5.4.3 (2 pts) Let \( g_m(x) = \sum_{n=0}^{m} \frac{1}{2^n} h(2^n x) \). We know that \( h \) is continuous so \( h(s^n x) \) is continuous since the composition of continuous functions is continuous. The finite sum of continuous functions is also continuous so \( g_m \) is continuous.

5.4.5(a) (6 pts) Let \( x_m = 1 - \frac{1}{2^m} \). Then
\[
h(2^n x_m) = h\left(2^n - \frac{2^n}{2^m}\right) = \begin{cases} 
0 & \text{for } n > m \\
1 & \text{for } n = m \\
h\left(\frac{2^n}{2^m}\right) & \text{for } n < m.
\end{cases}
\]

Then \( g(1) = 1 \) and
\[
g(x_m) = \sum_{n=0}^{m} \frac{1}{2^n} \frac{2^n}{2^m} = \frac{m + 1}{2^m}.
\]

So
\[
\frac{g(x_m) - g(1)}{x_m - 1} = 2^m - (m + 1).
\]

As \( m \to \infty \) we see that the previous expression goes to infinity.

For \( 1/2 \) we let \( x_m = 1/2 + 1/2^m \). Then we have \( g(1/2) = 1 \) and
\[
h(2^n x_m) = h\left(2^n - \frac{2^n}{2^m}\right) = \begin{cases} 
0 & \text{for } n > m \\
1 & \text{for } n = m \\
h\left(\frac{2^n}{2^m}\right) & \text{for } n < m.
\end{cases}
\]

As before we find that \( g'(1/2) \) does not exist.

5.4.5(b) (2 pts) For the general case we can simply construct points \( x_m \to p/2^k \) where the ratio
\[
\frac{g(x_m) - g(p/2^k)}{x_m - p/2^k}
\]

is arbitrarily large as \( m \to \infty \). The argument is similar to the one in part (a).

5.4.6(a) (2 pts) For \( x \) not a dyadic point we know that \( h_n(x) \in \mathbb{R} - \mathbb{Z} \) for all \( n \in \mathbb{N} \). So \( h'_n(x) \) exists for all \( n \).

5.4.6(b) (3 pts) We see that
\[
g'_{m+1}(x) - g'_m(x) = \frac{d}{dx} \left[ \frac{1}{2^{m+1}} h(2^{m+1}(x)) \right] = h'(2^{m+1}x).
\]

So \( |g'_{m+1}(x) - g'_m(x)| = 1 \). □