Exercise (2). Define \( F(x, y, z) = (xyz, x^2 + yz, 1 + 3x) \) for \((x, y, z) \in \mathbb{R}^3\). Find the derivative matrix of the mapping \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) at the points \((1, 2, 3), (0, 1, 0), \) and \((-1, 4, 0)\).

Solution (2). We let \( F(x, y, z) = (u(x, y, z, v(x, y, z), w(x, y, z)) \) where \( u = xyz, v = x^2 + yz, w = 1 + 3x \). Taking partial derivatives, we have \( u_x = yz, u_y = z, u_z = x, v_x = 2x, v_y = z, v_z = y, w_x = 3, w_y = 0, w_z = 0 \), so the general derivative matrix is

\[
DF(x, y, z) = \begin{pmatrix}
yz & xz & xy \\
2x & z & y \\
3 & 0 & 0
\end{pmatrix}
\]

Evaluating at the given points, we have

\[
DF(1, 2, 3) = \begin{pmatrix} 6 & 3 & 2 \\ 2 & 3 & 2 \\ 3 & 0 & 0 \end{pmatrix},
DF(0, 1, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix},
DF(-1, 4, 0) = \begin{pmatrix} 0 & 0 & -4 \\ -2 & 0 & 4 \\ 3 & 0 & 0 \end{pmatrix}
\]

Exercise (4). Suppose that \( A \) is an \( m \times n \) matrix. Define the mapping \( F : \mathbb{R}^n \to \mathbb{R}^m \) by \( F(x) = Ax \) for every \( x \in \mathbb{R}^n \). Prove that \( DF(x) = A \) for all \( x \in \mathbb{R}^n \).

Solution (4). We use Theorem 15.32. In order to do so, we must show that \( \lim_{h \to 0} \frac{|F(x+h) - [F(x) + Ah]|}{||h||} = 0 \). Using the fact that \( F(x) = Ax \forall x \in \mathbb{R}^n \), we have that \( \lim_{h \to 0} \frac{|F(x+h) - [F(x) + Ah]|}{||h||} = \lim_{h \to 0} \frac{|Ax + Ah - Ax - Ah|}{||h||} = \lim_{h \to 0} 0/||h|| = 0 \). Therefore, by the conclusion of Theorem 15.32, \( DF(x) = A \).

Exercise (6). Define the mapping \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( F(x, y) = (x^2 - y^2, 2xy) \) for \((x, y) \in \mathbb{R}^2\).
(a) Find the points \((x_0, y_0) \in \mathbb{R}^2\) at which the derivative matrix \(DF(x_0, y_0)\) is invertible.

(b) Find the points \((x_0, y_0) \in \mathbb{R}^2\) at which the differential \(dF(x_0, y_0) : \mathbb{R}^2 \to \mathbb{R}^2\) is invertible linear mapping.

**Solution (6).** The idea here is to use Theorem 15.33 to find the equivalent assertions that \(DF(x, y), dF(x, y)\) are invertible. We dive in and take derivatives:

(a) 

\[
DF(x, y) = \begin{pmatrix}
2x & -2y \\
2y & 2x
\end{pmatrix}
\]

By Theorem 15.33, this is invertible only when \(\det DF(x, y) \neq 0\). Well, \(\det DF(x, y) = 4x^2 + 2y^2 = 0\) iff \(x = y = 0\). This means that \(DF(x, y)\) is invertible on the set \(\mathbb{R}^2 \setminus \{0\}\).

(b) Now, Theorem 15.33 says that \(DF(x, y)\) invertible \(\iff dF(x, y)\) is an invertible linear mapping. So this means that \(dF(x, y)\) should be an invertible linear mapping on exactly the same set that \(DF(x, y)\) is invertible. Thus, \(dF(x, y)\) is an invertible linear map on \(\mathbb{R}^2 \setminus \{0\}\).

**Exercise (8).** Suppose that the mapping \(F : \mathbb{R}^n \to \mathbb{R}^n\) is continuously differentiable. Suppose also that \(F(0) = 0\) and that the derivative matrix \(DF(0)\) has the property that there is some number \(c > 0\) such that \(||DF(0)h|| \geq c||h||\) for all \(h \in \mathbb{R}^n\). Prove that there is some positive number \(r\) such that \(||F(h)|| \geq c/2||h||\) if \(||h|| \leq r\).

**Solution (8).** Choose \(\epsilon = c/2\). By Theorem 15.31, there exists \(r > 0\) such that \(||h|| \leq r\), then

\[
\frac{||F(h) - [F(0) + DF(0)h]||}{||h||} \leq \epsilon = \frac{c}{2}.
\]

We get by rearranging and the reverse triangle inequality that

\[
||DF(0)h|| - ||F(h)|| \leq ||F(h) - DF(0)h|| \leq \frac{c}{2}||h||,
\]

which gives us

\[
||F(h)|| \geq ||DF(0)h|| - \frac{c}{2}||h|| \geq c||h|| - \frac{c}{2}||h|| = \frac{c}{2}||h||,
\]

as desired.