1 Almost Periodic

Definition 1.1 A set $A \subset \mathbb{N}$ (or $\mathbb{Z}$) is relatively dense if there exists a $k > 0$ such that $[n, n + k] \cap A \neq \emptyset$ for all $n$ in $\mathbb{N}$ (or $\mathbb{Z}$). A point $x \in X$ is almost periodic if for any neighborhood $U$ of $x$, the set $\{ i \in \mathbb{N} \mid f^i(x) \in U \}$ is relatively dense in $\mathbb{N}$.

This says an iterate comes back by time $k$ depending on $U$.

Proposition 1.2 If $X$ is a compact Hausdorff space and $f : X \to X$ is continuous, then $\overline{O^+(x)}$ is minimal for $f$ if and only if $x$ is almost periodic.

Proof. Let $x$ be almost periodic and $y \in \overline{O^+(x)}$. Let $U$ be a neighborhood of $x$. The there exists an open set $U' \subset X$, $x \in U' \subset U$ and $V \subset X \times X$ containing the diagonal such that if $x_1 \in U'$ and $(x_1, x_2) \in V$, then $x_2 \in U$.

Since $x$ is almost periodic there exists a $K \in \mathbb{N}$ such that for all $j \in \mathbb{N}$ we have $f^{j+k}(x) \in U'$ for some $0 \leq k \leq K$. Let $V' = \bigcap_{i=0}^K f^{-i}(V)$. So $V'$ is open and contains the diagonal. There exists a neighborhood $W$ of $y$ such that $W \times W \subset V'$. Choose $n$ such that $f^n(x) \in W$ and $k$ such that $f^{n+k} \in U'$ with $0 \leq k \leq K$. Then $(f^{n+k}(x), f^k(y)) \in V$ and $f^k(y) \in U$. So $x \in \overline{O^+(y)}$ and $\overline{O^+(x)}$ is minimal.

Suppose that $x$ is not almost periodic. Then there exists a neighborhood $U$ of $x$ such that $A = \{ i \mid f^i(x) \in U \}$ is not relatively dense. Thus, there exist $a_i \in \mathbb{N}$ and $k_i \in \mathbb{N}$ such that $k_i \to \infty$ and $f^{a_i+j}$ is not in $U$ for $j = 0, ..., k_i$. Let $y$ be a limit point of $f^{a_i}(x)$. Fix $j \in \mathbb{N}$. Then $f^{a_i+j}(x) \to f^j(y)$ as $j \to \infty$ and $f^{a_i+j}(x)$ is not in $U$ for $i$ sufficiently large. Thus, $f^j(y)$ is not in $U$ for all $j \in \mathbb{N}$. So $x$ is not in $\overline{O^+(y)}$ and $\overline{O^+(x)}$ is not minimal. □
2 Topological Transitivity

For this section we assume that $X$ is a second countable metric space.

**Definition 2.1** A dynamical system $(f, X)$ is topologically transitive if there exists a point $x \in X$ such that $\overline{O^+(x)} = X$.

**Remark 2.2** If $X$ has no isolated points $f$ is topologically transitive if and only if there exists a point $x \in X$ such that $\omega(x) = X$.

Depending on the source there are different definitions of topological transitivity. For “nice” spaces the definitions are all equivalent.

**Proposition 2.3** Let $f : X \to X$ be continuous and $X$ be a locally compact Hausdorff space. Suppose for any open sets $U$ and $V$ of $X$ there exists an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Then $f$ is topologically transitive.

**Proof.** Note that $\bigcup_{n \in \mathbb{N}} f^{-n}(V)$ is dense since the set intersects every open set. Let $\{V_i\}$ be a countable basis for the topology of $X$. Then $Y = \bigcap_i \bigcup_{n \in \mathbb{N}} f^{-n}(V_i)$ is a countable intersection of open, dense sets and so non-empty. The forward orbit of any point $y \in Y$ enters $V_i$, and is then dense in $X$. $\square$

**Proposition 2.4** Let $f : X \to X$ be a homeomorphism of a compact metric space and suppose $X$ has no isolated points. If there exists a point whose orbit is dense in $X$, then there exists a point whose forward orbit is dense in $X$.

**Proof.** Let $x \in X$ such that $\overline{O(x)} = X$. We know that $O(x)$ hits every open set $U$ infinitely many times since $X$ has no isolated points. Hence, there exists a sequence $n_k$, with $|n_k| \to \infty$ such that $f^{n_k}(x) \in B_{1/k}(x)$ for $k \in \mathbb{N}$. So $f^{n_k+l}(x) \to f^l(x)$ for any $l \in \mathbb{Z}$ by the continuity of $f$. Then there are infinitely many $n_k$ that are either positive or negative. So $O(x) \subset \overline{O^+(x)}$ or $O(x) \subset \overline{O^-(x)}$. In the first case we are done.

Now suppose that the second case holds. Let $U$ and $V$ be open sets in $X$. Then there exists $i, j \in \mathbb{Z}^-$ such that $i < j$, $f^i(x) \in U$, and $f^j(x) \in V$. So $f^{j-i}(U) \cap V \neq \emptyset$ and by the previous proposition we are done. $\square$
3 Topological Mixing

Definition 3.1 A system \((f, X)\) is topologically mixing if given any two open sets \(U\) and \(V\) there exists an \(N \in \mathbb{N}\) such that \(f^n(U) \cap V \neq \emptyset\) for all \(n \geq N\).

From the previous proposition this is a stronger statement than topologically transitive.

Example 3.2 The map \(R_\alpha\) is topologically transitive, but not topologically mixing.

Proposition 3.3 Any hyperbolic toral automorphism is topologically mixing.

Proof. We now for each \(x \in \mathbb{T}^2\) that \(W^u(x)\) is dense in \(\mathbb{T}^2\). Then for all \(\epsilon > 0\) we have \(\bigcup_{y \in W^u(x)} B_\epsilon(y) = \mathbb{T}^2\), so there is a finite subcover. Hence, there exists a length \(L(\epsilon)\) such that every segment \(S\) of an unstable manifold is \(\epsilon\)-dense.

Let \(U\) and \(V\) be open, non-empty in \(\mathbb{T}^2\). Choose, \(y \in V\) and \(\epsilon > 0\) such that \(\overline{B_\epsilon(y)} \subset V\). For \(x \in U\) there exists a \(\delta > 0\) such that \(W^u_\delta(x) \subset U\). Let \(\lambda \in \mathbb{R}\) be the expanding eigenvector of \(A\). Hence, \(|\lambda| > 1\) and there exists an \(N > 0\) such that \(|\lambda|^N \delta \leq L(\epsilon)\). Therefore, for all \(n \geq N\), \(f^n_A(U)\) is \(\epsilon\)-dense in \(\mathbb{T}^2\) and intersects \(V\). \(\square\)

Proposition 3.4 \(\Sigma_m\) and \(\Sigma^+_m\) are topologically mixing.

Proof. A basis for \(\Sigma_m\) consists of all balls \(B_{2^{-l}}(w)\). Hence, it is sufficient to show for all \(B_{2^{-l_1}}(w)\) and \(B_{2^{-l_2}}(w')\) there exists some \(N > 0\) such that

\[
\sigma^n B_{2^{-l_1}}(w) \cap B_{2^{-l_2}}(w') \neq \emptyset
\]

for all \(n \geq N\).

The set \(\sigma^n B_{frm-\epsilon^{-l_1}}(w)\) consists of all sequence with \(-n - l_1, \ldots, -n + l_1\) specified by the word \(w\). Therefore, when \(-n + l_1 < -l_2\) the intersection is non-empty.

The proof for \(\Sigma^+_m\) is similar. \(\square\)

Corollary 3.5 The Horseshoe is topologically mixing.
4 Expansiveness

**Definition 4.1** A homeomorphism $f$ of a metric space $X$ to itself is expansive if there exists a $\delta > 0$ such that if $d(f^n(x), f^n(y)) < \delta$ for all $n \in \mathbb{Z}$, then $x = y$. A non-invertible map is positively expansive if there exists a $\delta > 0$ such that if $d(f^n(x), f^n(y)) < \delta$ for all $n \geq 0$, then $x = y$. The number $\delta$ is called an expansive constant.

**Remark 4.2** The constant $\delta$ depends on the metric, but for $X$ a compact metric space the system is expansive independent of the metric.

**Example 4.3** Full shifts, one-sided full shifts, Horseshoes, hyperbolic toral automorphisms, Solenoind, and quadratic maps for $\mu > 4$ are expansive. The map $E_m$ is positively expansive. Circle rotations are not expansive.

**Proposition 4.4** Let $X$ be a compact metric space, $|X| = \infty$, and $f$ be a homeomorphism of $X$ to itself. Then for all $\epsilon > 0$ there exist distinct points $x, y \in X$ such that $d(f^n(x), f^n(y)) \leq \epsilon$ for all $n \geq 0$.

**Proof.** Fix $\epsilon > 0$. Let $E = \{ m \in \mathbb{N} |$ there exist $x, y \in X, x \neq y, d(x, y) \geq \epsilon, d(f^n(x), f^n(y)) < \epsilon$ for all $n = 1, ..., m\}$. Let $M$ be the supremum over $E$ if $E \neq \emptyset$. If $E = \emptyset$, then define $M = 0$. If $M = \infty$, then for all $k \in \mathbb{N}$ there exists points $x_k, y_k \in X$ and $m_k \geq k$ such that $d(x_k, y_k) \geq \epsilon$ and $d(f^n(x_k), f^n(y_k)) < \epsilon$ for all $n = 1, ..., m_k$. Let $(x', y')$ be a limit point of the sequence $(x_k, y_k)$. So $d(x', y') \geq \epsilon$ and

$$d(f^j(x'), f^j(y')) = \lim_{k \to \infty} d(f^j(x_{m_k}), f^j(y_{m_k})) \leq \epsilon$$

for all $j \in \mathbb{N}$, since $f^j$ is continuous. Hence, $x_0 = f(x')$ and $y_0 = f(y')$ are the desired points.

Suppose $M < \infty$. We know there exists a $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < \epsilon$ for all $0 \leq n \leq M$. Be the definition of $M$ we know if $d(x, y) < \epsilon$, then $d(f^{-1}(x), f^{-1}(y)) < \epsilon$. Inductively, we have $d(f^{-j}(x), f^{-j}(y)) < \epsilon$ for all $j \in \mathbb{N}$.

Since $X$ is compact there exist finite $\delta/2$-balls covering $X$. Let $K$ be the number of balls. Fix $W$ a set of $K + 1$ distinct points. So for each $j \in \mathbb{Z}$ there exists $a_j, b_j \in W$ such that $f^j(a_j)$ and $f^j(b_j)$ belong to the same $\delta/2$ ball. Then $d(f^j(a_j), f^j(b_j)) < \delta$ and $d(f^n(a_j), f^n(b_j)) < \epsilon$ for all $-\infty < n \leq j$. Since $W$ is finite there exist distinct points $x_0, y_0 \in W$ such that $a_j = x_0$ and $b_j = y_0$ infinitely often. So $d(f^n(x_0), f^n(y_0)) < \epsilon$ for all $n \geq 0$. $\square$

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**Corollary 4.5** Let $f$ be an expansive homeomorphism of a compact metric space to itself, where $|X| = \infty$. Then there exists $x_0, y_0 \in X$ such that $d(f^n(x_0), f^n(y_0)) \to 0$ as $n \to \infty$.

**Proof.** Let $\delta > 0$ be an expansive constant for $f$. By the previous proposition there exist points $x_0, y_0 \in X$ such that $d(f^n(x_0), f^n(y_0)) < \delta$ for all $n \in \mathbb{N}$. Suppose that $d(f^n(x_0), f^n(y_0))$ does not go to zero as $n \to \infty$. Then by compactness there exists a limit point $(x', y')$ of $(f^n(x_0), f^n(y_0))$. For $k$ large we have $n_k + m > 0$ and $d(f^m(x'), f^m(y')) \leq \delta$ for all $m \in \mathbb{Z}$, a contradiction. 
\[\Box\]