ T_{j} \mid \text{there exists } T_{i} \in K(x) \text{ such that } T_{i} \cap T_{j} \neq \emptyset \},
\]

\[
R(x) = \bigcap \{ \text{int}(T_{ij}^{m}) \mid T_{i} \in K(x), T_{j} \in K^{*}(x), x \in T_{ij}^{m} \}.
\]

By construction \( R(x) \) is an open rectangle and \( \text{diam}(R(x)) < \rho \) for all \( x \in Z \).

**Lemma 0.15** If \( x, y \in Z \), then \( R(x) = R(y) \) or \( R(x) \cap R(y) = \emptyset \).

**Proof.** Let \( z \in Z \cap R(x) \). By the definition of \( R(x) \) we see that \( K(z) \supset K(x) \). If \( T_{j} \in K(z) \) and \( x \in T_{i} \), then \( T_{j} \) is in \( K^{*}(x) \) and \( x \in \text{int}(T_{ij}^{m}) \) for some \( 1 \leq m \leq 4 \). Since \( z \in \text{int}(T_{ij}^{l}) \) we know from the previous lemma that \( l = m = 1 \). Therefore, \( x \in T_{ij}^{l} \) and \( T_{j} \in K(x) \).

Hence, \( K(z) = K(x) \) and \( K^{*}(x) = K^{*}(z) \) and \( R(z) = R(x) \) for \( z \in R(x) \cap Z \). For \( x, y \in Z \) and \( R(x) \cap R(y) \neq \emptyset \) we know there exists some \( z \in R(x) \cap R(y) \cap Z \). So \( R(x) = R(z) = R(y) \). □

**Lemma 0.16** Let \( x, y \in Z \cap f^{-1}(Z) \), \( R(x) = R(y) \), and \( y \in W_{\epsilon}^{s}(x) \). Then \( R(f(x)) = R(f(y)) \).

**Proof.** If \( f(x) \in T_{i} \), then there exists some \( s \in \Sigma_{A} \) with \( s_{0} = i \) such that \( f(x) = \theta(s) \). Let \( j = s_{-1} \). Since \( x \in \theta \sigma^{-1}(s) \), and \( \theta(W^{s}(\sigma^{-1}(s), C_{j})) = W^{s}(x, T_{j}) \) we see that there exists some \( t \in \Sigma_{A} \) such that \( y = \theta(t) \) and \( t \in W^{s}(\sigma^{-1}(s), C_{j}) \). So \( t_{1} = s_{0} = i \) and \( f(y) = \theta \sigma(t) \in T_{i} \). Then \( K(f(x)) \subset K(f(y)) \). Exchanging \( x \) and \( y \) we see that \( K(f(x)) = K(f(y)) \) and \( K^{*}(f(x)) = K^{*}(f(y)) \).

Fix \( T_{i} \in K(f(x)) \) and \( T_{j} \in K^{*}(f(x)) \). Since \( W^{s}(f(x), T_{i}) = W^{s}(f(y), T_{i}) \) we have \( W^{s}(F(x), T_{i}) \cap T_{j} \neq \emptyset \). This implies that \( W^{s}(f(y), T_{i}) \cap T_{j} \neq \emptyset \). Similarly, for the unstable we see that if \( T_{m} \) is in \( K(x) \) where

\[
f^{-1}(W^{u}(f(x), T_{i})) \subset W^{u}(x, T_{m}) \text{ and } W^{u}(f(x), T_{i}) \cap T_{j}
\]

contains a point \( z \), then there exists \( T_{p} \) in \( K(f^{-1}(z)) \) such that

\[
f(W^{s}(f^{-1}(z), T_{p})) \subset W^{s}(z, T_{j}).
\]
Let $t = [z, f(y)]$. Then $t \in T_i$ since $z$ and $f(y)$ are in $T_i$ as well as $W^u(f(y))$.

From the choice of $z$ we see that $f^{-1}(z)$ is in $T_p \cap W^u(x, T_m)$. Then $x \in \tau^1_{mp} \cup \tau^3_{mp}$ and $y \in \tau^1_{mp} \cup \tau^3_{mp}$, so $W^u(y, T_m) \cap T_p \neq \emptyset$.

If $y \in W^u(y, T_m) \cap T_p$ we have $[f^{-1}(z), u] = [f^{-1}(z), y] \in W^s(f^{-1}(z), T_p)$.

So $t = [z, f(y)] = f([f^{-1}(z), y]) \in W^s(z, T_j) \subset T_j$, and if $W^u(f(x), T_i) \cap T_j \neq \emptyset$, then $W^u(y, T_i) \cap T_j \neq \emptyset$.

Exchanging $x$ and $y$ the converse holds. Then

$$W^s(f(x), T_i) \cap T_j \neq \emptyset \text{ if and only if } W^s(f(y), T_j) \cap T_i \neq \emptyset, \text{ and }$$

$$W^u(f(x), T_i) \cap T_j \neq \emptyset \text{ if and only if } W^u(f(y), T_i) \cap T_j \neq \emptyset.$$

Since $f(x), f(y) \in Z$ we have $R(f(x)) = R(f(y))$. □

Let $\mathcal{R} = \{R_1, ..., R_s\} = \{\overline{R(x)} | x \in Z\}$.

**Proposition 0.17** The collection $\mathcal{R}$ is a Markov partitions of $\Lambda$ for $f$.

**Proof.** We know by construction that $\bigcup_{i=1}^s R_i = \Lambda$, the diam$(R_i) < \rho$ for all $1 \leq i \leq s$, the rectangles are proper, and the interior of the rectangles are disjoint. The crossing property follows since the $T_i$ rectangles crossed properly and the fact that the $R(x)$ were open (i.e. if $x \in \text{int}(R_i)$ and $f(x) \in \text{int}(R_j)$, then $f(W^s(x, R_i)) \subset W^s(f(x), R_j)$, and if $x \in \text{int}(R_i)$ and $f^{-1}(x) \in \text{int}(R_j)$, then $f^{-1}(W^u(x, R_i)) \subset W^s(f^{-1}(x), R_j)$). □