Exercise (1). Define \( f(x, y) = e^{xy} + x^2 + 2xy \) for \((x, y) \in \mathbb{R}^2\).

(a) Define \( \phi : \mathbb{R} \to \mathbb{R} \) by \( \phi(t) = f(2t, 3t) \) for \( t \in \mathbb{R} \). Calculate \( \phi''(0) \) directly.

(b) Find the Hessian matrix of the function \( f : \mathbb{R}^2 \to \mathbb{R} \) at the point \((0, 0)\) and use formula (14.11) to calculate \( \phi''(0) = \left. \frac{d^2}{dt^2} [f(2t, 3t)] \right|_{t=0} \).

Solution (1). (a) Explicitly, \( \phi(t) = e^{6t^2} + 16t^2 \). The first derivative is \( \phi'(t) = 12te^{6t^2} + 32t \), and so the second derivative is \( \phi''(t) = 12e^{6t^2} + 144t^2e^{6t^2} + 32 \). Evaluating at \( t = 0 \), we get \( \phi''(0) = 12 + 32 = 44 \).

(b) To construct the Hessian matrix, we first take first-order partial derivatives of \( f \); \( \frac{\partial f}{\partial x}(x, y) = ye^{xy} + 2x + 2y \), \( \frac{\partial f}{\partial y}(x, y) = xe^{xy} + 2x \), now we calculate all second-order partials \( \frac{\partial^2 f}{\partial x^2}(x, y) = y^2e^{xy} + 2 \), \( \frac{\partial^2 f}{\partial y^2}(x, y) = x^2e^{xy} \), \( \frac{\partial^2 f}{\partial x \partial y}(x, y) = e^{xy} + xye^{xy} + 2 \), \( \frac{\partial^2 f}{\partial y \partial x}(x, y) = e^{xy} + xye^{xy} + 2 \). Thus the Hessian matrix is given by

\[
\nabla^2 f(x, y) = \begin{pmatrix}
y^2e^{xy} + 2 & e^{xy} + xye^{xy} + 2 \\
e^{xy} + xye^{xy} + 2 & x^2e^{xy}
\end{pmatrix}
\]

Now, using formula 14.11, we get \( \phi''(t) = \left. \frac{d^2}{dt^2} [f(0+2t, 0+3t)] \right|_{t=0} = \langle \nabla^2 f(2t, 3t)(2, 3)^T, (2, 3)^T \rangle = \langle (\frac{3}{3} \frac{3}{3}), (\frac{3}{3}) \rangle = 13(2) + 6(3) = 44 \).

Exercise (5). Let \( a, b, c \) be real numbers with \( a \neq 0 \), and define \( p(t) = at^2 + 2bt + c \).

(a) Show that \( p(t) > 0 \) for every number \( t \) if and only if \( a > 0 \) and \( ac - b^2 > 0 \).

(b) Show that \( p(t) < 0 \) for every number \( t \) if and only if \( a < 0 \) and \( ac - b^2 > 0 \).
Solution (5). Well, \( p(t) \) is a polynomial and is hence an analytic function (infinitely differentiable), and so we take the first derivative and get \( p'(t) = 2at + 2b \), and using Theorem 4.22 we set this equal to 0 in order to find minimizers and maximizers. Thus \( 2at + 2b = 0 \) implies \( t = -b/a \). Taking the second derivative we find that \( p''(t) = a \). So, if \( a > 0 \) \( t = -b/a \) is maximum, and if \( a < 0 \), \( t = -b/a \) is the minimum. We are now in a position to prove (a) and (b).

(a) \((\Rightarrow)\) Suppose \( p(t) > 0 \) for every \( t \in \mathbb{R} \). Also suppose by way of contradiction that \( a \leq 0 \). Then there is no minimum of \( p(t) \) and so for large \( t \), \( p(t) < 0 \), a contradiction. Also, \( p(-b/a) = a(-b/a)^2 + 2b(-b/a) + c = -b^2/a + c > 0 \), and so \( ac - b^2 > 0 \) since \( a \) is positive. \((\Leftarrow)\) Suppose \( a > 0 \), \( b^2 - ac > 0 \). Then the minimum of \( p(t) \) occurs at \(-b/a\), and since \( a \) is positive, \( ac - b^2 > 0 \) implies that \( 0 < c - b^2/a = p(-b/a) \leq p(t) \) for all \( t \), and hence \( p(t) > 0 \), as desired.

(b) \((\Rightarrow)\) Suppose \( p(t) > 0 \) for every \( t \in \mathbb{R} \). Also suppose by way of contradiction that \( a \geq 0 \). Then there is no maximum of \( p(t) \) and so for large \( t \), \( p(t) > 0 \), a contradiction. Also, \( p(-b/a) = a(-b/a)^2 + 2b(-b/a) + c = -b^2/a + c > 0 \), and so \( ac - b^2 > 0 \) since \( a \) is negative. \((\Leftarrow)\) Suppose \( a < 0 \), \( b^2 - ac > 0 \). Then the maximum of \( p(t) \) occurs at \(-b/a\), and since \( a \) is negative, \( ac - b^2 > 0 \) implies that \( 0 > c - b^2/a = p(-b/a) \geq p(t) \) for all \( t \), and hence \( p(t) < 0 \), as desired.

Exercise (7). For each of the following quadratic functions, find a 2 by 2 matrix with which it is associated.

(a) \( h(x, y) = x^2 - y^2 \) for \( (x, y) \in \mathbb{R}^2 \)

(b) \( g(x, y) = x^2 + 8xy + y^2 \) for \( (x, y) \in \mathbb{R}^2 \)

Solution (7). (a) The associated symmetric matrix is

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

(b) The associated symmetric matrix is

\[
\begin{pmatrix}
1 & 4 \\
4 & 1
\end{pmatrix}
\]

Exercise (8). By making a suitable choice of the matrix \( A \), show that the Generalized Cauchy-Schwarz Inequality contains the standard Cauchy-Schwarz Inequality as a special case.

Solution (8). Choose the \( n \times n \) matrix

\[
A = \begin{pmatrix}
v_1 & \cdots & v_n \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\]
Then the generalized Cauchy-Schwarz Inequality says that \( ||Au|| \leq ||A|| \cdot ||u|| \), but

\[
||Au|| = \begin{vmatrix} v_1 u_1 + \cdots + v_n u_n \\ 0 \\ \vdots \\ 0 \end{vmatrix} = \sqrt{(v_1 u_1 + \cdots + v_n u_n)^2} = \langle v, u \rangle
\]

and

\[
||A|| = \begin{vmatrix} v_1 & \cdots & v_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{vmatrix} = \sqrt{\sum_i v_i^2} = ||v||
\]

So, \( \langle u, v \rangle = ||Au|| \leq ||A|| \cdot ||u|| = ||v|| \cdot ||u|| \), as desired.