Midterm

February 28, 2008

The midterm is due on March 11th by the beginning of class. You may use notes and other books, but you may not discuss it with one another or ask any person (besides myself) for help.

1. For the weak* topology a basis is given by

\[ V_\mu(f_1, \ldots, f_k; \epsilon) = \{ m \in M(X) \mid |\int f_i dm - \int f_i d\mu| < \epsilon, 1 \leq i \leq k \} \]

where \( f_i \in C(X) \) and \( M(X) \) is the set of Borel probability measures on \( X \). Let \( X \) be a compact metric space and \( \{f_i\}_{i=1}^\infty \subset C(X) \) be dense and define \( d(\cdot, \cdot) : M(X) \times M(X) \to \mathbb{R} \) by

\[ d(m, \mu) = \sum_{n=1}^\infty \frac{|\int f_n dm - \int f_n d\mu|}{2^n \|f_n\|} \]

Show that \( d(\cdot, \cdot) \) is a metric on \( M(X) \) generating the weak* topology.

2. Computer \( h_{top}(\Sigma_A) \) where

(a) \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \),

(b) \( A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \), and

(c) \( A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \).
3. Let \((X,d)\) be a compact metric space and \(f: X \to X\) be an expansive homeomorphism with expansive constant \(\delta\). Prove that \(h(f) = h_\epsilon(f)\) for all \(\epsilon < \delta\).

4. Show the solenoid is expansive and compute an expansive constant.

5. Let \(T\) be a measure preserving transformation on \((X,\mathcal{B},\mu)\) and \(f \in L^1(X,\mu)\) such that \(f(T(x)) \leq f(x)\) for almost every \(x \in X\). Prove that \(f(T(x)) = f(x)\) for almost every \(x \in X\).

6. Let \(X\) be compact. Show that a measure preserving transformation \(T\) is mixing if and only if \(\lim_{n \to \infty} \int f(T^n(x)) \cdot g(x) d\mu = \int f(x) d\mu \cdot \int g(x) d\mu\) for any bounded measurable functions \(f\) and \(g\).

7. A continuous map \(T: X \to X\) of a compact metric space is uniquely ergodic if \(\mathcal{M}_T(X)\) consists of one point. Prove that this measure is ergodic. Also, prove that \(R_\alpha\) (the circle rotation) with \(\alpha\) irrational is uniquely ergodic.