Let $f_\mu(x) = \mu x(1-x)$ be defined on the reals. This is called the quadratic map. If $f_\mu(x) = x$, then $x(\mu x+(1-\mu)) = 0$. So the fixed points are $x = 0 = p_0$ and $x = (\mu - 1)/\mu = p_\mu$. The derivative is $f'_\mu(x) = \mu - 2\mu x$. Therefore, the origin is attracting for $\mu \in (0,1)$ and repelling for $\mu > 1$. The fixed point $p_\mu = (\mu - 1)/\mu$ is attracting for $\mu \in (1,3)$ and repelling for $\mu > 3$.

**Proposition 0.1** If $\mu > 1$ and $x$ is not in $[0,1]$, then $f^j_\mu(x) \to -\infty$ as $j \to \infty$.

**Proof.** For $x < 0$ we have $f'_\mu(x) = \mu - 2\mu x > 1$. So $0 > x > f_\mu(x) > ...$ is strictly decreasing. If this sequence converges, then it would converge to a fixed point. Therefore, the sequence tends to minus infinity. If $x > 1$, then $f_\mu(x) < 0$ and the previous argument applies. □

**Proposition 0.2** If $\mu \in (1,2]$ and $x \in (0,1)$, then $f^j_\mu(x) \to p_\mu$ as $j \to \infty$. Thus, $W^s(p_\mu) = (0,1)$.

**Proof.** For $\mu \in (1,2]$ the maximum of the graph is at $x = 1/2$ and $f_\mu(1/2) = \mu/4 \leq 1/2$. Hence, $p_\mu \leq 1/2$ and $f_\mu$ is strictly increasing on $(0, \mu)$ and the graph is above the line $y = x$. Hence, for $x \in (0, p_\mu)$ the sequence $f^j_\mu(x)$ is strictly increasing and converges to $p_\mu$.

For $x \in (\mu, 1/2)$ the function $f_\mu$ is strictly increasing and below the graph. Thus, $f^j_\mu(x)$ is strictly decreasing and convergent to $p_\mu$ for $x \in (p_\mu, 1/2)$.

If $x \in (1/2, 1)$, then $f_\mu(x) \in (0, 1/2)$ and the result follows. □
For $\mu \in [0, 4]$ the interval $I = [0, 1]$ is forward invariant. For $\mu > 4$, the interval
\[
\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}, \frac{1}{2} \right) + \sqrt{\frac{1}{4} - \frac{1}{\mu}}
\]
maps outside the interval $[0, 1]$. Let
\[
x_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}, \quad x_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\mu}},
\]
$I_1 = (0, x_1)$, and $I_2 = (x_2, 1)$. Let
\[
I_{i_0,i_1,...,i_n} = \bigcap_{k=0}^{n} f^{-k}(I_{i_k}) = \{x | f^k_\mu \in I_{i_k} \text{ for } 0 \leq k \leq n\}
\]
where $i_k = 1$ or $2$. Let
\[
S_n = \bigcap_{k=0}^{n} f^{-k}(I) = \bigcap_{k=0}^{n-1} f^{-k}(I_1 \cup I_2) = \bigcup_{i_0,...,i_{n-1} \in \{1,2\}^n} I_{i_0,...,i_{n-1}}.
\]

**Theorem 0.3** Assume $\mu > 4$ and
\[
\Lambda_\mu = \{x | f^n_\mu(x) \in [0, 1] \text{ for all } n \geq 0\}.
\]

Then $\Lambda_\mu$ is a Cantor set.

The proof will follow from the next 3 lemmas.

**Lemma 0.4** Assume $\mu > 4$. For all $n \in \mathbb{N}$ the following are true.

1. For all $i_0, ..., i_{n-1} \in \{1, 2\}^n$ the set $I_{i_0,...,i_{n-2}} \cap S_n = I_{i_0,...,i_{n-2,1}} \cup I_{i_0,...,i_{n-2,2}}$ is the union of two nonempty disjoint closed intervals.

2. For two labelings $i_0, ..., i_{n-1} \neq i'_0, ..., i'_{n-1}$, we have $I_{i_0,...,i_{n-1}} \cap I'_{i'_0,...,i'_{n-1}} = \emptyset$. Hence, $S_n$ is the union of $2^n$ disjoint intervals.

3. The function $f_\mu$ maps the interval $I_{i_0,...,i_{n-1}}$ homeomorphically onto $I_{i_1,...,i_{n-1}}$.  


Proof. The proof proceeds by induction. For \( n = 1 \) we have shown the result. Assume the results hold for \( n \). Let \( I_{i_0,\ldots,i_{n-1}} \) be in \( S_n \). Then \( f_{I_{i_0,\ldots,i_{n-1}}} = I_{i_1,\ldots,i_{n-1}} \) in \( S_{n-1} \) and

\[
I_{i_1,\ldots,i_{n-1}} \supset I_{i_1,\ldots,i_{n-1}} \cap S_n = I_{i_1,\ldots,i_{n-1},1} \cup I_{i_1,\ldots,i_{n-1},2}.
\]

Hence,

\[
I_{i_0,\ldots,i_{n-1}} \cap S_{n+1} = f_{\mu}^{-1}(S_n) \cap I_{i_0,\ldots,i_{n-1}} = f_{\mu}^{-1}(S_n \cap I_{i_0,\ldots,i_{n-1}}) \cap I_{i_0}
\]

\[
= \left[ f_{\mu}^{-1}(I_{i_1,\ldots,i_{n-1},1}) \cap f_{\mu}^{-1}(I_{i_1,\ldots,i_{n-1},2}) \right] \cap I_{i_0}
\]

is the union of two disjoint closed intervals. So (1) follows. Since there are \( 2^n \) choices for \( S_n \) we have \( S_{n+1} \) has \( 2(2^n) = 2^{n+1} \) intervals giving (2). Since \( f_{\mu} \) is monotone on \( I_{i_0,\ldots,i_{n-1},j} \) it is homeomorphic onto \( I_{i_1,\ldots,i_{n-1},j} \) where \( j = 1 \) or 2. \( \square \)

We now show the length of the components of \( S_n \) goes to zero as \( n \to \infty \). Note this is easier for \( \mu > 2 + \sqrt{3} \). For \( \mu \in (4, 2 + \sqrt{5}] \) see Robinson section 2.4.3.

Lemma 0.5 For all \( x \in I_1, \cup I_2, |f_{\mu}'(x)| > 1 \) if and only if \( \mu > 2 + \sqrt{5} \).

Proof. We know that \( f_{\mu}'(0x) = \mu - 2\mu x \) and \( f_{\mu}''(x) = -2\mu < 0 \) so the minimum of \( |f_{\mu}'(x)| \) on \( I_j \) is at \( x_1 \) or \( x_2 \). For \( i = 1 \) or 2 we have \( |f_{\mu}'(x_i)| = \sqrt{\mu^2 - 4\mu} \). Furthermore, for positive values of \( \mu \) we have \( \sqrt{\mu^2 - 4\mu} > 1 \) if and only if \( \mu > 2 + \sqrt{5} \). \( \square \)

Lemma 0.6 Let \( \lambda_\mu = \inf\{|f_{\mu}'(x)| \mid x \in I_1 \cup I_2\} \). Then the length of \( I_{i_0,\ldots,i_{n-1}} \) is bounded by \( \lambda_\mu^{-n} \).

Proof. For \( n = 1 \) we have \( f_{\mu}(I_{i_0}) = I \) by the Mean Value Theorem there exists some \( c \in I_{i_0} \) such that \( f_{\mu}'(b) = f_{\mu}'(a) = f_{\mu}'(c)(b - a) \) where \( I_{i_0} = [a, b] \). So \( 1 = |f_{\mu}'(b) - f_{\mu}'(a)| = |f_{\mu}'(c)||b - a| \geq \lambda_\mu|b - a| = \lambda_\mu L(I_{i_0}) \). Hence, \( L(I_{i_0}) < \lambda_\mu^{-1} \).

Assume the result holds for \( n-1 \). Let \( I_{i_0,\ldots,i_{n-1}} \in S_n \). Then \( f_{\mu}(I_{i_0,\ldots,i_{n-1}}) = I_{i_1,\ldots,i_{n-1}} \in S_{n-1} \) and \( L(I_{i_1,\ldots,i_{n-1}}) \leq \lambda_\mu^{-(n-1)} \). Using the Mean Value Theorem as above we see that \( L(I_{i_1,\ldots,i_{n-1}}) \geq \lambda_\mu L(I_{i_0,\ldots,i_{n-1}}) \). \( \square \)
0.1 Symbolic Dynamics for Quadratic Maps

Define $h : \Lambda_\mu \to \Sigma_2^+$ by $h(x) = t = (t_0, \ldots)$ where $f_{\mu}^k(x) \in I_{t_k}$. Then $x \in f_{\mu}^{-k}(I_{t_k})$ for all $k$ and $x \in \bigcap_{k=0}^n f_{\mu}^{-k}(I_{t_k}) = I_{t_0, \ldots, t_n} \in S_{n+1}$. This is called the itinerary map.

**Theorem 0.7** Let $\mu > 4$ and $f_\mu$ be the quadratic map. Then the itinerary map $h : \Lambda_\mu \to \Sigma_2^+$ is a topological conjugacy.

**Proof.** We first see that $\sigma \circ h = h \circ f_\mu$. Let $x \in \Lambda_\mu$ so $s = h(x)$ and $t = h(f_\mu(x))$. Then $f_\mu^k(f_\mu(x)) \in I_{t_k}$ and $f_\mu^k(f_\mu(x)) = f_\mu^{k+1}(x) \in I_{s_{k+1}}$. Hence, $t_k = s_{k+1}$.

Next we show that $h$ is onto. Let $s \in \Sigma_2^+$. We know that $I_{s_0, \ldots, s_n} = \bigcap_{k=0}^n f_{\mu}^{-k}(I_{s_k})$ is non-empty, closed, and nested as $n$ increases. Therefore, $f_{\mu}^n(x) \in I_{s_k}$ for $0 \leq k < \infty$ and $h$ is onto. To see that $h$ is one-to-one we use the fact that the length of the intervals goes to zero as $n$ goes to infinity. Therefore, the $x$ above is unique.

To show that $h$ is continuous we take $x \in \Lambda_\mu$ and $t = h(x)$. Fix $\epsilon > 0$ and $n \in \mathbb{N}$ such that $1/2^n < \epsilon$. For $\delta > 0$ small enough if $y \in \Lambda_\mu$ and $0 < |y - x| < \delta$, then $y \in I_{t_0, \ldots, t_n}$. So if $s = h(y)$ we have $d(s, t) \leq 2^{-n} < \epsilon$. □