\textbf{Math 342}

\textbf{Homework 16}

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14.1: 1, 2, 6, 12, 15

**Exercise (1).** Define \( f(x, y) = e^{2x+4y+1} \) for \((x, y) \in \mathbb{R}^2\). Find the equation of the tangent plane to the graph of the function \( f : \mathbb{R}^2 \to \mathbb{R} \) at the point \((0, 0, e)\).

**Solution (1).** We take the first-order partials of \( f \) and get \( \frac{\partial f}{\partial x}(0, 0) = 2, \frac{\partial f}{\partial y}(0, 0) = 4 \), so by Corollary 14.3, the tangent plane at \((0, 0, e)\) is \( \psi(x, y) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)(x - 0) + \frac{\partial f}{\partial y}(0, 0)(y - 0) = e + 2ex + 4ey \).

**Exercise (2).** Define \( f(x, y) = x^2 - xy + 2y^2 + x \) for \((x, y) \in \mathbb{R}^2\). At what points on the graph of the function \( f : \mathbb{R}^2 \to \mathbb{R} \) is the tangent plane parallel to the \( xy \) plane?

**Solution (2).** We take the first-order partials of \( f \) and get \( \frac{\partial f}{\partial x}(x, y) = 2x - y + 1, \frac{\partial f}{\partial y}(x, y) = 4y - x \). The tangent plane at \((x_0, y_0)\), as given by Corollary 14.3, is \( \psi(x, y) = x_0^2 - x_0y_0 + 2y_0^2 + (2x - y + 1)(x - x_0) + (4y - x)(y - y_0) \). The \( xy \) plane is given by \( \theta(x, y) = 0 \), so any parallel plane will have a normal vector that is parallel to \((0, 0, 1)\). The normal vector of the tangent plane of \( f \) at \((x, y)\) is given by \( \nabla f(x, y, z) = (2x - y + 1, 4y - x, 1) \), so these two vectors will be parallel when they differ by a scalar multiple. We express this as \((2x - y + 1, 4y - x, 1) = (0, 0, 1)\), which becomes a system of equations

\[
\begin{align*}
2x - y + 1 &= 0 \\
4y - x &= 0
\end{align*}
\]

which, solving, gives us \( y = -1/7, x = -4/7 \). So, the only point where the tangent plane of \( f \) is parallel to the \( xy \) plane is at \((-4/7, -1/7)\).

**Exercise (6).** Define \( f(x, y, z) = x^2 + y^2 + z \) for \((x, y, z) \in \mathbb{R}^3\). Find the affine function that is a first-order approximation to the function \( f : \mathbb{R}^3 \to \mathbb{R} \) at the point \((0, 0, 0)\).

**Solution (6).** By Corollary 14.4, there is an affine function that is a first-order approximation of \( f \) defined by \( g(u) = f(x) + (\nabla f(x), u - x) = x^2 + y^2 + z + 2x(u_1 - x) + 2y(u_2 - y) + (u_3 - z) = 0 + 0 + 0 + 2(0)(u_1 - x) + 2(0)(u_2 - 0) + (u_3 - 0) = u_3\), so \( g(u_1, u_2, u_3) = u_3 \).
Exercise (12). Prove that \( \lim_{(x,y) \to (0,0)} \frac{(1+2x+y^2)^{3/2}-1-3x}{\sqrt{x^2+y^2}} = 0 \)

Solution (12). Let \( f(x, y) = (1 + 2x + y^2)^{3/2} \) and let \( h = (x, y) \). Then \( f(0) = 1 \), \( \frac{\partial f}{\partial x}(x, y) = 3(1 + 2x + y^2)^{1/2} \), \( \frac{\partial f}{\partial y}(x, y) = 3y(1 + 2x + y^2)^{1/2} \), so \( \nabla f(0) = (3, 0) \), which makes \( \lim_{(x,y) \to (0,0)} (1+2x+y^2)^{3/2}-1-3x \sqrt{x^2+y^2} = \lim_{(x,y) \to (0,0)} f(0+h) - f(0) - \langle \nabla f(0), h \rangle = 0 \) since the first-order partials of \( f \) are a composition of polynomials and square roots which are continuous.

Exercise (15). Suppose that the function \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous. Let \( a \) and \( b \) be any real numbers. Prove that \( \lim_{(x,y) \to (0,0)} [f(x,y) - (f(0,0) + ax + by)] = 0 \).

Is it true that \( \lim_{(x,y) \to (0,0)} f(x,y) - \sqrt{x^2+y^2} = 0 \)?

Solution (15). Well, since \( f \) is continuous, we can use the algebraic limit theorem to get \( \lim_{(x,y) \to (0,0)} f(x,y) - (f(0,0) + ax + by) = \lim_{(x,y) \to (0,0)} f(x,y) - \lim_{(x,y) \to (0,0)} f(0,0) - \lim_{(x,y) \to (0,0)} ax - \lim_{(x,y) \to (0,0)} by = f(0,0) - f(0,0) - 0 - 0 = 0 \), as desired.

To demonstrate that \( \lim_{(x,y) \to (0,0)} \frac{f(x,y)-[f(0,0)+ax+by]}{\sqrt{x^2+y^2}} \neq 0 \) in general we consider the case of the continuous function \( f(x,y) = \sqrt{x^2+y^2} + x + y, a = 1, b = 1 \), so \( \lim_{(x,y) \to (0,0)} \frac{f(x,y)-[f(0,0)+ax+by]}{\sqrt{x^2+y^2}} = \lim_{(x,y) \to (0,0)} \frac{\sqrt{x^2+y^2} + x + y - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y) \to (0,0)} \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{(x,y) \to (0,0)} 1 = 1 \).