1 Horseshoe

At the turn of the 20th century Poincare investigated what happens when a stable and unstable set for a fixed point have a transverse intersection. This is called a transverse homoclinic point. The dynamics of such a system are very complicated. We will examine this in more detail later in the semester.

In the 1960’s Smale described a simple system with similar behavior called a horseshoe. Let \( D \) be a region in \( \mathbb{R}^2 \) consisting of 2 semi-circles and the unit square. Let \( R = [0, 1] \times [0, 1] = D_2 \cup D_3 \cup D_4 \) be the unit square. Suppose that \( f : D \to D \) stretches \( D_2 \) and \( D_4 \) uniformly in the vertical direction by some constant \( \mu > 2 \) and contract in the horizontal direction by \( \lambda < 1/2 \), \( f(D_1) \cup f(D_3) \subset D_1 \), and \( f(D_3) \subset D_5 \). So the shape of \( f(D) \) is a horseshoe.

Let \( R_0 = f(D_2) \cap R \) and \( R_1 = f(D_4) \cap R \). Note that \( f(R) \cap R = R_0 \cup R_1 \). Furthermore, the set \( f^2(R) \cap f(R) \cap R = f^2(R) \cap R \) consists of 4 vertical rectangles \( R_{ij} \), \( i, j \in \{0, 1\} \), of length \( \lambda_2 \) such that \( R_{ij} = R_i \cap f(R_j) \). For \( i_0, ..., i_n \in \{0, 1\} \) we let

\[
R_{i_0, ..., i_n} = R_{i_0} \cap f(R_{i_1}) \cap \cdots \cap f^n(R_{i_n}).
\]

This will be a vertical rectangle of length \( \lambda^n \) and \( f^n(R) \cap R \) consists of \( 2^n \) such rectangles.

Similar to the construction of \( \Lambda_{\mu} \) for quadratic maps we see that if \( t \in \Sigma_2^+ \), then \( R_t = \bigcap_{i=0}^{\infty} f^i(R_{i_t}) \) consists of \( x \times [0, 1] \) for some \( x \in [0, 1] \). Let \( H^+ = \bigcup_{t \in \Sigma_2^+} R_t \). Then \( H^+ \) consists of a Cantor set, \( C^+ \), cross the interval \([0, 1]\).
Using the preimage of $R$ we see that $f^{-1}(R_0) = f^{-1}(R) \cap D_2$ and $f^{-1}(R_2) = f^{-1}(R) \cap D_4$ are horizontal rectangles of width $\mu^{-1}$. For any sequence $i_{-m}, \ldots, i_{-1} \in \{0, 1\}$ we have $R_{i_{-m}, \ldots, i_{-1}} = \bigcap_{k=0}^{m} f^{-k}(R_{i_k})$ is a horizontal rectangle of width $\mu^{-m}$ and $H^- = \bigcap_{i=1}^{\infty} f^{-i}(R)$ is the product of the interval $[0, 1]$ with a Cantor set, $C^-$. The set $H = H^+ \cap H^- = \bigcap_{i=-\infty}^{\infty} f^i(R)$ is $C^+ \times C^-$. The map $\phi : \Sigma_2 \rightarrow H$ defined by $\phi(t) = \bigcap_{k=-\infty}^{\infty} f^k(R_{i_k})$ is a conjugacy.

Then the set $H$ has the same dynamic complexity as $\Sigma_2$. Specifically, we know that periodic points are dense, $\text{Per}_n(f|_H) = 2^n$, and there is a point with a dense orbit (so $f|_H$ is topologically transitive).

## 2 Solenoid

**Definition 2.1** A set $X$ is an attractor for a map $f$ if there exists a neighborhood $U$ of $X$ such that $f(U) \subset U$ and $X = \bigcap_{n \geq 0} f^n(U)$. The set $U$ is called an attracting set.

The solenoid is an attractor. Take $N = S^1 \times D_2$ to be the solid torus. Define $g : S^1 \rightarrow S^1$ by $g(t) = 2t \mod 1$ and

$$f(t, x, y) = (2t, \frac{x + \cos 2\pi t}{2}, \frac{y + \sin 2\pi t}{2}).$$

Let $N_k = \bigcap_{j=0}^{k} f^j(N)$.

**Claim 2.2** For all $t \in S^1$, the set $N_k \cap D(t)$ is the union of $2^k$ disks of radius $(1/4)^k$.

**Proof.** True for $k = 0$ and $k = 1$. Suppose it is true for $k - 1$. The set

$$N_k \cap D(t) = f(N_{k-1} \cap D(t/2)) \cup f(N_{k-1} \cap D(t/2 + 1/2)).$$

By induction each of these are $2^{k-1}$ disks of radius $(1/4)^{k-1}$. Since $f$ contracts by $1/4$ we have the desired result. □

Let $\Lambda = \bigcup_{j=0}^{\infty} f^j(N) = \bigcap_{j=0}^{\infty} N_j$.

**Proposition 2.3** The set $\Lambda$ has the following properties:

1. $\Lambda$ is connected.
2. $\Lambda$ is not locally connected.

3. $\Lambda$ is not path connected.

4. $\Lambda$ has topological dimension one.

**Proof.** For (1) we notice that the sets $N_k$ are compact, connected, and nested. Hence, so is $\Lambda$.

For (2) we take $0 < t_2 - t_1 < 1$. Then $D[t_1, t_2] \cap N_k$ is the union of $2^k$ twisted tubes. For any neighborhood $U$ of a point $p \in \Lambda$ there exist choices of $t_1$, $t_2$, an $k$ such that $U$ contains 2 tubes. Therefore, $\Lambda$ is not locally connected.

For (3) fix $p = (t_0, x_0, y_0) \in \Lambda$. Then there exists $q_k \in \Lambda \cap D(t_0)$ for each $k$ such that

1. For $k \geq 2$, the point $q_k$ is in the same component of $N_{k-1} \cap D(t_0)$ as $q_{k-1}$, and

2. any path from $p$ to $q_k$ that is in $N_k$ must go around $S^1$ at least $2^{k-1}$ times.

The sequence $\{q_k\}$ converges to a point $q \in \Lambda$, since $\Lambda$ is closed. We claim there is no path from $p$ to $q$. The point $q$ is in the same component of $N_k \cap D(t_0)$ as $q_k$ for any $k \in \mathbb{N}$. Hence, any path from $p$ to $q$ must intersect $N_k \cap D(t_0)$ at least $2^{k-1}$ and any path from $p$ to $q$ in $\Lambda$ must intersect $D(t_0)$ infinitely often and traverse $S^1$ between each intersection, a contradiction.

For (4) we notice that the set $\Lambda \cap D(t_0)$ is totally disconnected so has topological dimension 0. The set $\Lambda \cap D([t_1, t_2])$ is a product of $\Lambda \cap D(t_1)$ with an interval. So the topological dimension of $\Lambda$ is 1. $\square$

**Proposition 2.4** The map $f$ restricted to $\Lambda$ has the following properties:

1. The periodic points are dense.

2. There is a point with a dense orbit.

**Proof.** For (1) we note that $g^k(t_0) = t_0$ if and only if $2^k t_0 = t_0 + j$ so $t_0 = j/2^{k-1}$. So the periodic points of $g$ are dense in $S^1$. If $g^k(t_0) = t_0$, then $f^k(D(t_0)) \subset D(t_0)$. The set $D(t_0)$ is a disk so has a fixed point in $D(t_0)$. So there is a periodic point for $f$ in $D(t_0)$. 

3
Now take \( p \in \Lambda \) and let \( U \) be a neighborhood of \( p \). There exists a \( k \in \mathbb{N} \) and \( t_1, t_2 \in S^1 \) such that \( f^k(D([t_1, t_2])) \subset U \). By above \( f \) has a periodic point in \( D([t_1, t_2]) \) so in \( f^k(D([t_1, t_2])) \) and we are done.

We will see (2) at a later time. \( \square \)

2.1 Conjugacy of the Solenoid to an inverse limit

Let

\[
\Sigma^- = \{ s \in (S^1)^\mathbb{N} \mid g(s_{j+1}) = s_j \}.
\]

Define \( \sigma \) on \( \Sigma^- \) by \( \sigma(s) = t \) if

\[
t_j = \begin{cases} 
    s_{j-1} & \text{if } j \geq 1 \\
    g(s_0) & \text{if } j = 0.
\end{cases}
\]

If \( s \in \Sigma^- \), then \( g(s_{j+1}) = s_j \). So \( s_{j+1} \in g^{-1}(s_j) \) is one of the two preimages of \( s_j \). The pair \((\Sigma^-, \sigma)\) is called the inverse limit of \( g \).

Define the map \( h : \Lambda \to (S^1)^\mathbb{N} \) by \( h(p) = s \) where \( f^i(p) \in D(s_j) \) with \( s_j \in S^1 \) for all \( j \geq 0 \).

Theorem 2.5 The map \( h \) is a conjugacy from \((\Lambda, f)\) to \((\Sigma^-, \sigma)\).

Proof. We first show that \( h(\Lambda) \subset \Sigma^- \). Let \( h(p) = s \). Then \( f^{-j}(p) \in D(s_j) \) and \( f^{j-1}(p) \in D(s_{j+1}) \). So \( f(D(s_{j+1})) \cap D(s_j) \neq \emptyset \) and \( f(D(s_{j+1})) \subset D(s_j) \). Thus, \( g(s_{j+1}) = s_j \) for all \( j \) and \( s \in \Sigma^- \).

We now show that \( h \circ f = \sigma \circ h \). For \( p \in \Lambda \), \( h(p) = s \), and \( h(f(p)) = t \), we have \( f^{-(j+1)}(f(p)) = f^{-j}(p) \) and so is in \( D(t_{j+1}) \) and \( D(s_j) \). Therefore, \( t_{j+1} = s_j \) for all \( j \geq 0 \). Similarly, \( f(p) \) is in \( D(t_0) \) and \( f(D(s_0)) \) so \( t_0 = g(s_0) \). Hence, \( \sigma(s) = t \).

We show \( h \) is one-to-one. If \( h(p) = h(g) = s \), then \( p, q \in \bigcap_{i=0}^k f^i(D(s_j)) \) this is nested with radii going to zero. So we have \( p = q \).

To show that \( h \) is onto we see that if \( s \in \Sigma^- \), then \( g(s_{j+1}) = s_j \). So \( f(D(s_{j+1})) \subset D(s_j) \),

\[
f^j(D(s_j)) \subset f^{j-1}(D(s_{j-1})) \subset \ldots \subset D(s_0),
\]

and \( \bigcap_{j=0}^k f^j(D(s_j)) \) is a nested sequence of disks with nonempty intersection. \( \square \)