Final

April 11, 2008

The final is due on April 18th by 10pm. You may use notes and other books, but you may not discuss it with one another or ask any person (besides myself) for help.

1. Let \( p_0, p_1, p_2 \) be points such that \( f(p_0) = p_1, f(p_1) = p_2, \) and \( f(p_2) = p_0 \) for a diffeomorphism \( f \). Prove that \( Df_{p_0}, Df_{p_1}, \) and \( Df_{p_2} \) are similar. (Hint: use the chain rule.) Generalize this to show that if \( p_0, ..., p_{n-1} \) is a periodic orbit for a diffeomorphism, then \( Df_{p_0}, ..., Df_{p_{n-1}} \) are all similar.

2. Let \( f \in \text{Diff}(\mathbb{R}^2) \) and \( x \in \text{Fix}(f) \) such that

\[
Df_x = \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}
\]

where \( \lambda \in (0,1) \).

(a) Prove that \( \{x\} \) is a hyperbolic set for \( f \).

(b) Describe how you could generalize this to show that a hyperbolic fixed point is a hyperbolic set. Then explain how this could be used with the first problem in the test to show that a periodic orbit is a hyperbolic set if and only if \( Df^n_p \) is a hyperbolic matrix.

3. Let \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) be transitive and Anosov and \( A \) be a transition matrix for a Markov partition of \( f \). Prove that \( A \) is irreducible.

4. Let \( f \) be a \( C^2 \) orientation preserving diffeomorphism of \( S^1 \). Prove that \( h(f) = 0 \).
5. Prove that \( f : [0, 1] \to [0, 1] \) given by
\[
f(x) = \begin{cases} 
\frac{x}{2} & \text{when } 0 < x \leq 1 \\
1 & \text{when } x = 0
\end{cases}
\]
has no invariant Borel probability measure.

6. Prove that the “tent” map \( g : [0, 1] \to [0, 1] \) given by
\[
g(x) = \begin{cases} 
2x & \text{when } 0 \leq x \leq \frac{1}{2} \\
2 - 2x & \text{when } \frac{1}{2} \leq x \leq 1
\end{cases}
\]
is transitive and has periodic points dense.

7. Let \( F : \mathbb{T}^2 \to \mathbb{T}^2 \) be given by \( F(x, y) = (x + \alpha_1, y + \alpha_2) \mod 1 \) where \( c_1\alpha_1 + c_2\alpha_2 \neq 0 \) unless \( c_1 = c_2 = 0 \). Prove that \( F \) is ergodic but not mixing.

8. Construct a map of \([0, 1]\) with points of all periods except 3.

9. Construct a map of \([0, 1]\) with a point of period 10 and all periods implied by 10 by the Sharkovskii ordering, but no others. (Hint: Take the double of the map in the previous problem.)

10. Let \( f_0 = 1/3 \) for all \( x \in [0, 1] \). Let \( f_n \) be the double map of \( f_{n-1} \) for all \( n \in \mathbb{N} \). So
\[
f_n = \mathcal{D}(f_{n-1}) = \begin{cases} 
\frac{2}{3} + \frac{1}{3}f_{n-1}(3x) & \text{when } 0 \leq x \leq \frac{1}{2} \\
(2 + f_{n-1}(1))(\frac{2}{3} - x) & \text{when } \frac{1}{2} \leq x \leq \frac{2}{3} \\
x - \frac{2}{3} & \text{when } \frac{2}{3} \leq x \leq 1
\end{cases}
\]
Define \( f_\infty \) by \( f_\infty(x) = \lim_{n \to \infty} f_n(x) \).

(a) Prove that \( f_\infty \) is continuous.
(b) Prove that the periods of \( f_\infty \) are exactly \( \{2^i \mid 0 \leq i < \infty \} \).
(c) Prove that for each \( n \), \( f_\infty \) has exactly one periodic orbit of each period \( 2^n \), that it is repelling, and that the points of this orbit lie in the gaps \( G_{n,j} \) which define the middle - (1/3) Cantor set.
(d) Let
\[ S_n = [0, 1] - \left( \bigcup_{1 \leq k \leq n} \bigcup_{1 \leq j \leq 2^{k-1}} G_{k,j} \right) \]
be the union of the \(2^n\) intervals used to define the middle-(1/3) Cantor set. Prove that \(f_\infty(S_n) = S_n\).

(e) Let \(\Lambda = \bigcap_{n \geq 1} S_n\). Prove that \(\Lambda\) is invariant for \(f_\infty\).

(f) Let \(\Sigma^+_2\) be the set of all sequences of 0’s and 1. Define \(A : \Sigma^+_2 \to \Sigma^+_2\) by
\[ A(s_0s_1s_2...) = (s_0s_1s_2...) + (100000) \mod 2. \]
The map \(A\) is called the adding machine. Define \(h : \Lambda \to \Sigma^+_2\) by \(h(p) = s\) where \(s_k = 1\) if \(p\) belongs to the left hand choice for the interval in \(S_{n-1}\). Prove that \(h\) is a topological conjugacy from \(f_\infty\) on \(\Lambda\) to \(A\) on \(\Sigma^+_2\).

(g) Prove that the adding machine \(A\) on \(\Sigma^+_2\) has no periodic points, and every forward orbit is dense in \(\Sigma^+_2\).