4.4.1(b) (3 pts) Fix $\epsilon_0 = 1$. Let $x_n = n + \frac{1}{n}$ and $y_n = n$. Then $|x_n - y_n| = \frac{1}{n}$ which converges to 0 as $n$ goes to infinity. However,

$$|f(x_n) - f(y_n)| = |(n + \frac{1}{n})^3 - n^3| = n^3 + 3n + \frac{3}{n} - n^3 - n^3 > 3.$$  

4.4.6(a)(b) (4 pts) (a) Let $x$ and $y$ be such that if $0 < |x - y| < \delta_1$ and $x, y \in [0, b]$, then $|f(x) - f(y)| < \epsilon/2$. There also exists some $\delta_2 > 0$ such that if $0 < |x - y| < \delta_2$ and $x, y \in [b, \infty)$, then $|f(x) - f(y)| < \epsilon/2$. Also, there exists some $\delta_3 > 0$ such that if $0 < |x - b| < \delta_3$, then $|f(x) - f(b)| < \epsilon/2$. Fix $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. For $x, y \in [0, \infty)$, $x < y$ and $0 < |x - y| < \delta$ we have the following cases.

CASE 1: Let $x, y \in [0, b]$. Then we know $|f(x) - f(y)| < \epsilon$ since $\delta \leq \delta_1$.

CASE 2: Let $x, y \in (b, \infty)$. Then we know that $|f(x) - f(y)| < \epsilon$ since $\delta \leq \delta_2$.

CASE 3: Let $x < b < y$. Then $0 < |x - b| < \delta$ and $0 < |y - b| < \delta$. Since $\delta < \delta_3$ we know that $|f(x) - f(y)| = |f(x) - f(b) + f(b) - f(y)| \leq |f(x) - f(b)| + |f(y) - f(b)| < \epsilon/2 + \epsilon/2 = \epsilon$.

4.4.8(b) (4 pts) We know that $\sqrt{x}$ is continuous on all of $[0, \infty)$. On $[0, 1]$ we that $f$ is uniformly continuous. We want to show that $f$ is uniformly continuous on $[1, \infty)$. Fix $\epsilon > 0$ and let $\delta = \epsilon$. If $x, y \in [1, \infty)$ and $0 < |x - y| < \delta$, then

$$|\sqrt{x} - \sqrt{y}| = \left|\frac{x - y}{\sqrt{x} + \sqrt{y}}\right| < \frac{x - y}{2} < \frac{\epsilon}{2} < \epsilon.$$  

So from part (a) we know that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

4.4.9 (5 pts) (a) Fix $\epsilon > 0$. Let $M$ be the Lipschitz constant and fix $\delta = \epsilon/M$. If $0 < |x - y| < \delta$ we have $|f(x) - f(y)| \leq M|x - y| < M(\epsilon/M) = \epsilon$. (b) No. We know that $\sqrt{x}$ is uniformly continuous on $[0, 1]$, but it is not Lipschitz. To see this we notice that

$$\frac{1}{2^n} - 0 = 2^n.$$