Lecture 6
Hyperbolic Toral Automorphisms

January 14, 2008

Let $A \in \text{GL}_n(\mathbb{Z})$ with $|\det(A)| = 1$. Since $A$ has integer entries we know that $A$ preserves the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, and since $|\det(A) = 1$ we know that $A^{-1} \in \text{GL}_n(\mathbb{Z})$. Then $A$ induces a map of $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, the $n$-torus. The map is denoted $f_A$, where

$$f_A(x) = (A_1x \mod 1, \ldots, A_nx \mod 1)$$

and

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}.$$ 

Such maps are called toral automorphisms. These are $C^\infty$ diffeomorphisms of $\mathbb{T}^n$. A toral automorphism is called a hyperbolic toral automorphism if none of the eigenvalues for $A$ lie on the unit circle.

**Example 0.1** The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has determinant 1 and eigenvalues $\lambda_1 = (3 + \sqrt{5})/2 > 1$ and $\lambda_2 = 1/\lambda_1$. Eigenvectors of $A$ are $v_{\lambda_1} = (1 + \sqrt{5})/2, 1)$ and $v_{\lambda_2} = ((1 - \sqrt{5})/2, 1)$. Since $A$ is symmetric we know that $v_{\lambda_1}$ and $v_{\lambda_2}$ are perpendicular.

**Theorem 0.2** For $A$ a hyperbolic toral automorphism of $\mathbb{T}^2$ we have the following:

1. Periodic points are dense, and
2. The eigenvalues are irrational and eigenvectors have irrational slopes.

Proof. (Exercise)

Remark 0.3 The above shows that the projections of $E^s$ and $E^u$ of $\mathbb{R}^2$ onto $T^2$ are each dense at every point. Where the tangent space $T_{T^2}x = E^s \oplus E^u$, $E^s = \{tv_2 \mid t \in \mathbb{R}\}$, and $E^u = \{tv_1 \mid t \in \mathbb{R}\}$. The unstable set $W^u(x)$ is then dense in $T^2$ for every $x$ and $f_A(W^u(x)) = W^u(f_A(x))$. Similarly, the stable set $W^s(x)$ is dense in $T^2$ and $f_A(W^s(x)) = W^s(f_A(x))$. Furthermore, for $x \in T^2$ and $v \in E^s_x$ we have $\|D(f_A^k)xv\| = \|A^kv\| = \lambda^kv$ and for $v \in E^u_x$ we have $|D(f_A^k)xv| = |A^{-k}v| = (1/\lambda^k)v = \lambda^kv$. Hence, vectors in $E^s_x$ are uniformly contracted under $f_A$ and vectors in $E^u_x$ are uniformly expanded under $f_A$. These will be characteristics of a general class of dynamical systems called hyperbolic dynamical systems.

0.1 Markov partitions

Markov partitions are a way to code a system to a symbolic system in a dynamic way. Using the stable and unstable sets through $(0, 0)$ we divide $T^2$ into 2 rectangles with disjoint interiors. The set $f_A(R_1)$ has 3 components, 2 in $R_1$ and one in $R_2$. The set $f_A(R_2)$ has two components one in $R_1$ and one in $R_2$. We code by the crossings of the components. We then have the following matrix.

\[
B = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

Where there is a 1 if $\Delta_i$ crosses $\Delta_j$ so the image of $\Delta_0$, $\Delta_1$, and $\Delta_3$ stretch across $\Delta_1$, $\Delta_1$, and $\Delta_3$, whereas each of $\Delta_3$ and $\Delta_4$ stretch across $\Delta_2$ and $\Delta_4$. (See figure on page 86 of Katok and Hasselblatt). We let $\Sigma_B$ be the subshift of finite type associated with the transition matrix $B$. Define $h : \Sigma_B \to T^2$ by

\[
h(s) = \bigcap_{n=0}^{\infty} (\bigcap_{j=-n}^{n} f_A^{-j}(\text{int}(R_{s_j}))).
\]

By uniform contraction and expansion estimates we see that $h(s)$ is well-defined. Furthermore, $f_A \circ h = h \circ \sigma_B$ so $h$ is a semi-conjugacy and almost
every point has a unique pre-image (those that avoid the boundaries of the rectangles), and $h$ is at most 4 to 1.

For $R_i$ a rectangle we set $W^\sigma(z, R_i)$ to be the connected component of $W^\sigma(z)$ in $R_i$, and $W^\sigma(z, \text{int} R_i) = W^\sigma(z, R_i) \cap \text{int} R_i$, where $\sigma = u$ or $s$.

**Definition 0.4** Let $f_A$ be a hyperbolic toral automorphism. A Markov partition for $f$ is a finite collection of rectangles $\mathcal{R} = \{R_j\}_{j=1}^m$ satisfying the following:

1. $\mathbb{T}^n = \bigcup_{j=1}^m R_j$.
2. If $i \neq j$, then $\text{int}(R_i) \cap \text{int}(R_j) = \emptyset$.
3. If $z \in \text{int}(R_i)$ and $f(z) \in \text{int}(R_j)$, then $f_A(W^u(z, R_i)) \supset W^u(f_A(z), R_j)$ and $f_A(W^s(z, R_i)) \subset W^s(f_A(z), R_j)$.
4. If $z \in \text{int}(R_i) \cap f^{-1}(\text{int}(R_j))$, then $\text{int}(R_j) \cap f_A(W^u(z, \text{int}(R_i))) = W^u(f_A(z), \text{int}(R_j))$ and $\text{int}(R_i) \cap f_A^{-1}(W^s(f_A(z), \text{int}(R_i))) = W^s(z, \text{int}(R_j))$.

**Theorem 0.5** Let $\mathcal{R} = \{R_j\}_{j=1}^m$ be a Markov partition for a hyperbolic toral automorphism on $\mathbb{T}^2$ with transition matrix $B$. Let $(\Sigma_B, \sigma)$ be the shift space associated with $B$ and $h : \Sigma_B \to \mathbb{T}^2$ be defined by

$$h(s) = \bigcap_{n=0}^{\infty} \left( \bigcap_{j=-n}^{n} f_A^{-j}(\text{int}(R_{s_j})) \right).$$

Then $h$ is a semi-conjugacy from $\sigma$ to $f_A$. In fact, $h$ is at most $m^2$-to-one.

**Proof.** By (4) in the definition of a Markov partition we know that $\bigcap_{j=k+1}^{k+1} f^{-j}(\text{int}(R_{s_j}))$ is a subrectangle that stretches across in the stable directions and stable width decreases exponentially. Then

$$\bigcap_{n=0}^{\infty} \left( \bigcap_{j=0}^{n} f_A^{-j}(\text{int}(R_{s_j})) \right) = W^s(p_s, R_{s_0})$$

for some $p_s \in R_{s_0}$. Similarly,

$$\bigcap_{n=0}^{-\infty} \left( \bigcap_{j=n}^{0} f_A^{-j}(\text{int}(R_{s_j})) \right) = W^u(p_u, R_{s_0}).$$
So $h(s)$ is a unique point.

To show continuity, onto, semi-conjugacy is similar to what we did for expansive endomorphisms.

To show $m^2$-to-one we look at points on the boundary. If $f^n(p) \in \partial^s(R_{s_n})$, then $f^j(p) \in \partial^s(R_{s_j})$ for all $j \geq n$. So there are at most $m$ choices for $s_n$. Since transitions in interior are unique, a choice of $s_n$ determines $s_j$ for all $j \geq n$. Similarly, for $\partial^u$ so there are $m^2$ choices. □