4.3.1 (5 pts)

a) For $x = 0$ we know that $g(x) = 0$. Fix $\epsilon > 0$. Let $\delta = \epsilon^3$. Then for $0 < |x - 0| < \delta$ we have $|g(x) - g(0)| = |\sqrt[3]{x}| < (\sqrt[3]{\epsilon})^3 = \epsilon$. So $g$ is continuous at $x = 0$.

b) For $x = c$ we know that $g(c) = \sqrt[3]{c}$. Fix $\epsilon > 0$

If $c \geq 0$ we let

$$\delta = \min\left\{\frac{c}{2}, \epsilon((\frac{c}{2})^{2/3} + \frac{3\sqrt[3]{c^2}}{2} + (\sqrt[3]{c})^2)\right\}.$$ 

If $0 < |x - c| < \delta$ we have

$$|\sqrt[3]{x} - \sqrt[3]{c}| = \left|\frac{x-c}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}}\right| < \epsilon.$$ 

If $c < 0$ we repeat the same as above with

$$\delta = \min\left\{\frac{c}{2}, \epsilon((\frac{3c}{2})^{2/3} + \frac{3\sqrt[3]{c^2}}{2} + (\sqrt[3]{c})^2)\right\}.$$ 

4.3.4(a) (3 pts) Let $f : \mathbb{Z} \to \mathbb{R}$. Let $c \in \mathbb{Z}$ and $\epsilon > 0$. For $\delta = 1/2$ we know that if $0 < |x - c| < \delta$ and $x \in \mathbb{Z}$, then $x = c$. So $|f(c) - f(c)| = 0 < \epsilon$.

4.3.7 (2 pts) Let $K = \{x : h(x) = 0\}$, and $\{x_n\} \subset K$ such that $x_n \to x$ as $n \to \infty$. Then $h(x_n) = 0$ for all $n \in \mathbb{N}$ and $h(x_n) \to h(x)$. Hence, $h(x) = 0$, $x \in K$, and $K$ is closed.

4.3.8(b) (2 pts) Yes. Let $x \in \mathbb{R}$. Then there exists $q_n \in \mathbb{Q}$ such that $q_n \to x$ as $n \to \infty$. Since $f$ is continuous we know that $f(q_n) \to f(x)$. Similarly, $g(q_n) \to g(x)$. Since $g(q_n) = f(q_n)$ for all $n \in \mathbb{N}$ we know that $f(x) = g(x)$.

4.3.9 (8 pts)

a) Fix $\epsilon > 0$. Let $\delta = \frac{\epsilon}{c}$. Then if $0 < |x - y| < \delta$ we have

$$|f(x) - f(y)| \leq c|x - y| < c\frac{\epsilon}{c} = \epsilon.$$ 

So $f$ is continuous.

b) We know that $|y_{n+1} - y_n| = |f(y_n) - f(y_{n-1})| \leq c|y_n - y_{n-1}|$ for all $n \geq 2$. Inductively, we then have $|y_{n+1} - y_n| \leq c^{n-1}|y_2 - y_1|$ for all $n \in \mathbb{N}$. Fix $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} c^i |y_2 - y_1| < \epsilon$. For $n > m \geq N$ we have

$$|y_n - y_m| \leq |y_n - y_{n-1}| + \cdots + |y_{m+1} - y_m| \leq \sum_{i=m}^{n-1} c^i |y_2 - y_1| \leq \sum_{i=N}^{\infty} c^i |y_2 - y_1| < \epsilon.$$ 

So the sequence is Cauchy and $\lim y_n$ exists and equals some point $y \in \mathbb{R}$. 

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c) We know that
\[ y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} f(y_n) = f(y) \]

since \( f \) is continuous. Now suppose that \( z = f(z) \). Then \( |z - y| = |f(z) - f(y)| \leq c|z - y| \). So \( y = z \).

d) Let \( x_0 \in \mathbb{R} \) and let \( x_n \) be the image of \( x_0 \) under \( n \) compositions of \( f \). Then from above we know that \( x_n \) converges to some points \( x \in \mathbb{R} \) and that \( x = f(x) \). Since there is only one fixed point we know that \( x = y \) from above.