2.5.3(a) (2 pts) Let $a_n = \frac{1}{n+1}$ for $n$ odd and $a_n = 1 - \frac{1}{n}$ for $n$ even.

2.5.3(b) (2 pts) Not possible. If $a_n$ is monotone (say increasing) and not convergent then $a_n$ is not bounded so for all $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $a_n \geq M$ for all $n \geq N$. So there cannot exist a convergent subsequence.

2.5.3(c) (2 pts) $a_1 = 1, a_2 = \frac{1}{2}, a_3 = 1, a_4 = \frac{1}{2}, a_5 = \frac{1}{3}, a_6 = 1,$...

2.5.3(d) (2 pts) Let $a_n = n$ for $n$ odd and $a_n = \frac{1}{2^n}$ for $n$ even.

2.5.3(e) (2 pts) Not possible. Every bounded sequence has a convergent subsequence.

2.5.4 (5 pts) Let $\{a_n\}$ be a bounded sequence and every convergent subsequence converge to $a \in \mathbb{R}$. Suppose that $\{a_n\}$ does not converge. For some $\epsilon > 0$ we know that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - a| \geq \epsilon$. Let $\{a_{n_j}\}$ be a subsequence of $\{a_n\}$ such that each point is not contained in $(a - \epsilon, a + \epsilon)$. Hence, $a_{n_j}$ has a convergent subsequence converging to $a$, a contradiction. Hence, $\{a_n\}$ is convergent. □

2.5.6 (5 pts) Let $\{a_n\}$ be a bounded sequence and

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many } a_n\}.$$ 

We know that sup $S \leq \sup \{a_n\}$. So $S$ has a least upper bound we denote as $s$. Then there exists $a_{n_1}$ such that $a_{n_1} > s - 1$ since $s - 1$ is not an upper bound for $S$ and then also not an upper bound for $\{a_n\}$. Let $a_{n_{j+1}} \in \{a_n\}$ such that $n_{j+1} > n_j$ and $a_{n_{j+1}} > s - \frac{1}{2^j}$ for all $j \in \mathbb{N}$.

Suppose that $\{a_{n_k}\}$ does not converge to $s$. Then there exists some $\epsilon : > 0$ such that for all $N \in \mathbb{N}$ there exists a $k \geq N$ such that $|a_{n_k} - s| \geq \epsilon$. So there exists a subsequence of $\{a_{n_k}\}$ such that no term is contained in $(s - \epsilon, s + \epsilon)$. This implies that $s + \epsilon \in S$, a contradiction. So $\{a_n\}$ has a subsequence converging to $s$. □