Math 112 Midterm 1 - Review

September 30, 2014

Midterm 1 covers Section 1.1 - 1.6 and 2.1-2.8. Primarily the content will come from chapter 2, but you need to have the material mastered from chapter 1. The midterm will have both short answer (multiple choice) and long answer portions. Roughly half of the exam for each. For the multiple choice only the answers marked on the bubble sheet will be accepted. Make sure to mark the answers clearly and fill in completely. The long answer will be given partial credit. You must show your work and it needs to be easy to follow. If there is something you don’t want scored you should cross it out. If two solutions are given the first will be graded.

The concepts in this review are not exhaustive, but represent the most important concepts from the chapters. Mastering the topics listed will result in a good, but not necessarily great grade on the exam.

1.1 Four Ways to Represent a Function

A function is a rule that assigns to each element $x$ in a set $D$ exactly one element, called $f(x)$, in a set $E$.

There are four ways to represent a function.

- verbally - by a description in words
- numerically - by a table of values
- visually - by a graph
- algebraically - by an explicit formula

In this class the last three are the most frequent.
Remark 0.1 (The vertical line test) A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersect the curve more than once.

A function $f$ is increasing on an interval $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in $I$. A function $f$ is decreasing on $I$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in $I$.

1.2 Mathematical Models: A Catalog of Essential Functions

This section primarily introduced a number of functions we will use throughout the class.

A linear function is expressed in the form $y = f(x) = mx + b$ where $m$ is the slope of the line ($m = \delta y/\delta x$) and $b$ is the $y$-intercept. To find the equation of a line one needs either two points, or a point and the slope. In this class we will often find the equation of a line using the point slope form. So if $(x_0, y_0)$ is a point on the line with slope $m$, then we have

$$y - y_0 = m(x - x_0) \text{ or } y = m(x - x_0) + y_0.$$  

A function $P$ is a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$  

The constants $a_0, ..., a_n$ are called the coefficients and the degree of the polynomial is the power $n$. A quadratic polynomial takes the form $P(x) = ax^2 + bx + c$.

This section also briefly introduced $f(x) = a^x$ and $f(x) = \sin x$, other trigonometric functions, rational functions, and root functions.

Remark 0.2 You should be able to compute all of the trig functions from the value of one of them.

1.3 New Functions from Old Functions

In this section it is shown how to obtain more general functions from basic functions such as $f(x) = x$, $f(x) = \sin x$, $f(x) = e^x$, etc...

The first topic is vertical and horizontal shifts. Let $c > 0$. Then

- $y = f(x) + c$ shifts the graph of $y = f(x)$ a distance $c$ units upwards
• \(y = f(x) - c\) shifts the graph of \(y = f(x)\) a distance \(c\) units downwards

• \(y = f(x - c)\) shifts the graph of \(y = f(x)\) a distance \(c\) units to the right

• \(y = f(x + c)\) shifts the graph of \(y = f(x)\) a distance \(c\) units to the left

**Warning:** Students often mistake the last two horizontal shifts on a multiple choice question. If you are in doubt plug in some numbers (this is generally a good idea anyway).

The second topic is *vertical and horizontal stretching and reflecting.* Suppose \(c > 1\). To obtain the graph of

• \(y = cf(x)\) stretch the graph of \(y = f(x)\) vertically by a factor of \(c\)

• \(y = (1/c)f(x)\) contract the graph of \(y = f(x)\) vertically by a factor of \(1/c\)

• \(y = f(cx)\) shrinks the graph of \(y = f(x)\) horizontally by a factor of \(c\)

• \(y = f(x/c)\) stretches the graph of \(y = f(x)\) horizontally by a factor of \(1/c\)

• \(y = -f(x)\) reflect the graph of \(y = f(x)\) about the \(x\)-axis

• \(y = f(-x)\) reflect the graph of \(y = f(x)\) about the \(y\)-axis

As in the last topic these make good multiple choice problems. Again if in doubt plug in some values and check for the correct response.

Given two functions \(f\) and \(g\) the *composite function* \(f \circ g\) (also called the *composition* of \(f\) and \(g\)) is defined by \((f \circ g)(x) = f(g(x))\).

**Remark 0.3** *Watch the order on the composition. If this is multiple choice both orders will be given and only one correct.*

1.4 **Graphing Calculators and Computers**

Nothing is needed from this section on the exam.

1.5 **Exponential Functions**
An exponential function is a function of the form $f(x) = a^x$ or $f(x) = Ca^x$ where $a$ is positive and $C$ is constant. You should know the general shape of an exponential function for $a > 1$ and $a < 1$. Also, you should be able to find the equation for $f(x) = Ca^x$ given two points on the graph. We did this in class.

You should have memorized the laws of exponents. If $a$ and $b$ are positive and $x$ and $y$ are real numbers, then

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = \frac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$.

The main examples we will use in the course is $f(x) = e^x$ where $e$ is a constant between 2 and 3 that we will explicitly state later.

### 1.6 Inverse Functions and Logarithms

A function $f$ is called one-to-one if it never takes on the same value twice. A function that is one-to-one will pass the horizontal line test which states that a horizontal line will not intersect the graph in more than one point.

For $f$ a one-to-one function from a set $A$ to a set $B$ the inverse function $f^{-1}$ has domain $B$ and range $A$ and is defined by $f^{-1}(y) = x$ if and only if $f(x) = y$.

**Remark 0.4** Do not mistake $f^{-1}$ with $1/f(x)$.

To find an inverse function you should write $y = f(x)$, solve for $x$ if possible, and then express $f^{-1}$ as a function of $x$ by interchanging $x$ and $y$.

To graph $f^{-1}$ you need only reflect the graph of $f$ about the line $y = x$.

The function $\log_a x = y$ is the inverse function for $f(x) = a^x$. From this we know that $\log_a(a^x) = x$ and $a^{\log_a x} = x$ when the expressions are defined.

From the laws for exponents we have similar laws for logarithms. If $x$ and $y$ are positive numbers, then

- $\log_a (xy) = \log_a x + \log_a y$
- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a (x^r) = r \log_a x$
If we let $a = e$, then we write $\log_e x = \ln x$. We know that $\ln e = 1$. For any other base we have the following:

$$\log_a x = \frac{\ln x}{\ln a}.$$ 

You should also know the graphs and definitions of the inverse trigonometric functions.

**Chapter 1 review problems:** True-False Quiz: 1, 3, 5, 9; Exercises: 3, 5, 7, 19, 25

**Chapter 2: Limits and Derivatives**

The vast majority of the problems will come from this chapter although you will need to use the tools from chapter 1 in most of the computations.

**2.1 The Tangent and Velocity Problems**

This section is introductory. It explains the concept of a secant and tangent line as well as the concept of a limit. It is good to know the ideas, but most of the concepts are made clearer in later sections.

**2.2 Limit of a function**

Let $f$ be defined when $x$ is near the number $a$. Then $\lim_{x \to a} f(x) = L$ if $f(x)$ takes on values arbitrarily close to $L$ as $x$ gets closer to $a$. We can also look at one-sided limits the right hand limit $\lim_{x \to a^+} f(x) = L$ means that $f(x)$ takes on values arbitrarily close to $L$ as $x$ get closer to $a$ where $x > a$. Similarly, the left hand limit $\lim_{x \to a^-} f(x) = L$ means that $f(x)$ takes on values arbitrarily close to $L$ as $x$ get closer to $a$ where $x < a$.

This section again introduces the notion of a limit through graphs and tables. It is important that you can look at a graph and explain if the limit and/or one-sided limits exist on points for the graph.

The last part of this section deals with infinite limits. For $f$ a function defined on both sides of $a$, except possibly at $a$ we say $\lim_{x \to a} f(x) = \infty$ if $f(x)$ takes on arbitrarily large values as $x$ is close to $a$, but not equal to $a$. Similarly, we can define $\lim_{x \to a} f(x) = -\infty$ if $f(x)$ takes on arbitrarily large negative values as $x$ is close to $a$, but not equal to $a$. We can also define these notions for one-sided limits.
A line \( x = a \) is a \textit{vertical asymptote} of the curve \( y = f(x) \) if the one sided limits are both infinity or minus infinity.

### 2.3 Calculating Limits Using the Limit Laws

In this section we move from conceptual and graphical analysis to computation of limits. The main tool is the following limit laws.

**Theorem 0.5** *Suppose that \( c \) is a constant the limits\* \[
\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M
\]
*exist. Then*  
1. \( \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M \)
2. \( \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M \)
3. \( \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x) = cL \)
4. \( \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = LM \)
5. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M} \) provided \( M \neq 0 \).

In this section it is also shown that \[
\lim_{x \to a} x = a \quad \text{and} \quad \lim_{x \to a} c = c.
\] Using the limit laws we learn that if \( f(x) \) is a polynomial or rational function that \( \lim_{x \to a} f(x) = f(a) \).

**Theorem 0.6** *If \( f(x) = g(x) \) except when \( x \neq a \), then \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) \) provided the limit exists.*

This result is used throughout the chapter to find the limit of functions not defined at \( a \) including derivatives later in the chapter. The idea is that if we have an indeterminate form, for instance one of one of the following types \[
\begin{array}{cccc}
0 & \infty \\
\infty & 0, \infty
\end{array}
\]
then we use the theorem and manipulate the expression so it is no longer indeterminate. This was the technique used in most of the homework for this section.
Theorem 0.7  \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x) \).

The above theorem was used to compute limits for piecewise defined functions.

Theorem 0.8  If \( f(x) \leq g(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and the limits of \( f \) and \( g \) both exist at \( a \), then

\[
\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).
\]

The next theorem is used frequently to compute the limit of trigonometric functions especially involving \( \sin \) and \( \cos \).

Theorem 0.9 (Squeeze Limit Theorem)  If \( f(x) \leq g(x) \leq h(x) \) for all \( x \) near \( a \) (except possibly at \( a \)) and

\[
\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)
\]

then

\[
\lim_{x \to a} g(x) = L.
\]

2.4 Precise Definition of a Limit

Let \( f \) be a function defined on some open interval that contains the number \( a \), except possibly at \( a \) itself. Then the limit of \( f(x) \) at \( a \) is \( L \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( 0 < |x-a| < \delta \) then \( |f(x) - L| < \epsilon \).

We also have definitions for the one-sided limits. \( \lim_{x \to a^+} f(x) = L \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( a < x < a+\delta \) then \( |f(x) - L| < \epsilon \), and \( \lim_{x \to a^-} f(x) = L \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( a - \delta < x < a \) then \( |f(x) - L| < \epsilon \).

It is possible that there be a multiple choice question asking you to pick out the correct definition. It is unlikely there will be a problem asking you to prove the limit exists using the definition, but if there it should be a linear function as discussed in class. A likely problem involves a graph and you use the graph to determine \( \delta \) given \( \epsilon \) as in the homework.

Let \( f \) be function defined on some open interval that contains the number \( a \) except possibly at \( a \) itself. Then \( \lim_{x \to a} f(x) = \infty \) if for every \( M > 0 \) there exists some \( \delta > 0 \) such that if \( 0 < |x-a| < \delta \), then \( f(x) > M \). We can
similarly define the limit as negative infinity as in the book.

2.5 Continuity

A function \( f \) is continuous at \( a \) if \( \lim_{x \to a} f(x) = f(a) \). Sometimes we also refer to a continuous function if it is continuous over its domain.

**Theorem 0.10** If \( f \) and \( g \) are continuous at \( a \) and \( c \) is a constant, then the following functions are continuous at \( a \): 1. \( f + g \), 2. \( f - g \), 3. \( cf \), 4. \( fg \), and \( f/g \) provided \( g(a) \neq 0 \).

The section discusses that the following functions are continuous over their domains: polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.

You should also be able to determine if piecewise functions are continuous. To do this you may need to compute and compare one-sided limits.

**Theorem 0.11** If \( f \) is continuous at \( b \) and \( \lim_{x \to a} g(x) = g(a) = b \), then \( \lim_{x \to a} f(g(x)) = f(b) \).

The above theorem states that the composition of continuous functions is continuous.

**Theorem 0.12** (The Intermediate Value Theorem) Suppose that \( f \) is continuous on the closed interval \( [a, b] \) and let \( N \) be any number between \( f(a) \) and \( f(b) \) where \( f(a) \neq f(b) \). Then there exists a number \( c \) in \( (a, b) \) such that \( f(c) = N \).

2.6 Limits at Infinity: Horizontal Asymptotes

Let \( f \) be a function defined on some interval \( (a, \infty) \). Then \( \lim_{x \to \infty} f(x) = L \) means that the values of \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) sufficiently large. We can similarly define the limit and negative infinity.

The line \( y = L \) is a horizontal asymptote if either

\[
\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.
\]

The main tool we used is the following:
Theorem 0.13 If \( r > 0 \), then
\[
\lim_{x \to \infty} \frac{1}{x^r} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^r} = 0.
\]

We used this with rational functions. As a reminder if \( f(x) = \frac{P(x)}{Q(x)} \) is a rational function then:

- \( \lim_{x \to \infty} f(x) = 0 = \lim_{x \to -\infty} f(x) \) if \( \deg P < \deg Q \),
- \( \lim_{x \to \infty} f(x) = \pm \infty = \lim_{x \to -\infty} f(x) \) if \( \deg P > \deg Q \),
- and \( \lim_{x \to \infty} f(x) \) is the ratio of the leading coefficients if \( \deg p = \deg Q \).

You should also know the behavior at infinity for functions such as exponential, logarithmic, trigonometric, and inverse trigonometric functions.

2.7 Derivatives and Rates of Change

The slope of the tangent line of \( f(x) \) at a point \( a \) or derivative of \( f \) at \( a \) is
\[
m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)
\]
provided the limit exists.

These limits are inherently indeterminate so to compute the limit you need to factor a power of \( x - a \) from the numerator for the first expression or \( h \) for the second expression in order to compute the limit. Remember on the midterm you must compute the limit from the definition.

The equation for the tangent line is: \( y = f'(a)(x - a) + f(a) \).

The derivative represents the instantaneous rate of change and is useful in many applications.

2.8 The Derivative as a Function

If the derivative of \( f \) exists at all points in the domain then we can let \( f'(x) \) be the derivative of \( f \) at any point \( x \). This is now a function of \( x \). We write this as
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\]
Remark 0.14 Remember in computing this limit that $h$ is the variable and $x$ is held fixed as some value in the domain.

You should be able to compute this function for some simple polynomials.

Theorem 0.15 If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

A function can fail to be differentiable if it is not continuous, but even more if there are any cusps or corner at the point.

If $f'(x)$ now exists for all $x$, then this is a function we can try to compute the derivative. This is called the second derivative and denoted $f''(x)$. We will see many applications of this later. You should be able to do a simple calculation for a second derivative, but more importantly for this exam you should be able to look at a graph of three functions and decide which one is $f$, which one is $f'$, and which one is $f''$.

Chapter 2 review problems True-False Quiz: 3, 9; Exercises: 1, 3, 5, 7, 13, 17, 29, 33, 39 (a)(b), 47, 48

Good Luck!!!