Exercise (1). Let $X$ be a sequentially compact metric space. Prove that the subspace $K$ of $X$ is sequentially compact if and only if $K$ is a closed subset of $X$.

Solution (1). ($\Rightarrow$) Suppose that $K$ is sequentially compact. Take a sequence $\{u_j\} \subseteq K$ converging to a point in $X$. By sequential compactness, $\exists$ a subsequence $u_{j_i}$ converging to a point in $K$. Since the sequence and subsequence must converge to the same thing, $u_j$ converges to the same point in $K$. ($\Leftarrow$)

Suppose that $K$ is closed in $(X,d)$. Take a sequence $\{u_i\} \subseteq K$. Since $X$ is sequentially compact, there exists a subsequence $\{u_{i_j}\} \subseteq K$ converging to a point in $X$. Now, $\{u_{i_j}\}$ is contained in $K$ and is convergent, and since $K$ is closed, $\{u_{i_j}\}$ converges to a point in $K$.

Exercise (3). Show that if $X$ is a metric space and $U$ and $V$ are open subsets of $X$ that separate $X$, then both $U$ and $V$ are closed in $X$.

Solution (3). Take a sequence $\{u_k\} \subseteq U$ converging to a point $u$ in $X$. Suppose by way of contradiction that $u$ is in $V$. Since $V$ is open, $\exists$ an open ball $B_r(u)$ with radius $r > 0$ about $u$ containing points only in $V$. Now, since $\{u_k\} \to u$, we can find a $K \in \mathbb{N}$ such that for $k \geq K$, $d(u_k, u) < r$. This means that there are points $u_k \in B_r(u)$ and $\in U$. Then $u_k \in V \cap U$, a contradiction of $U$ and $V$ separating $X$. Thus, $u$ must be in $U$, making $U$ closed. By the symmetry of the problem, switching $U$ and $V$ and doing the same argument gives us $V$ is closed.

Exercise (5). Let $X$ be a sequentially compact metric space. Prove that $X$ is disconnected if and only if there are nonempty subsets $A$ and $B$ of $X$ and a positive number $\epsilon$, with $A \cap B = \emptyset, A \cup B = X$, and $d(p,q) > \epsilon$ for all $p \in A, q \in B$. Is sequential compactness necessary?

Solution (5). ($\Rightarrow$) Suppose that $X$ is disconnected. Then there exist open sets $A$ and $B$ such that $A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset, A \cup B = X$. Suppose by way of contradiction that for all $\epsilon > 0$, there exists a $p \in A, q \in B$ such that $d(p,q) \leq \epsilon$. We construct two sequences by letting $\epsilon_1 = 1$, and taking...
$p_1, q_1$ such that $d(p_1, q_1) < \epsilon$, which we are guaranteed to find by assumption. In general, let $p_k, q_k$ be the points such that $d(p_k, q_k) < \epsilon_k$, where $\epsilon_k = \frac{1}{k}$. Since $X$ is sequentially compact, we can take subsequences of each of these sequences $p_{k_l}, q_{k_l}$ so that $p_{k_l}$ converges to some point $r \in X$. Take another subsequence of these subsequences $p_{k_{l_n}}, q_{k_{l_n}}$ so that $q_{k_{l_n}}$ converges. We show these subsequences converge to the same thing. Let $\epsilon > 0$. We can find some $L \in \mathbb{N}$ such that for all $n \geq L, d(p_{k_{l_n}}, r) < \frac{\epsilon}{2}$, and we can find some $M \in \mathbb{N}, M > \frac{2}{\epsilon}$ such that for $n \geq M, d(p_{k_{l_n}}, q_{k_{l_n}}) < \frac{1}{M} < \frac{1}{\frac{2}{\epsilon}} = \frac{\epsilon}{2}$. Let $N = \max(L, M)$, and so for $n \geq N$,

$$d(q_{k_{l_n}}, r) \leq d(p_{k_{l_n}}, r) + d(p_{k_{l_n}}, q_{k_{l_n}}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $q_{k_{l_n}} \to r$. Now, by Exercise 3, $A$ and $B$ are closed, and $\{p_{k_{l_n}}\} \subseteq A$ and $\{q_{k_{l_n}}\} \subseteq B$, so $r \in A \cap B$, a contradiction since $A \cap B = \emptyset$.

$(\Rightarrow)$ Suppose there are nonempty subsets $A$ and $B$ of $X$ and a positive number $\epsilon$, with $A \cap B = \emptyset, A \cup B = X$, and $d(p, q) > \epsilon$ for all $p \in A, q \in B$. We only need to show that $A$ and $B$ are open to get that $X$ is disconnected. Choose $r = \frac{\epsilon}{2}$. Let $p \in A$. Then $B_r(p)$ does not contain points in $B$, and since $X \setminus B = A, B_r(p) \subseteq A$, so $A$ is open. Switching $B$ for $A$ using this argument gives that $B$ is open as well. Thus, $X$ is disconnected.

Our argument in the “$\Rightarrow$” argument was dependent on $X$ being sequentially compact, and we demonstrate why without it the theorem is false in general with a simple example: Let $X = \mathbb{R} \setminus \{0\}$, which is not sequentially compact. We separate $X$ with $\mathbb{R}^+ = (0, \infty), \mathbb{R}^- = (-\infty, 0)$. Then $\mathbb{R}^+ \cap \mathbb{R}^- = \emptyset, \mathbb{R}^+ \cup \mathbb{R}^- = X$, and they are both open intervals, so they separate $X$, so $X$ is disconnected. However, we can let $\epsilon > 0$ and pick points $-\frac{\epsilon}{2} \in \mathbb{R}^-, \frac{\epsilon}{2} \in \mathbb{R}^+$, and $d\left(-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) = \epsilon$. Conversely, the “$\Leftarrow$” argument did not require sequential compactness, and therefore remains valid without it.